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Higher derivations of algebraic function fields

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HIGHER DERIVATIONS OF ALGEBRAIC FUNCTION FIELDS

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Let A be a commutative k-algebra with unity 1 where k is a field of characteristic zero. A higher derivation on A is a sequence $\underline{D} = \{D_0, D_1, D_2, \dots\}$ of endomorphisms of the k-module A where D_0 is the identity map of A and $D_n(ab) = \sum_{m=0}^{n} D_m(a)D_{n-m}(b)$ for every $a, b \in A$. Let $\phi_{\underline{D}}$ be the embedding of A into $A[[t]]$ the ring of formal power series in indeterminate t induced by the higher derivation $\underline{D}$ and defined by $\phi_{\underline{D}}(a) = \sum_{n=0}^{\infty} D_n(a)t^n$ for every $a \in A$. We say that two higher derivations $\underline{D} = \{D_n \colon n \geq 0\}$, and $\underline{E} = \{E_n \colon n \geq 0\}$ are equivalent if and only if $\sigma\phi_{\underline{E}} = \phi_{\underline{D}}$ for some A-automorphism σ of $A[[t]]$. In a previous paper [1] the author proved that there is a one to one correspondence between the set of all higher derivations on A and the set of ordered sequences of first order derivations on A. In this paper we prove that every higher derivation $\underline{D} = \{D_n \colon n \geq 0, D_1 \neq 0\}$ of an algebraic function field K of transcendency degree one over a field k of characteristic zero is equivalent to the higher derivation $\underline{\nabla} = \{(1/n!)\delta^n \colon n \geq 0\}$ where δ is the unique extension to K of the ordinary derviation d/dx of $k(x)$.

Our terminology is essentially the same as that of [1].

Definition. *Two higher derviations* $\underline{D} = \{D_n \colon n \geq 0\}$ *and* $\underline{E} = \{E_n \colon n \geq 0\}$ *on* A *are said to be equivalent if and only if there exists an* A*-automorphism* σ *of* $A[[t]]$ *such that* $\sigma\phi_{\underline{E}} = \phi_{\underline{D}}$ *where* $\phi_{\underline{E}}$, $\phi_{\underline{D}}$ *are the embeddings of* A *into* $A[[t]]$ *induced respectively by* $\underline{E}$ *and* $\underline{D}$.

Theorem 1([1]). *There is a one to one correspondence between the set of ordered sequences of* k*-derivations on* A *of order one and the set of higher derivations on* A *in such a way that if* $\{\delta_n \colon n \geq 0, \delta_0$ *identity,* δ_n *is a* k*-derivation of order one*$\}$ *and the higher derivation* $\underline{D} = \{D_n \colon n \geq 0\}$ *correspond to each other, then*

$$D_n = \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1,\dots,n_r) \in P_{n,r}} \delta_{n_1}\delta_{n_2}\cdots\delta_{n_r}$$

and

$$\delta_n = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{(n_1,\dots,n_r)\in P_{n,r}} D_{n_1} D_{n_2} \cdots D_{n_r}$$

for every $n \geq 1$ and $D_0 = \delta_0$, where $P_{n,r}$ stands for the set of ordered partitions of n into r-positive integers.

Theorem 2. *If K is an algebraic function field of transcendency degree one over a field k of characteristic zero, then every higher derivation $\underline{D} = \{D_n: n \geq 0,\ D_1 \neq 0\}$ on K is equivalent to the higher derivation $\underline{\nabla} = \{(1/n!)\delta^n: n \geq 0\}$ where δ is the unique extension to K of the ordinary derivation d/dx of $k(x)$.*

Proof. By Theorem 1, let $\{\delta_n: n \geq 0\}$ be the sequence of k-derivations on K of order one associated to $\underline{D}$. Hence

$$D_n = \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1,\dots,n_r)\in P_{n,r}} \delta_{n_1} \delta_{n_2} \cdots \delta_{n_r}$$

but since K is a simple separable extension of $k(x)$ we have by [2, Theorem 4.3.10]

$$\delta_{n_i} = C_{n_i}\delta \text{ for some } C_{n_i} \in K$$

then by Leibniz Rule we get for $r \geq 1$,

$$\delta_{n_1}\delta_{n_2}\cdots\delta_{n_r} = \sum_{q=1}^{r} \Big[\sum_{(q_{0,1},q_{0,2},\dots,q_{0,r})\in B_{r,q}} \times Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) C_{n_1}^{q_{0,1}} C_{n_2}^{q_{0,2}} C_{n_r}^{q_{0,r}}\Big]\delta^q$$

where $B_{r,q} = \{(q_{0,1}, q_{0,2}, \dots, q_{0,r}): q_{0,i}$'s are non-negative integers such that $q_{0,1}+q_{0,2}+\cdots+q_{0,r} = r-q$ and $q_{0,1}+\cdots+q_{0,i} \leq \min(i-1, r-q)$ for every $1 \leq i \leq r\}$ and $Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) = \binom{0}{q_{0,1}}\binom{1-q_{0,1}}{q_{0,2}}\binom{2-q_{0,1}-q_{0,2}}{q_{0,3}}\cdots \binom{r-1-q_{0,1}-\cdots-q_{0,r-1}}{q_{0,r}}$ and $C_{n_i}^{q_{0,i}} = \delta^{q_{0,i}}(C_{n_i})$, $\delta^0(C_{n_i}) = C_{n_i}$.

Notice that $q_{0,1} = 0$ and if $\lambda = (q'_{0,1}, \dots, q'_{0,m_0}, \dots, q'_{s,1}, \dots, q'_{s,m_s}, \dots, q'_{t,1}, \dots, q'_{t,m_t})$ is a permutation of $(q_{0,1}, q_{0,2}, \dots, q_{0,r})$ such that $\sum_{s=0}^{t} m_s = r$, $q'_{s,j} = q'_{s,j+1}$ and $q'_{s,j} < q'_{s+1,j}$ for every $0 \leq s \leq t$ then also $\lambda \in B_{r,q}$.

Let

$$a_n = \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1,\dots,n_r)\in P_{n,r}} \sum_{(q_{0,1},q_{0,2},\dots,q_{0,r})\in B_{r,1}} \times Q(q_{0,1}, q_{0,2}, \dots, q_{0,r}) C_{n_1}^{q_{0,1}} C_{n_2}^{q_{0,2}} \cdots C_{n_r}^{q_{0,r}}$$

for $n \geq 1$. Then we get

$$D_n = \alpha_n\delta + \sum_{q=2}^{n}\left[\sum_{r=q}^{n}\frac{1}{r!}\sum_{(n_1,\dots,n_r)\in P_{n,r}}\sum_{(q_{0,1},q_{0,2},\dots,q_{0,r})\in B_{r,q}} \times Q(q_{0,1},q_{0,2},\dots,q_{0,r})C_{n_1}^{q_{0,1}}C_{n_2}^{q_{0,2}}\cdots C_{n_r}^{q_{0,r}}\right]\delta^q$$

for $n \geq 2$.

Define

$$\sigma(t) = \sum_{n=1}^{\infty}\alpha_n t^n.$$

Since $D_1 = \delta_1 = c_1\delta = \alpha_1\delta \neq 0$ hence $\alpha_1 \neq 0$. It is easily seen that σ can be extended to a K-automorphism of $K[t]$, and that, for such σ, we have $\sigma\phi_{\underline{\nabla}} = \phi_{\underline{D}}$ if and only if

$$D_n = \sum_{q=1}^{n}\frac{1}{q!}\left[\sum_{(n_1,\dots,n_q)\in P_{n,q}}\alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_q}\right]\delta^q.$$

Hence to prove the assertion it is sufficient to show that

$$\sum_{r=q}^{n}\frac{1}{r!}\sum_{(n_1,\dots,n_r)\in P_{n,r}}\sum_{(q_{0,1},q_{0,2},\dots,q_{0,r})\in B_{r,q}} \times Q(q_{0,1},q_{0,2},\dots,q_{0,r})C_{n_1}^{q_{0,1}}C_{n_2}^{q_{0,2}}\cdots C_{n_r}^{q_{0,r}} = \frac{1}{q!}\sum_{(n_1,\dots,n_q)\in P_{n,q}}\alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_q} \text{ for every } q \geq 1.$$

For convenience let $B_{i_j,1} = \{(q'_{j,1},\dots,q'_{j,i_j}) : q'_{j,e}$'s are non-negative integers such that $q'_{j,1}+\cdots+q'_{j,i_j} = i_j - 1$ and $q'_{j,1}+\cdots+q'_{j,e} \leq \min(e-1, i_j-1)$ for every $1 \leq e \leq i_j\}$ and

$$Q_{i_j} = Q(q'_{j,1},\dots,q'_{j,i_j})$$

$$R_{i_j} = C_{n_{j1}}^{q'_{j,1}}C_{n_{j2}}^{q'_{j,2}}\cdots C_{n_{ji_j}}^{q'_{j,i_j}}.$$

It is easily seen that for every $q \geq 1$, $n \geq 2$ we have

$$\sum_{(n_1,\dots,n_q)\in P_{n,q}}\alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_q} = \sum_{(n_1,\dots,n_q)\in P_{n,q}}C_{n_1}C_{n_2}\cdots C_{n_q} + \sum_{r=q+1}^{n}\sum_{(n_1,\dots,n_q)\in P_{n,q}}\sum_{(i_1,\dots,i_q)\in P_{r,q}} \times \frac{1}{i_1!\,i_2!\cdots i_q!}\sum_{\substack{(n_{j1},\dots,n_{ji_j})\in P_{n_j,i_j} \\ (q'_{j,1},\dots,q'_{j,i_j})\in B_{i_j,1} \\ \text{for every } 1\leq j\leq q}}(Q_{i_1}\cdots Q_{i_q}R_{i_1}\cdots R_{i_q}).$$

Hence it is sufficient to show that

$$\frac{1}{r!}\sum_{(n_1,\dots,n_r)\in P_{n,r}}\sum_{(q_{0,1},\dots,q_{0,r})\in B_{r,q}} \times Q(q_{0,1},\dots,q_{0,r})C_{n_1}^{q_{0,1}}C_{n_2}^{q_{0,2}}\cdots C_{n_r}^{q_{0,r}}$$
$$=\frac{1}{q!}\sum_{(n_1,\dots,n_q)\in P_{n,q}}\sum_{(i_1,\dots,i_q)\in P_{r,q}} \times\sum_{\substack{(n_{j1},\dots,n_{ji_j})\in P_{n_j,i_j}\\ (q'_{j,1},\dots,q'_{j,i_j})\in B_{i_j,1}\\ \text{for every } 1\le j\le q}}\left(\frac{1}{i_1!\cdots i_q!}Q_{i_1}\cdots Q_{i_q}R_{i_1}\cdots R_{i_q}\right)$$

for every $1\le q\le r\le n$.

It is clear that both sides of this relation have the same number of terms and that they are equal for $q=1$, $r\ge 1$ and for $q=r$.

Let

$$\lambda=(q_{0,1},\dots,q_{0,m_0},\dots,q_{s,1},\dots,q_{s,m_s},\dots,q_{t,1},\dots,q_{t,m_t})\in B_{r,q}$$

such that $\sum_{s=0}^{t}m_s=r$, $q_{s,j}=q_{s,j+1}$ and $q_{s,j}<q_{s+1,j}$ for every $0\le s\le t$, and

$$[\lambda]=\{\mu\in B_{r,q}:\ \mu \text{ is a permutation of } \lambda\}.$$

Then for any $(n_{0,1},\dots,n_{t,m_t})\in P_{n,r}$ it is seen without essential difficulty that the coefficient of $C_{n_{0,1}}^{q_{0,1}}C_{n_{0,2}}^{q_{0,2}}\cdots C_{n_{t,m_t}}^{q_{t,m_t}}$ in the left side of this relation after collecting similar terms is

$$\frac{1}{r!}\sum_{\mu\in[\lambda]}L_1L_2\cdots L_tQ(\mu)$$

where L_s is the number of permutations of $(n_{s,1},\dots,n_{s,m_s})$ for every $0\le s\le t$, and the coefficient of the same expression in the right side of this relation is

$$\frac{1}{q!}\left[\sum_{\substack{(q'_{j,1},\dots,q'_{j,i_j})\in B_{i_j,1}\\ \text{for every } 1\le j\le q \text{ such that}\\ (q'_{1,1},\dots,q'_{1,i_1},\dots,q'_{q,1},\dots,q'_{q,i_q})\in[\lambda]\\ \text{and } (i_1,\dots,i_q)\in P_{r,q}}}\left(L_1L_2\cdots L_t\cdot\frac{Q_{i_1}Q_{i_2}\cdots Q_{i_q}}{(i_1)!(i_2)!\cdots(i_q)!}\right)\right].$$

Let $\mu\in[\lambda]$ which can be written in the form $\mu=(\mu_{i_1},\dots,\mu_{i_p})$ for some positive integers, $i_1<i_2<\cdots<i_p$ where $\mu_{i_j}=(\lambda^j_{1,i_j}$ repeated n_{j1} times,..., $\lambda^j_{s_j,i_j}$ repeated n_{js_j} times) and

$$\lambda^j_{k,i_j}=(q^j_{k,1},\dots,q^j_{k,i_j})\in B_{i_j,1} \text{ for every } 1\le j\le p \text{ and } 1\le k\le s_j$$

such that $\sum_{j=1}^{p}\sum_{k=1}^{s_j} n_{jk} = q$, $\sum_{j=1}^{p}[\sum_{k=1}^{s_j} n_{jk}]\, i_j = r$ and $\lambda^j_{k_1, i_j} \neq \lambda^j_{k_2, i_j}$ for every $1 \leq j \leq p$ and $1 \leq k_1 \neq k_2 \leq s_j$ and such that $q^j_{k,e}$'s in each λ^j_{k,i_j} are ordered in the same way as in λ.

Let $\bar{\mu} = \{\mu' \in [\lambda] : \mu'$ is obtained from μ by a permutation of λ^j_{k,i_j}'s in $\mu\}$.

It is clear that such a μ exists and $|\bar{\mu}| = \dfrac{q!}{\prod_{j=1}^{p}\prod_{k=1}^{s_j}(n_{jk})!}$. For simplicity we set

$$f(r, q;\ \lambda) = \sum_{\mu \in |\lambda|} Q(\mu)$$

and

$$N_j = \sum_{k=1}^{s_j} n_{jk}$$

and

$$g(r, q;\ \lambda) = \frac{r!}{q!} \sum_{\substack{(q'_{j,1}, \ldots, q'_{j,i_j}) \in B_{i_j,1} \\ \text{for every } 1 \leq j \leq q \\ \text{such that} \\ (q'_{1,1}, \ldots, q'_{1,i_1}, \ldots, q'_{q,i_q}) \in |\lambda|}} \left(\frac{1}{(i_1)! \cdots (i_p)!} \cdot Q_{i_1} Q_{i_2} \cdots Q_{i_q} \right).$$

Then it is easily seen that

$$g(r, q;\ \lambda) = \frac{r!}{q!} \sum_{\substack{\mu \in |\lambda| \\ \text{corresponding to} \\ \text{distinct classes } \bar{\mu}}} \left[\frac{1}{(i_1!)^{N_1} \cdots (i_p!)^{N_p}} \cdot \frac{q!}{\prod_{j=1}^{p}\prod_{k=1}^{s_j}(n_{jk})!} \cdot \prod_{k=1}^{p}\prod_{k=1}^{s_j} (f(i_{j,1};\ \lambda^j_{k,i_j}))^{n_{jk}} \right].$$

Hence to prove the assertion it is sufficient to show that $f(r, q;\ \lambda) = g(r, q;\ \lambda)$ for every $r \geq q \geq 1$ and $\lambda \in B_{r,q}$.

We prove this equality by induction on r.

(1) It is clear that the equality holds for $q = 1$, $r \geq 1$ and for $q = r$ and that $f(r, r;\ \lambda) = g(r, r;\ \lambda) = 1$.

(2) We prove the following two lemmas.

Lemma 1. *Let $\lambda_s \in B_{r-1, q_{s,1}+q-1}$ obtained by deleting $q_{s,1}$ from λ, then we have*

$$f(r, q;\ \lambda) = \sum_{s=0}^{t} \binom{q_{s,1}+q-1}{q_{s,1}} f(r-1, q_{s,1}+q-1;\ \lambda_s)$$

for every $q \geq 2$ and $f(r, 1;\ \lambda) = \sum_{s=1}^{t} f(r-1, q_{s,1};\ \lambda_s)$ for $r > 1$.

Proof. It follows easily from the definition of $Q(\lambda_s)$.

Lemma 2. *Let* $\gamma^e=(q^e_{0,1},\dots,q^e_{0,h_0},\dots,q^e_{s',1},\dots,q^e_{s',h_{s'}},\dots,q^e_{\iota,1},\dots,q^e_{\iota,h_\iota})$ $\in B_{e,q-1}$ *and* $\gamma^{r-e}=(q^{r-e}_{0,1},\dots,q^{r-e}_{0,z_0},\dots,q^{r-e}_{s',1},\dots,q^{r-e}_{s',z_{s'}},\dots,q^{r-e}_{\nu,1},\dots,q^{r-e}_{\nu,z_\nu})\in$ $B_{r-e,1}$ *where* $r>e\geq q-1$ *and the* $q^e_{0,i}$*'s in* γ^e *and* $q^{r-e}_{0,i}$*'s in* γ^{r-e} *are ordered in the same way as in* λ *and such that* $(\gamma^e,\gamma^{r-e})=(q^e_{0,1},\dots,q^e_{\iota,h_\iota},$ $q^{r-e}_{0,1},\dots,q^{r-e}_{\nu,z_\nu})\in[\lambda]$. *Then we have*

$$g(r,q;\lambda)=\sum_{\substack{(e,r-e)\in P_{r,2}\\ \text{such that there exist}\\ \gamma^e\in B_{e,q-1}\text{ and}\\ \gamma^{r-e}\in B_{r-e,1}\text{ with }(\gamma^e,\gamma^{r-e})\in[\lambda]}} \times\sum_{\substack{\gamma^e\in B_{e,q-1}\\ \gamma^{r-e}\in B_{r-e,1}\\ \text{such that}\\ (\gamma^e,\gamma^{r-e})\in[\lambda]}}\binom{r-1}{e}g(e,q-1;\gamma^e)f(r-e,1;\gamma^{r-e}).$$

Proof. Notice that the expression $\prod_{j=1}^p\prod_{k=1}^{s_j}[f(i_j,1;\lambda^j_{k,i_j})]^{n_{jk}}$ appears in the right side when $r-e=i_j$ and $\gamma^{r-e}=\lambda^j_{k,i_j}$ and γ^e is obtained from μ by deleting λ^j_{k,i_j} for every $1\leq j\leq p$ and $1\leq k\leq s_j$. Hence its coefficient is

$$\sum_{j=1}^p\frac{(r-1)!}{(r-i_j)!(i_j-1)!}\cdot\frac{(r-i_j)!}{(q-1)!}\cdot\frac{(i_j)!}{(i_1!)^{N_1}\cdots(i_p!)^{N_p}}\cdot\frac{\sum_{k=1}^{s_j}n_{jk}\cdot(q-1)!}{\prod_{j=1}^p\prod_{k=1}^{s_j}(n_{jk})!}$$
$$=\frac{(r-1)!}{(i_1!)^{N_1}\cdots(i_p!)^{N_p}\prod_{j=1}^p\prod_{k=1}^{s_j}(n_{jk})!}\cdot\sum_{j=1}^p\left[\sum_{k=1}^{s_j}n_{jk}\right]i_j$$
$$=\frac{r!}{(i_1!)^{N_1}\cdots(i_p!)^{N_p}\prod_{j=1}^p\prod_{k=1}^{s_j}(n_{jk})!}$$

$=$the coefficient of the same expression in $g(r,q;\lambda)$.

(3) By Lemma 1 and induction hypothesis we have

$$f(r,q;\lambda)=\sum_{s=0}^t\binom{q_{s,1}+q-1}{q_{s,1}}g(r-1,q_{s,1}+q-1;\lambda_s).$$

On the other hand by Lemma 2 we have

$$g(r,q;\lambda)=\sum_{\substack{(e,r-e)\in P_{r,2}\\ \text{such that there exist}\\ \gamma^e\in B_{e,q-1}\text{ and}\\ \gamma^{r-e}\in B_{r-e,1}\text{ with }(\gamma^e,\gamma^{r-e})\in[\lambda]}} \times\sum_{\substack{\gamma^e\in B_{e,q-1}\\ \gamma^{r-e}\in B_{r-e,1}\\ \text{such that}\\ (\gamma^e,\gamma^{r-e})\in[\lambda]}}\binom{r-1}{e}g(e,q-1;\gamma^e)f(r-e,1;\gamma^{r-e}).$$

Since the term in $f(r, q; \lambda)$ corresponding to $s=0$ is equal to the term in $g(r, q; \lambda)$ corresponding to $r-e=1$ and by Lemma 2 and induction hypothesis we have

$$f(r-e, 1; \gamma^{r-e}) = \sum_{S'=1}^{\nu} g(r-e-1, q_{S',1}^{r-e}; \gamma_{S'}^{r-e}) \text{ for } r-e>1$$

where $\gamma_{S'}^{r-e}$ is obtained from γ^{r-e} by deleting $q_{S',1}^{r-e}$.

Hence it is sufficient to show that

$$\sum_{S=1}^{t} \binom{q_{s,1}+q-1}{q_{s,1}} g(r-1, q_{s,1}+q-1; \lambda_s)$$
$$= \sum_{\substack{(e,\, r-e-1) \in P_{r-1,2} \\ \text{such that there exist } \gamma^e \in B_{e,q-1} \text{ and} \\ \gamma_{S'}^{r-e} \in B_{r-e-1,\, q_{S',1}^{r-e}} \text{ with } (\gamma^e,\, \gamma^{r-e}) \in |\lambda|}} \times \sum_{\substack{\gamma^e \in B_{e,q-1} \\ \gamma_{S'}^{r-e} \in B_{r-e-1,\, q_{S',1}^{r-e}} \\ \text{such that } (\gamma^e,\, \gamma^{r-e}) \in |\lambda|}} \left[\sum_{S'=1}^{\nu} \binom{r-1}{e} g(e, q-1; \gamma^e) \cdot g(r-e-1, q_{S',1}^{r-e}; \gamma_{S'}^{r-e}) \right].$$

Since

$$g(r-1, q_{s,1}+q-1; \lambda_s) = \frac{(r-1)!}{(q_{s,1}+q-1)!} \sum_{\substack{\mu \in |\lambda_s| \\ \text{corresponding to} \\ \text{distinct classes } \bar{\mu}}} \frac{1}{(i_1!)^{M_1} \cdots (i_{p'}!)^{M_{p'}}} \cdot \frac{(q_{s,1}+q-1)!}{\prod_{j=1}^{p'} \prod_{k=1}^{s'_j} (n'_{jk})!} \cdot \prod_{j=1}^{p'} \prod_{k=1}^{s'_j} [f(i_j, 1; \Omega_{k,i_j}^{j})]^{n'_{jk}}$$

where $\sum_{j=1}^{p'} \sum_{k=1}^{s'_j} n'_{jk} = q_{s,1}+q-1$, $\sum_{j=1}^{p'} [\sum_{k=1}^{s'_j} n'_{jk}] i_j = r-1$ and $M_j = \sum_{k=1}^{s'_j} n'_{jk}$.

Let T be the collection of all factors in the expression $\prod_{j=1}^{p'} \prod_{k=1}^{s'_j} [f(i_j, 1; \Omega_{k,i_j}^{j})]^{n'_{jk}}$ and for each $1 \leq s \leq t$ let $T_{q_{s,1}}$ and T_{q-1} be a partition of T into two collections containing respectively $q_{s,1}$ and $q-1$ factors. It is clear that if $P(T)$ is the number of permutations of T then we have

$$P(T) = \sum_{T_{q_{s,1}} \cup T_{q-1} = T} P(T_{q_{s,1}}) \cdot P(T_{q-1}) = \frac{(q_{s,1}+q-1)!}{\prod_{j=1}^{p'} \prod_{k=1}^{s'_j} (n'_{jk})!}.$$

Next the assertion follows easily from the fact that for each $1 \leq s \leq t$ the coefficient of the expression $\prod_{j=1}^{p} \prod_{k=1}^{s'_j} [f(i_j, 1; \Omega_{k,i_j}^{j})]^{n'_{jk}}$ in the left side is

$$= \frac{(r-1)!}{(q-1)!(q_{s,1})!} \cdot \frac{1}{(i_1!)^{M_1} \cdots (i_{p'}!)^{M_{p'}}} \cdot P(T).$$

While the coefficient of the same expression in the right side is

$$= \sum_{T_{q_{s,1}} \cup T_{q-1} = T} \left[\frac{(r-1)!}{e!(r-e-1)!} \cdot \frac{e!}{(q-1)!} \cdot \frac{(r-e-1)!}{(q_{s,1})!} \cdot \frac{1}{(i_1!)^{M_1} \cdots (i_{p'}!)^{M_{p'}}} \cdot P(T_{q_{s,1}}) \cdot P(T_{q-1}) \right]$$

$$= \frac{(r-1)!}{(q-1)!(q_{s,1})!} \cdot \frac{1}{(i_1!)^{M_1} \cdots (i_{p'}!)^{M_{p'}}} \sum_{T_{q_{s,1}} \cup T_{q-1} = T} P(T_{q_{s,1}}) \cdot P(T_{q-1}).$$

Which is obtained by noticing that for each possible value of e we have $q_{s,1} = q_{s',1}^{r-e}$ for some s'.

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