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Almost Hermitian Geometry, Geodesic Spheres and Symmetries

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ALMOST HERMITIAN GEOMETRY, GEODESIC SPHERES AND SYMMETRIES

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1. Introduction. The geometric properties of small geodesic spheres on a Riemannian manifold (M, g) influence strongly the geometry of this ambient space, and conversely. Many examples are known and we refer to [1], [9] for a collection of results. The main purpose of this paper is to give some new examples for the case where (M, g) is an almost Hermitian manifold (M, g, J). More precisely, let G be a small geodesic sphere and let N be a unit normal vector field of G. Then JN is a distinguished tangent vector field of G. Our aim is to investigate some of the properties of this vector field. For example, we will consider the integral curves of JN and also the action of the shape operator and the Ricci operator of G on the field JN. This leads to new characterizations of locally Hermitian symmetric spaces, nearly Kähler manifolds of constant holomorphic sectional curvature and complex space forms. Moreover, the investigation of several symmetric properties gives rise to some new theorems about 3-symmetric spaces and s-regular manifolds.

To derive our results we mainly work with Jacobi vector fields and their use to study the geometry of a normal neighborhood. For a detailed survey about this theory we refer to [9] and the detailed reference list included there may serve to get more information. In particular we also refer to [1]. From all this information and from the results we will prove in this paper, it follows that there is a great analogy between the properties of the shape operator (extrinsic geometry) and the properties of the Ricci operator (intrinsic geometry) of a geodesic sphere.

In the next section we collect some formulas and results which will be needed to prove our theorems in almost Hermitian geometry and for s-regular manifolds.

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2. Preliminaries. Let (M, g) be an n-dimensional Riemannian manifold of class C^∞ which we suppose to be connected. Denote by $\mathfrak{T}_q^p$ the algebra of smooth tensor fields on M with contravariant and covariant orders p and

q, respectively. In particular, we put $\mathfrak{T}_q^0 = \mathfrak{T}_q$ and $\mathfrak{T}_0^p = \mathfrak{T}^p$. Further, let ∇ denotes the Riemannian connection and R the corresponding Riemannian curvature tensor field. The curvature operator R_{XY} is defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all $X, Y \in \mathfrak{T}^1$. ρ denotes the $(0, 2)$-Ricci tensor field and τ is the associated scalar curvature.

Let $m \in M$ and γ a geodesic parametrized by arc length r such that $\gamma(0) = m$, $\gamma'(0) = u$. Moreover, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_mM at m with $u = e_1$ and extend this basis to a parallel basis $\{E_1, \ldots, E_n\}$ along the geodesic γ. In the rest of the paper we will always suppose that we work in a sufficiently small neighborhood of m in order to have a diffeomorphic exponential map $\exp_m$ centered at m. Often we will work in a neighborhood of m which is a normal neighborhood of each of its points, without mentioning it explicitly.

A vector field Y along γ is called a *Jacobi vector field* if and only if

$$Y'' + R_{\gamma' Y}\gamma' = 0. \tag{1}$$

We consider the $n-1$ Jacobi vector fields Y_a, $a = 2, \ldots, n$, along γ, determined by the initial conditions

$$Y_a(0) = 0, \ Y_a'(0) = e_a, \ a = 2, \ldots, n. \tag{2}$$

For sufficiently small r, the vectors $Y_a(r)$ determine a basis for the space $\{\gamma'(r)\}^\perp$. Next, put

$$Y_a(r) = (AE_a)(r). \tag{3}$$

Then $r \mapsto A(r)$ is an endomorphism-valued function. Each $A(r)$ is an endomorphism of the space $\{\gamma'(r)\}^\perp$ and these spaces may be identified via the parallel translation along γ by using the parallel basis $\{E_i\}$. We will often do this without mentioning it explicitly. Substituting (3) in (1) and using (2), we obtain the following matrix-valued *Jacobi differential equation*

$$A'' + R \circ A = 0 \tag{4}$$

with initial conditions

$$A(0) = 0, \ A'(0) = I. \tag{5}$$

Here R denotes the *Jacobi endomorphism* or the *Jacobi operator* $Y \mapsto R_{\gamma' Y}\gamma'$ along γ.

Let $G_m(r) = \exp_m S_m(r)$, where $S_m(r)$ denotes the sphere with radius r and center m in T_mM. $G_m(r)$ is the *geodesic sphere* with center m and radius r. For small r it is a nice hypersurface of (M, g). The extrinsic geometry of $G_m(r)$ is described by the *shape operator* T_m. Since $\frac{\partial}{\partial r}(\gamma(r))$ is a unit normal vector of $G_m(r)$ at $p = \gamma(r) = \exp_m(ru)$, T_m is defined by

$$(6) \qquad T_m(p)X = \left(\nabla_X \frac{\partial}{\partial r}\right)(p)$$

where X is tangent to $G_m(r)$ at p. Next, since the Jacobi vector fields Y_a, $a = 2, \ldots, n$, are tangent to the geodesic spheres $G_m(r)$, (6) leads to

$$(7) \qquad T_m Y_a = \nabla_{Y_a} \frac{\partial}{\partial r} \quad .$$

But we also have

$$(8) \qquad \nabla_{Y_a} \frac{\partial}{\partial r} = \nabla_{\frac{\partial}{\partial r}} Y_a = Y_a'$$

and hence, using (3), (7) and (8), we obtain

$$(9) \qquad T_m(p) = (A'A^{-1})(r).$$

The key method to derive our results will be the use of power series expansions which may be derived from (4) and (5). From these formulas one gets easily

$$(10) \qquad A(r) = rI - \frac{r^3}{6} R_m - \frac{r^4}{12} R_m' + 0(r^5),$$

$$(11) \qquad A^{-1}(r) = \frac{1}{r} I + \frac{r}{6} R_m + \frac{r^2}{12} R_m' + 0(r^3),$$

where $R_m = R(m) = R_u.u$ and $R_m' = R'(m) = (\nabla_u R)_u.u$. This yields from (9) and for $p = \exp_m(ru)$:

$$(12) \qquad T_m(p) = \frac{1}{r} I - \frac{r}{3} R_m - \frac{r^2}{4} R_m' + 0(r^3).$$

Using the Gauss equation for the hypersurface $G_m(r)$ we obtain the following power series expansion for the Ricci endomorphism $\widetilde{Q}_m(p)$ (of type (1, 1)) of the geodesic sphere $G_m(r)$ at p (see for example [1]):

$$(13)\quad \tilde{Q}_m(p) = \frac{n-2}{r^2}I + \{Q - \rho(u,\cdot)u - \frac{1}{3}\rho(u,u)I - \frac{n}{3}R\}(m)$$
$$+ r\{\nabla_u Q - (\nabla_u\rho)(u,\cdot)u - \frac{1}{4}(\nabla_u\rho)(u,u)I - \frac{n+1}{4}\nabla_u R\}(m) + 0(r^2),$$

where Q denotes the Ricci operator for the ambient space (M, g).

Next, we treat some aspects about almost Hermitian geometry. Let (M, g, J) be an almost Hermitian manifold, that is $J \in \mathfrak{T}^1_1$ and

$$J^2 = -I,\ g(JX, JY) = g(X, Y)$$

for all $X, Y \in \mathfrak{T}^1$. (M, g, J) is said to be a *Kähler manifold* if $\nabla J = 0$ and a *nearly Kähler manifold* if

$$(\nabla_X J)X = 0$$

for all $X \in \mathfrak{T}^1$. For a unit vector X, $R_{XJXXJX} = g(R_{XJX}X, JX)$ is called the *holomorphic sectional curvature* associated with X. A Kähler manifold of constant holomorphic sectional curvature is called a *complex space form*. As is well-known, such a manifold is locally isometric to complex Euclidean space, a complex projective space or a complex hyperbolic space. We will need the following theorems:

Theorem 1 [8]. *Let* (M, g, J), $\dim M \geq 4$, *be a nearly Kähler manifold. Then* (M, g, J) *is of constant holomorphic sectional curvature if and only if* $R_{XJX}X$ *is proportional to* JX *for any* $X \in \mathfrak{T}^1$.

Theorem 2 [3]. *Let* (M, g, J), $\dim M \geq 4$, *be a nearly Kähler manifold of constant holomorphic sectional curvature. Then it is a complex space form or it is locally isometric to the sphere* $S^6(\lambda)$ *of constant sectional curvature* $\lambda > 0$.

For real and complex space forms one can solve explicitly the Jacobi equation. First let (M, g) be a space of constant curvature $\alpha > 0$. Then we get, since $R_m = \alpha I$ for all $m \in M$,

$$(14)\quad A(r) = \frac{\sin\sqrt{\alpha}\, r}{\sqrt{\alpha}}I,\ T_m(p) = \sqrt{\alpha}\cot\sqrt{\alpha}\, rI,\ p = \exp_m(ru).$$

For $\alpha < 0$ one has to replace the trigonometric functions by hyperbolic functions and the case $\alpha = 0$ may be obtained by taking the limit for $\alpha \to 0$. Next, let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature. We suppose again that $\alpha > 0$. Then we have, for all $m \in M$,

$$R_m = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{\alpha}{4} I \end{pmatrix} \tag{15}$$

where α is the eigenvalue corresponding to the eigenvector Ju at m. Further we obtain

$$A(r) = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} \sin \sqrt{\alpha}\, r & 0 \\ 0 & \frac{2}{\sqrt{\alpha}} \sin \frac{\sqrt{\alpha}}{2} r I \end{pmatrix} \tag{16}$$

and

$$T_m(p) = \begin{pmatrix} \sqrt{\alpha} \cot \sqrt{\alpha}\, r & 0 \\ 0 & \frac{\sqrt{\alpha}}{2} \cot \frac{\sqrt{\alpha}}{2} r I \end{pmatrix} \tag{17}$$

where $p = \exp_m(ru)$. Finally, it follows easily from the expressions for the curvature tensor of (M, g) and from the Gauss equation that $\widetilde{Q}_m(p)$ has similar expressions as $T_m(p)$ in (14) and (17).

Next, we note that for a Kähler and a nearly Kähler manifold, the holomorphic plane $\{\gamma', J\gamma'\}$ is parallel along γ. When we take the restriction of the holomorphic sectional curvature function to this plane field, then it follows directly that this restriction is a constant function if and only if

$$(\nabla_{\gamma'} R)_{\gamma' J\gamma' \gamma' J\gamma'} = 0$$

and this holds for any geodesic if and only if

$$(\nabla_X R)_{XJXXJX} = 0 \tag{18}$$

for all $X \in \mathfrak{X}^1$. (18) is a very useful condition to characterize some particular classes of almost Hermitian manifolds. Indeed, we will use the following results:

Theorem 3 [2], [7], [10]. *A Kähler manifold is locally symmetric if and only if* $\nabla_X R_{XJXXJX} = 0$ *for all* $X \in \mathfrak{X}^1$.

Theorem 4 [2]. *Let* (M, g, J) *be an analytic nearly Kähler manifold. Then it is a locally* 3*-symmetric space with canonical almost complex structure*

J if and only if $\nabla_X R_{XJXXJX} = 0$ for all $X \in \mathfrak{X}^1$.

We note that a *locally* 3-*symmetric space* is a C^∞ Riemannian manifold (M, g) together with a family of local cubic diffeomorphisms $m \mapsto s_m$, $m \in M$, such that each s_m is a holomorphic isometry in a neighborhood of m with respect to the canonical almost complex structure J of the family, determined by

$$S_m = s_{m*}(m) = -\frac{1}{2} I_m + \frac{\sqrt{3}}{2} J_m. \tag{19}$$

By a family of local cubic diffeomorphisms we mean a differentiable function $m \to s_m$ which assigns to each $m \in M$ a diffeomorphism s_m on a neighborhood $U(m)$ of m such that $s_m^3 =$ identity and m is the unique fixed point of s_m. We refer to [2] for more information.

The notion of a locally 3-symmetric space is a special case of a more general concept due to Graham and Ledger [4]. We will now treat this in view of our results in section 5. First, let $P \in \mathfrak{X}^p_q$ and $S \in \mathfrak{X}^1_1$. Then the tensor field P is said to be S-*invariant* if for all $\omega_1, \dots, \omega_p \in \mathfrak{X}_1$ and $X_1, \dots, X_q \in \mathfrak{X}^1$

$$P(\omega_1 S, \dots, \omega_p S, X_1, \dots, X_q) = P(\omega_1, \dots, \omega_p, SX_1, \dots, SX_q)$$

where $(\omega S)(X) = \omega(SX)$ for $\omega \in \mathfrak{X}_1$ and $X \in \mathfrak{X}^1$. Next, S is called a *symmetry tensor field* if $I - S$ is non-singular and g is S-invariant. In particular, if ∇S and $\nabla^2 S$ are S-invariant, then we say that S is *regular*.

For any symmetry tensor field S on M we define a local symmetry s_m at each point $m \in M$ by

$$s_m = \exp_m \circ S_m \circ \exp_m^{-1} \tag{20}$$

on a sufficiently small neighborhood of m. Then s_m is a local diffeomorphism. We denote by s the map $m \mapsto s_m$ so defined on M and note that, for each $m \in M$,

$$s_{m*} | T_m = S_m.$$

Hence, S is determined uniquely by s.

Now, let S be a regular symmetry tensor field. Then (M, g) together with s is called a *Riemannian locally* s-*regular manifold* if each s_m is also a local isometry which preserves S, that is

$$s_{m*}(SX) = S(s_{m*}X)$$

for all $m \in M$ and each vector field X defined on some neighborhood of m. In this case s is called a *local regular s-structure* on M. Finally, (M, g) is said to be a *Riemannian k-symmetric space* if there exists a local regular s-structure on it such that $s^k =$ identity. For $k = 2$ we obtain the well-known locally symmetric spaces and for $k = 3$ we have a locally 3-symmetric space. For more details we refer to [2], [4], [5]. Here we just state the following characterization which will be needed in section 5:

Theorem 5 [4]. *Let S be a regular symmetry tensor field on a Riemannian manifold (M, g). Then (20) defines a local regular s-structure on (M, g) if and only if R and ∇R are S-invariant.*

We are now ready to state and prove our results about the properties of the shape operator, the Ricci operator and the integral curves of the vector field $J\frac{\partial}{\partial r}$.

3. The shape operator of a geodesic sphere. In this section we consider the characterization theorems related to the shape operator.

Theorem 6. *Let (M, g, J) be a nearly Kähler manifold with* $\dim M \geq 4$. *Then (M, g, J) has constant holomorphic sectional curvature if and only if $J\frac{\partial}{\partial r}$ is an eigenvector of the shape operator T_m of the geodesic sphere $G_m(r)$ for all $m \in M$ and all sufficiently small r.*

Proof. We use the notations of section 2 and put $u = \gamma' = \frac{\partial}{\partial r}$ along γ. Since (M, g, J) is nearly Kählerian, Ju is parallel along γ. Then (12) yields

$$(21) \qquad T_m(p)Ju = \frac{1}{r}(Ju)(m) - \frac{r}{3}(RJu)(m) - \frac{r^2}{4}(R'Ju)(m) + 0(r^3).$$

Next, let x be a parallel vector field orthogonal to the parallel plane $\{u, Ju\}$ along γ. Then it follows that Ju is an eigenvector of T_m at p if and only if

$$(22) \qquad g(T_m(p)Ju, x) = 0$$

for all such x. Using (21), this gives

$$g(RJu, x)(m) = R_{uJuux}(m) = 0$$

for all $x \in T_mM$, orthogonal to $\{u, Ju\}$ at m. This means that $R_{uJu}u$ must be proportional to Ju for all u and all $m \in M$. Then the result follows from Theorem 1.

Conversely, let (M, g, J) has constant holomorphic sectional curvature. Then the classification theorem (Theorem 2) and the explicit expressions (14) and (17) show at once that Ju is an eigenvector of T_m.

Theorem 7. *Let (M, g, J) be a nearly Kähler manifold with* $\dim M \geq 4$. *Then (M, g, J) has constant holomorphic sectional curvature if and only if each integral curve of the vector field $J\frac{\partial}{\partial r}$ on the geodesic sphere $G_m(r)$ is a geodesic for all $m \in M$ and all sufficiently small r.*

Proof. Let (M, g, J) be a nearly Kähler manifold and denote by $\tilde{\nabla}$ the induced Riemannian connection on the geodesic sphere $G_m(r)$. Then, with $u = \frac{\partial}{\partial r}$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)u \tag{23}$$

for all X, Y tangent to $G_m(r)$. σ denotes the second fundamental form. Hence, the integral curves of the vector field Ju are geodesics on $G_m(r)$ if and only if

$$\tilde{\nabla}_{Ju}(Ju) = 0$$

that is, if and only if

$$g(\nabla_{Ju}(Ju), X) = 0 \tag{24}$$

for all vectors X tangent to $G_m(r)$. Now, (24) is equivalent to

$$g((\nabla_{Ju}J)u, X) + g(J\nabla_{Ju}u, X) = 0.$$

Since (M, g, J) is nearly Kählerian, the first term vanishes and so we get as condition

$$g(J\nabla_{Ju}u, X) = 0.$$

This implies that the integral curves of Ju are geodesics if and only if T_mJu is proportional to Ju, that is if and only if Ju is an eigenvector of T_m. Then the result follows from Theorem 6.

But we have even a stronger result:

Theorem 8. *Let (M, g, J) be a nearly Kähler manifold with* $\dim M \geq 4$. *Then (M, g, J) has constant holomorphic sectional curvature if and only if the vector field $J\frac{\partial}{\partial r}$ on $G_m(r)$ is a Killing vector field for all $m \in M$ and all sufficiently small r.*

Proof. First, let Ju be a Killing vector field. Then the integral curves are geodesics and the result follows from Theorem 7.

Conversely, let (M, g, J) be a nearly Kähler manifold of constant holomorphic sectional curvature. Ju is a Killing vector field if and only if

$$g(\tilde{\nabla}_X(Ju), Y)+g(\tilde{\nabla}_Y(Ju), X) = 0$$

for all X, Y tangent to $G_m(r)$, or by using (23), if and only if

$$g(\nabla_X(Ju), Y)+g(\nabla_Y(Ju), X) = 0. \tag{25}$$

Since (M, g, J) is nearly Kählerian, it follows easily, using $g(JX, Y)+g(X, JY) = 0$, that (25) is equivalent to

$$g(J\nabla_X u, Y)+g(J\nabla_Y u, X) = 0$$

or, equivalently,

$$g(T_m X, JY)+g(T_m Y, JX) = 0 \tag{26}$$

for all X, Y tangent to $G_m(r)$. Hence the result follows again using the classification theorem (Theorem 2) and (14), (17).

Now we will prove some applications of Theorem 3 for Kähler manifolds. Again let $p = \exp_m(ru)$ and consider the geodesic of $G_m(r)$ tangent to Ju at p. Then the curvature $\varkappa_m(p)$, at p, of this geodesic is given by

$$\varkappa_m(p) = g(T_m(p)Ju, Ju). \tag{27}$$

Further, let $s_m : p = \exp_m(ru) \rightarrow s_m(p) = \exp_m(-ru)$ denote the geodesic reflection centered at m. Then we have

Theorem 9. *Let (M, g, J) be a Kähler manifold and let $\varkappa_m(p)$ denote the curvature at $p =$ $\exp_m(ru)$ of the geodesic of $G_m(r)$ tangent to Ju. Then, for all $m \in M$ and all p with r sufficiently small,*

$$\varkappa_m(p) = \varkappa_m(s_m(p)) \tag{28}$$

if and only if (M, g, J) is locally isometric to a Hermitian symmetric manifold.

Proof. First, let (M, g, J) be locally isometric to a Hermitian symmetric manifold. Then the result follows easily from the fact that the geodesic reflections are holomorphic isometries.

Conversely, suppose (28) holds. Using (12) we get for (27):

$$\varkappa_m(p) = \frac{1}{r}g(Ju, Ju)(m) - \frac{r}{3}R_{uJuuJu}(m) - \frac{r^2}{4}(\nabla_u R)_{uJuuJu}(m) + 0(r^3). \tag{29}$$

Hence (28) and (29) imply

$$(\nabla_u R)_{uJuuJu} = 0$$

and so the result follows from Theorem 3.

Theorem 10. *Let (M, g, J) be a Kähler manifold and put $p = \exp_m(ru)$. Denote by $\varkappa_p(m)$ the curvature of the geodesic, at m, of the geodesic sphere $G_p(r)$ tangent to Ju. Then (M, g, J) is locally isometric to a Hermitian symmetric manifold if and only if*

$$\varkappa_p(m) = \varkappa_{s_m(p)}(m). \tag{30}$$

for all $m \in M$ and all p with r sufficiently small.

Proof. Using (12) we get

$$T_p(m) = \frac{1}{r}I - \frac{r}{3}R_p + \frac{r^2}{4}R'_p + 0(r^3)$$

and hence

$$T_p(m) = \frac{1}{r}I - \frac{r}{3}R_m - \frac{r^2}{12}R'_m + 0(r^3).$$

So we have

$$\varkappa_p(m) = \frac{1}{r}g(Ju, Ju)(m) - \frac{r}{3}R_{uJuuJu}(m) - \frac{r^2}{12}(\nabla_u R)_{uJuuJu}(m) + 0(r^3). \tag{31}$$

The desired result follows now by proceeding in the same way as in the proof of Theorem 9.

4. The Ricci operator of a geodesic sphere. In this section we will consider the Ricci operator of the small geodesic spheres and prove similar theorems as for the shape operator. We start with

Theorem 11. *Let (M, g, J) be a nearly Kähler Einstein manifold with $\dim M \geq 4$. Then (M, g, J) has constant holomorphic sectional curvature if and only if $J\frac{\partial}{\partial r}$ is an eigenvector of the Ricci operator of $G_m(r)$ for all $m \in M$ and all sufficiently small r.*

Proof. We proceed as in the proof of Theorem 6 but we use now the power series expansion (13) for the Ricci operator $\widetilde{Q}_m(p)$ of $G_m(r)$ where $p = \exp_m(ru)$. Then Ju is an eigenvector of $\widetilde{Q}_m(p)$ if and only if

$$g(\widetilde{Q}_m(p)Ju, x) = 0 \tag{32}$$

with the same convention for x as in Theorem 6. Further, (13) yields

$$\begin{aligned}\widetilde{Q}_m(p)Ju = {} & \frac{n-2}{r^2}(Ju)(m)+\left\{QJu-\rho(u, Ju)u-\frac{1}{3}\rho(u, u)Ju\right.\\ & \left.-\frac{n}{3}RJu\right\}(m)+r\left\{(\nabla_u Q)Ju-(\nabla_u\rho)(u, Ju)u\right.\\ & \left.-\frac{1}{4}(\nabla_u\rho)(u, u)Ju-\frac{n+1}{4}R'Ju\right\}(m)+0(r^2)\end{aligned} \tag{33}$$

and hence, from (32) and (33), we get the condition

$$3g(QJu, x) = nR_{uJuux} \tag{34}$$

for all $m \in M$, all unit $u \in T_mM$ and all x orthogonal to the plane $\{u, Ju\}$. Since (M, g) is an Einstein manifold, (34) becomes

$$R_{uJuux} = 0$$

and so, our result follows again from Theorem 1.

The converse follows easily by using Theorem 2 and the explicit expressions for $\widetilde{Q}_m$.

We have a better result when the ambient space (M, g, J) is a Kähler manifold but the proof is more complicated.

Theorem 12. *Let (M, g, J) be a Kähler manifold with* $\dim M \geq 4$. *Then (M, g, J) is a complex space form if and only if $J\frac{\partial}{\partial r}$ is an eigenvector of the Ricci operator of $G_m(r)$ for all $m \in M$ and all sufficiently small r.*

Proof. Suppose first that (M, g, J) is a complex space form. Then the result is proved in Theorem 11.

Now, we prove the converse. In this case (34) also holds for all u orthogonal to Jx, since $g(QJx, x) = 0$ for a Kähler manifold. Put $u = \alpha y + \beta z$ for $y, z \in \{Jx\}^{\perp}$ in (34) and write down the coefficient of $\alpha\beta^2$. Using the Kähler and the first Bianchi identity we get

$$(35) \qquad 3g(z, z)\rho(Jy, x)+6g(z, y)\rho(Jz, x) = 3nR_{Jyzxz}+nR_{yzxJz}.$$

Now, put $z = e_i$, $i = 1, \dots, n-1$ and $Jx = e_n$, where $\{e_i, i = 1, \dots, n\}$ is an orthonormal basis, and sum with respect to i. Then we get

$$(36) \qquad (n-3)\rho(Jy, x) = 3nR_{yxxJx}.$$

This, together with (34), implies

$$(n-12)\rho(Jy, x) = 0$$

and so, for $n \neq 12$, we have

$$\rho(Jy, x) = 0$$

which, with (36), yields

$$R_{yxxJx} = 0$$

for all $y \in \{Jx\}^{\perp}$. Hence, for $n \neq 12$, the result follows from Theorem 1.

Next, let $n = 12$. Then, with a similar choice for x, we get from (33):

$$(37) \qquad 4(\nabla_u Q)(Ju, x) = (n+1)(\nabla_u R)_{uJuux}.$$

Differentiation of (34) and (37) then leads to

$$(38) \qquad (\nabla_u R)_{uJuux} = 0$$

for all $u \in \{Jx\}^{\perp}$.

Now, let $u \in \{x, Jx\}^{\perp}$. Then (38) also implies

$$(39) \qquad (\nabla_{Ju} R)_{JuuJux} = 0,$$

and the use of the Kähler and the second Bianchi identity in (39) gives

$$(\nabla_u R)_{Juuxu}+(\nabla_{Jx} R)_{JuuJuu} = 0$$

and hence, from (38), we obtain

(40) $$(\nabla_x R)_{uJuuJu} = 0$$

for any $x \in \{u, Ju\}^\perp$. Since (M, g) is Kählerian, this implies

$$(\nabla_x R)_{uvwt} = 0$$

for all $u, v, w, t \in \{x, Jx\}^\perp$ and so, we have

(41) $$(\nabla_x R)_{uvuv} = 0$$

for all $u, v \in \{x, Jx\}^\perp$. Again, let $\{e_i, i = 1, \dots, n\}$ be an orthonormal basis with $x = e_{n-1}$, $Jx = e_n$. Put $v = e_i$ in (41) and sum with respect to i. Then we get

(42) $$(\nabla_x \rho)(u, u) = \nabla_x R_{uxux}+\nabla_x R_{uJxuJx}.$$

Doing the same for u in (42), we obtain

(43) $$\nabla_x \tau-4(\nabla_x \rho)(x, x)+2(\nabla_x R)_{xJxxJx} = 0$$

for any unit vector x, that is, for an arbitrary vector x we must have

(44) $$2(\nabla_x R)_{xJxxJx}-4g(x, x)(\nabla_x \rho)(x, x)+g(x, x)g(x, x)\nabla_x \tau = 0.$$

Next, we linearize (44) to get, for $y \in \{x\}^\perp$:

(45) $$2(\nabla_y R)_{xJxxJx}+8(\nabla_x R)_{yJxxJx} = 4(\nabla_y \rho)(x, x)g(x, x) \\ +8(\nabla_x \rho)(x, y)g(x, x)-(\nabla_y \tau)g(x, x)g(x, x).$$

Now, suppose $y \in \{x, Jx\}^\perp$ and substitute Jx for x in (45). Adding the result up with (45) gives

(46) $$12(\nabla_y R)_{xJxxJx} = 8(\nabla_y \rho)(x, x)g(x, x)+8(\nabla_x \rho)(x, y)g(x, x) \\ +8(\nabla_{Jx} \rho)(Jx, y)g(x, x)-2(\nabla_y \tau)g(x, x)g(x, x).$$

Then (40) and (46) yield

$$4(\nabla_y \rho)(x, x)+4(\nabla_x \rho)(x, y)+4(\nabla_{Jx} \rho)(Jx, y) = (\nabla_y \tau)g(x, x)$$

for all $y \in \{x, Jx\}^\perp$.

We use again the summation procedure to get

(47) $$16(\nabla_y \rho)(y, y) = (10-n)(\nabla_y \tau)g(y, y).$$

Finally, linearization of (47) yields

$$16\{(\nabla_z\rho)(y, y)+2(\nabla_y\rho)(z, y)\} = (10-n)\{(\nabla_z\tau)g(y, y)+2(\nabla_y\tau)g(y, z)\}$$

and using again summation, we obtain

$$(n-2)(n-6)\nabla_z\tau = 0.$$

So, since $n = 12$, $\nabla_z\tau = 0$ and hence (47) yields

$$(\nabla_y\rho)(y, y) = 0.$$

These relations and (44) yield

$$(\nabla_y R)_{yJyyJy} = 0 \tag{48}$$

and so, from Theorem 3, we conclude that (M, g, J) is locally symmetric.

When (M, g, J) is irreducible, then it is an Einstein space and hence the result follows from Theorem 11. Next, if (M, g, J) is reducible, it is locally a product $M_1\times M_2\times\cdots\times M_k$ of Kählerian Einstein spaces. On the other hand (34) implies that each factor M_i, with $\dim M \geq 4$, is a space of constant holomorphic sectional curvature and (48) implies the same result for a two-dimensional factor.

Next, (34) may be written in the form

$$3QJu-nR_{uJu}u = \alpha Ju$$

are projecting this onto the factor M_1, we get

$$3(QJu)_1-n(R_{uJu}u)_1 = \alpha(Ju)_1. \tag{49}$$

Since this factor is a space of constant holomorphic sectional curvature, say c_1, we have

$$(QJu)_1 = \frac{\tau_1}{n_1}(Ju)_1,\ (R_{uJu}u)_1 = (c_1\cos^2\alpha_1)(Ju)_1 \tag{50}$$

where $n_1 = \dim M_1$ and τ_1 denotes the scalar curvature of M_1, that is

$$4\tau_1 = n_1(n_1+2)c_1. \tag{51}$$

Moreover, $g(u_1, u_1) = \cos^2\alpha_1$. Then, using (49), (50) and (51), we obtain

$$c_1\{3(n_1+2)-4n\cos^2\alpha_1\} = 4\alpha. \tag{52}$$

Projecting onto the second factor gives

$$c_2\{3(n_2+2)-4n\cos^2\alpha_2\} = 4\alpha$$

and hence, we have

(53) $$c_1\{3(n_1+2)-4n\cos^2\alpha_1\} = c_2\{3(n_2+2)-4n\cos^2\alpha_2\}.$$

By taking different possible values for α_1 and α_2 in (53) and by using a similar method for the coefficient of r^2 in (13)(See [1]. We delete the details.), we obtain finally $c_1 = c_2 = 0$. Proceeding in the same way for the other factors implies that (M, g, J) is flat. This completes the proof of our theorem.

In what follows we consider the Ricci curvature of $G_m(r)$ with respect to $Ju = J\frac{\partial}{\partial r}$, that is $\tilde{\rho}_m(Ju, Ju)$. Then we have

Theorem 13. *A Kähler manifold of dimension ≥ 4 is locally isometric to a Hermitian symmetric space if and only if the Ricci curvature of the geodesic spheres $G_m(r)$ satisfies*

(54) $$\tilde{\rho}_m(Ju, Ju)(p) = \tilde{\rho}_m(Ju, Ju)(s_m(p)), \qquad p = \exp_m(ru),$$

for all $m \in M$ and all sufficiently small r. Here s_m denotes the geodesic reflection centered at m.

Proof. First, let (M, g, J) be locally isometric to a Hermitian symmetric space. Then the result follows again from the fact that the geodesic reflections are holomorphic isometries.

Conversely, suppose (54) holds. Using (13) we get

(55) $$\tilde{\rho}_m(Ju, Ju)(p) = \frac{n-2}{r^2} + \left\{\rho(Ju, Ju) - \frac{1}{3}\rho(u, u) - \frac{n}{3}R_{uJuuJu}\right\}(m) + r\left\{(\nabla_u\rho)(Ju, Ju) - \frac{1}{4}(\nabla_u\rho)(u, u) - \frac{n+1}{4}(\nabla_u R)_{uJuuJu}\right\}(m) + 0(r^2).$$

So (54) and (55) imply

(56) $$4(\nabla_u\rho)(Ju, Ju) - (\nabla_u\rho)(u, u) - (n+1)(\nabla_u R)_{uJuuJu} = 0,$$

and since $\rho(Ju, Ju) = \rho(u, u)$, (56) becomes

(57) $$3(\nabla_u\rho)(u, u) = (n+1)(\nabla_u R)_{uJuuJu}.$$

Now, a similar linearization and summation procedure as in Theorem 12 (we omit the details) yields

(58) $$(2-3n)(\nabla_u \rho)(u, u)+2\nabla_u \tau = 0.$$

Doing this again for (58), we obtain

(59) $$\nabla_u \tau = 0.$$

So, (57), (58) and (59) yield

$$\nabla_u R_{uJuuJu} = 0$$

and the desired result follows then from Theorem 3.

Theorem 14. *Let (M, g, J) be a Kähler manifold of dimension ≥ 4 and let $\tilde{\rho}_p$ denote the Ricci tensor of the geodesic sphere $G_p(r)$ with center $p = \exp_m(ru)$. Then (M, g, J) is locally isometric to a Hermitian symmetric space if and only if*

(60) $$\tilde{\rho}_p(Ju, Ju)(m) = \tilde{\rho}_{s_m(p)}(Ju, Ju)(m)$$

for all $m \in M$ and all sufficiently small r. (s_m denotes again the geodesic reflection centered at m.)

Proof. The proof is similar to that of Theorem 13 but now we use

(61) $$\begin{aligned}\tilde{\rho}_p(Ju, Ju)(m)\\ &= \frac{n-2}{r^2}+\left\{\rho(Ju, Ju)-\frac{1}{3}\rho(u, u)-\frac{n}{3}R_{uJuuJu}\right\}(p)\\ &\quad -r\left\{(\nabla_u\rho)(Ju, Ju)-\frac{1}{4}(\nabla_u\rho)(u, u)-\frac{n+1}{4}(\nabla_u R)_{uJuuJu}\right\}(p)\\ &\quad +0(r^2)\\ &= \frac{n-2}{r^2}+\left\{\rho(Ju, Ju)-\frac{1}{3}\rho(u, u)-\frac{n}{3}R_{uJuuJu}\right\}(m)\\ &\quad -\frac{r}{12}\{(\nabla_u\rho)(u, u)+(n-3)(\nabla_u R)_{uJuuJu}\}(m)+0(r^2).\end{aligned}$$

(We omit the rest of the proof.)

5. 3-Symmetric spaces and s-regular manifolds. The aim of this final section is to extend some of the theorems of section 3 and section 4. We start with a theorem for the Ricci operator. The similar theorem for the shape operator has been proved in [6] (see also [10]).

Theorem 15. *Let S be a regular symmetry tensor field on (M, g) where* $\dim M > 3$. *Then (M, g) is a locally s-regular manifold with associated symmetric tensor field S if and only if*

$$\widetilde{Q}_{s_m(p)}(m)\circ S_m = S_m\circ\widetilde{Q}_p \tag{62}$$

for all m and all $p = \exp_m(ru)$ *with r sufficiently small (s_m is the local symmetry determined by S via* (20)).

Proof. (62) holds trivially when (M, g) is a locally s-regular manifold since s_m is an isometry.

To prove the converse we use (13) to determine $\widetilde{Q}_p(m)$. Then we proceed as for $T_p(m)$ in the proof of Theorem 10. This gives

$$\begin{aligned}\widetilde{Q}_p(m) = \frac{n-2}{r^2}I+\left\{Q-\rho(u,\cdot)u-\frac{1}{3}\rho(u,u)I-\frac{n}{3}R\right\}(m)\\ -\frac{r}{12}\left\{(\nabla_u\rho)(u,u)I+(n-3)\nabla_u R\right\}(m)+0(r^2).\end{aligned}$$

Then (62) and the expression for $\widetilde{Q}_p(m)$ yield the conditions

$$\begin{aligned}\rho(Sx,Sx)-\frac{1}{3}\rho(Su,Su)-\frac{n}{3}R_{SuSxSuSx}\\ =\rho(x,x)-\frac{1}{3}\rho(u,u)-\frac{n}{3}R_{uxux}\end{aligned} \tag{63}$$

and

$$\begin{aligned}(\nabla_{Su}\rho)(Su,Su)+(n-3)(\nabla_{Su}R)_{SuSxSuSx}\\ =(\nabla_u\rho)(u,u)+(n-3)(\nabla_u R)_{uxux}\end{aligned} \tag{64}$$

for all $m\in M$, all unit $u\in T_mM$ and all unit x orthogonal to u. Next, let $\{e_i,\ i=1,\dots,n\}$ be an orthonormal basis such that $u=e_n$. Put $x=e_i$ in (63) and sum with respect to i. This gives

$$\rho(u,u)=\rho(Su,Su)$$

which implies that ρ is S-invariant. Then (63) takes the form

$$R_{SuSxSuSx}=R_{uxux}\ . \tag{65}$$

Further, put

$$T_{xyzw}=R_{xyzw}-R_{SxSySzSw}. \tag{66}$$

Then $T \in \mathfrak{T}_4$ satisfies the algebraic identities of a Riemannian curvature tensor and moreover, (65) implies

$$T_{uxux} = 0.$$

Hence, $T = 0$ and so we see from (66) that R is S-invariant.

Similarly, (64) yields

$$(\nabla_u \rho)(u, u) = (\nabla_{Su} \rho)(Su, Su) \tag{67}$$

and since $\dim M > 3$, (64) and (67) give

$$(\nabla_{Su} R)_{SuSxSuSx} = (\nabla_u R)_{uxux}.$$

Now, put

$$B_{uxyzw} = \nabla_{Su} R_{SxSySzSw} - \nabla_u R_{xyzw}.$$

Then B is a (0, 5)-tensor which satisfies all the identities of the covariant derivative ∇R, and moreover,

$$B_{uuxux} = 0.$$

This yields (see for example [3], [10], [1]) $B = 0$ and so, ∇R is also S-invariant. Then the desired result follows from Theorem 5.

Since we obtain the usual locally symmetric spaces for $S = -I$, we have

Corollary 16. *A Riemannian manifold M of dimension > 3 is locally symmetric if and only if*

$$\tilde{Q}_{\exp_m(ru)}(m) = \tilde{Q}_{\exp_m(-ru)}(m)$$

for all $m \in M$, all unit $u \in T_m M$ and all sufficiently small r.

Now we turn to the characterizations of 3-symmetric nearly Kähler manifolds. We use the conventions and notations as before.

Theorem 17. *Let (M, g, J) be an analytic nearly Kähler manifold and let s_m be the local symmetry defined by (20) for $S_m = -\frac{1}{2} I_m + \frac{\sqrt{3}}{2} J_m$. Further let $\varkappa_p(m)$ be the curvature at m of the geodesic of $G_p(r)$ tangent to Ju and let $\varkappa_{s_m(p)}(m)$ be the curvature at m of the geodesic of $G_{s_m(p)}(r)$ tangent to $JS_m u$. Then (M, g, J) is a locally 3-symmetric space with J as canonical* *almost complex structure if and only if for all $m \in M$ and all $p = \exp_m(ru)$ (r sufficiently small) we have*

(68) $$\varkappa_p(m) = \varkappa_{s_m(p)}(m).$$

Proof. Let (M, g, J) be such that (68) holds. Then (31) and the corresponding expression for the second member of (68) imply

(69) $$(\nabla_{Su} R)_{SuSJuSuSJu}(m) = (\nabla_u R)_{uJuuJu}(m).$$

Using the explicit expression for S_m in (69) we obtain

(70) $$\sqrt{3}\,(\nabla_u R)_{uJuuJu} = (\nabla_{Ju} R)_{uJuuJu}.$$

Replacing u by Ju in (70) yields

(71) $$\sqrt{3}\,(\nabla_{Ju} R)_{uJuuJu} = -(\nabla_u R)_{uJuuJu}.$$

Hence, from (70) and (71), we obtain

$$(\nabla_u R)_{uJuuJu} = 0$$

and so the result follows from Theorem 4.

The converse is trivial since s_m is then a holomorphic isometry.

Proceeding similarly, but now using (29), we have

Theorem 18. *Let (M, g, J) be an analytic nearly Kähler manifold and let s_m and S_m be as in Theorem 17. Further let $\varkappa_m(p)$, $p = \exp_m(ru)$, denote the curvature at p of the geodesic of $G_m(r)$ tangent to Ju and $\varkappa_m(s_m(p))$ the curvature at $s_m(p)$ of the geodesic of $G_m(r)$ tangent to $JS_m u$. Then (M, g, J) is a locally 3-symmetric space with canonical almost complex structure J if and only if*

$$\varkappa_m(p) = \varkappa_m(s_m(p))$$

for all m and all $p = \exp_m(ru)$ with r sufficiently small.

Finally, using the power series expansions for the Ricci operators we may prove the following theorem (we omit the details since the proof is similar to that for Theorems 13, 14, 17 and 18):

Theorem 19. *Let (M, g, J) be an analytic nearly Kähler manifold of dimension ≥ 4 and let s_m and S_m be as in Theorem 17. Then (M, g, J) is* *a locally* 3-*symmetric space with canonical almost complex structure J if and only if*

$$\tilde{\rho}_m(Ju, Ju)(p) = \tilde{\rho}_m(JS_m u, JS_m u)(s_m(p))$$

or, if and only if

$$\tilde{\rho}_p(Ju, Ju)(m) = \tilde{\rho}_{s_m(p)}(JS_m u, JS_m u)(m)$$

for all $m \in M$ *and all* $p = \exp_m(ru)$ *with* r *sufficiently small.*

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