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Some Riemannian manifolds admitting a concircular scalar field

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SOME RIEMANNIAN MANIFOLDS ADMITTING A CONCIRCULAR SCALAR FIELD

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Introduction

Recently, R. S. Kulkarni [2] and others have dealt with Riemannian manifolds admitting a concircular scalar field in a theory of curvature-preserving mappings or in connection with the so-called Nomizu's conjecture. Properties of concircular scalar fields, of special ones or in Einstein manifolds etc, were studied by Y. Tashiro [1] or other authers. However, we have less knowledge of concircular scalar fields in manifolds of constant scalar curvature, even in case of low-dimensional manifolds.

In this paper, we shall discuss properties of Riemannian manifolds admitting a concircular scalar field by using adapted coordinate systems for concircular scalar fields as tools. We refer to [1] as to notations and terminologies.

In 1, formulas with respect to an adapted coordinate system will be stated as preliminaries. In 2, we shall determine the structure of a 4-dimensional Riemannian manifold of constant scalar curvature admitting a concircular scalar field. We shall prove, in 3, that a 4-dimensional Einstein manifold admitting a concircular scalar field is of constant curvature, and slightly generalize Kulkarni's theorem concerning conformal map. It will be proved, in 4, that a manifold having the vanishing tensor $H_{\sigma\omega\nu\mu\lambda}{}^{\chi}$ defined by E. Cartan and admitting a concircular scalar field is of constant curvature, and, in 5, that a manifold satisfying $\overset{*}{H}_{\sigma\omega\mu\lambda} = H_{\sigma\omega\nu\mu\lambda}{}^{\nu} = 0$ is an Einstein manifold, unless the gradient vector field of the concircular scalar field is concurrent.

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1. Preliminaries

We shall assume, throughout this paper, that a Riemannian manifold M is connected, differentiable and of dimension n, and the metric tensor $g_{\mu\lambda}$ of M is positive definite. Two kinds of indices run on the ranges

$$\chi, \lambda, \mu, \nu, \omega, \sigma, \tau = 1, 2, 3, \cdots\cdots n$$
$$h, i, j, k = 2, 3, \cdots\cdots n$$

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respectively. We denote the Christoffel symbol, the curvature tensor, the Ricci tensor and the scalar curvature of M by $\{^{\kappa}_{\mu\lambda}\}$, $K_{\nu\mu\lambda}{}^{\kappa}$, $K_{\mu\lambda}$ and κ respectively, where the scalar curvature defined by $\kappa=\frac{1}{n(n-1)}K_{\mu\lambda}g^{\mu\lambda}$.

The scalar field ρ is said to be *concircular* if it satisfies the equation

$$\nabla_{\mu}\nabla_{\lambda}\rho=\phi g_{\mu\lambda}, \tag{1.1}$$

∇ indicating covariant differentiation and ϕ being a scalar field, and to be *special concircular* if it satisfies the equation

$$\nabla_{\mu}\nabla_{\lambda}\rho=(-k\rho+b)g_{\mu\lambda} \tag{1.2}$$

k and b being constant.

Along any geodesic with arc-length u, the equation (1.1) becomes to the ordinary equation

$$\frac{d^2\rho}{du^2}=\phi. \tag{1.3}$$

We put $\rho_{\lambda}=\delta_{\lambda}\rho$. A point P is said to be *ordinary* or *stationary* according as $\rho_{\lambda}(\mathrm{P})\neq 0$ or $\rho_{\lambda}(\mathrm{P})=0$. Stationary points of a concircular scalar field ρ are isolated and there exist at most two in M, see [1]. In a neighborhood U of an ordinary point of ρ, we can choose an adapted coordinate system (u^{κ}) having the following properties: the first coordinate u^1 is the arc-length u of ρ-curves, trajectories of ρ^{κ}, the coordinate hypersurfaces $u^1=$constant are ρ-hypersurfaces defined by $\rho=$ constant, the field ρ is a function of $u^1=u$ only, and the metric form ds^2 of M is given in the from

$$ds^2=du^2+\rho'(u)^2\overline{ds^2}, \tag{1.4}$$

where the prime indicates ordinary derivative with respect to u and $\overline{ds^2}=f_{ji}du^jdu^i$ is a metric form of an $(n-1)$-dimensional manifold $\overline{M}$. We indicate quantities of $\overline{M}$ by barring. With respect to an adapted coordinate system (u^{κ}), the metric tensor has components

$$g_{11}=1,\quad g_{j1}=g_{1i}=0,\quad g_{ji}=\rho'^2f_{ji}, \tag{1.5}$$

the curvature tensor $K_{\nu\mu\lambda}{}^{\kappa}$ of M has components

$$\begin{aligned} &K_{1j1}{}^{h}=-K_{j11}{}^{h}=\frac{\rho'''}{\rho'}\delta_j^h,\quad K_{1ji}{}^{h}=-K_{j1i}{}^{h}=-\rho'\rho'''f_{ji},\\ &K_{kji}{}^{h}=\overline{K}_{kji}{}^{h}-\rho''^2(\delta_k^hf_{ji}-\delta_j^hf_{ki}), \end{aligned} \tag{1.6}$$

the other components being zero, the Ricci tensor has components

$$K_{11}=-(n-1)\frac{\rho'''}{\rho'},\quad K_{j1}=K_{1i}=0 \tag{1.7}$$
$$K_{ji}=\bar{K}_{ji}-[(n-2)\rho''^2+\rho'\rho''']f_{ji},$$

and the scalar curvature κ of M is equal to

$$\kappa=\frac{1}{n(\rho')^2}[(n-2)(\bar{\kappa}-\rho''^2)-2\rho'\rho'''], \tag{1.8}$$

where $\bar{\kappa}$ is the scalar curvature of $\bar{M}$ defined by $\bar{\kappa}=\frac{1}{(n-1)(n-2)}\bar{K}_{ji}f^{ji}$.

2. 4-dimensional Riemannian manifolds of constant scalar curvature

Let M be a 4-dimensional Riemannian manifold of constant scalar curvature κ and ρ a concrcular scalar field. For $n=4$, the equation (1.8) is reduced to

$$2\kappa\rho'^2+\frac{1}{2}(\rho'^2)''=\bar{\kappa}. \tag{2.1}$$

Since the left hand side depends on u only and $\bar{\kappa}$ is independent of u, $\bar{\kappa}$ is also a constant. According the signature of the constant scalar curvature κ, we put

$$\kappa=\begin{cases}\text{(I)} & 0\\ \text{(II)} & -c^2\\ \text{(III)} & c^2\end{cases} \tag{2.2}$$

c being a positive constant. By a suitable choice of the arclength u, the general solution of (2.1) is given by one of

$$\rho'^2=\begin{cases}\text{(I, A)} & au & (\bar{\kappa}=0)\\ \text{(I, B)} & \bar{\kappa}u^2+a & (\bar{\kappa}\neq 0)\\ \text{(II, A}_0\text{)} & a\exp 2cu-\frac{\bar{\kappa}}{2c^2} & \\ \text{(II, A}_-\text{)} & a\sinh 2cu-\frac{\bar{\kappa}}{2c^2} & \\ \text{(II, B)} & a\cosh 2cu-\frac{\bar{\kappa}}{2c^2} & \\ \text{(III)} & a\cos 2cu+\frac{\bar{\kappa}}{2c^2}\,. & \end{cases} \tag{2.3}$$

Therefore the manifold M has a local structure such that the metric form is given by (1. 4) substituted with (2. 3) for ρ'^2.

Next we suppose that M is complete. Then the arc-length of any geodesic is extendable to the infinities. Since ρ-curves are geodesic, the cases (I, A) and (II, A_-) do not occur, and in the other cases, the inequalities

$$(2.4)\qquad \begin{cases} \text{(I, B)} & \bar{\kappa}>0, \quad a\geqq 0 \\ \text{(II, A}_0\text{)} & a>0, \quad \bar{\kappa}<0 \\ \text{(II, B)} & a>0, \quad \bar{\kappa}\leqq 2ac^2 \\ \text{(III)} & \bar{\kappa}>0, \quad \bar{\kappa}\geqq 2ac^2 \end{cases}$$

should be satisfied respectively, because $\rho'^2\geqq 0$.

Moreover, in order that there exists no stationary point of ρ in a complete manifold M, it is necessary and sufficient that the equalities in (2. 4) do not appear in all cases. Then the manifold M is topologically the direct product $I\times\overline{M}$ of a straight line I and a 3-dimensional complete manifold $\overline{M}$. By transferring the factor $\bar{\kappa}$ in the case (I, B) or a in the cases (II, A_0), (II, B) and (III) into the metric tensor f_{ji} of $\overline{M}$, in other words, applying a homothety to $\overline{M}$, the metric form of M is given by

$$(2.5)\qquad ds^2=\begin{cases} \text{(I, B)} & du^2+(u^2+a)\overline{ds}^2 \qquad (a>0) \\ \text{(II, A}_0\text{)} & du^2+\left(\exp 2cu-\dfrac{\bar{\kappa}}{2c^2}\right)\overline{ds}^2 \qquad (\bar{\kappa}<0) \\ \text{(II, B)} & du^2+\dfrac{1}{2}\left(\cosh 2cu-\dfrac{\bar{\kappa}}{c^2}\right)\overline{ds}^2 \quad (\bar{\kappa}<c^2) \\ \text{(III)} & du^2+\dfrac{1}{2}\left(\dfrac{\bar{\kappa}}{c^2}-\cos 2cu\right)\overline{ds}^2 \quad (\bar{\kappa}>c^2) \end{cases}$$

in the whole manifold M, respecitively. On the other hand, the existence of a stationary point of ρ is possible in the cases (I, B), (II, B) and (III). Then M is of constant curvature and the scalar curvature is equal to $\bar{\kappa}=1$ in (I, B), or $\bar{\kappa}=c^2$ in (II, B) and (III). There is one stationary point corresponding to $u=0$ in (I, B) and (II, B) and are two corresponding to $u=0$ and $u=\dfrac{\pi}{c}$ in (III). The metric form of M is given by

$$(2.6)\qquad ds^2=\begin{cases} \text{(I, B)} & du^2+u^2\overline{ds}^2 \\ \text{(II, B)} & du^2+(\sinh cu)^2\overline{ds}^2 \\ \text{(III)} & du^2+(\sin cu)^2\overline{ds}^2. \end{cases}$$

These are the polar forms of the metrics of (I, B) a Euclidean space, (II, B) a hyperbolic space and (III) a sphere, respectively. Thus we have established the following

Theorem 1. *Let M be a 4-dimensional complete Riemannian manifold of constant scalar curvature κ and suppose that M admits a concircular scalar field ρ. If there exists no stationary point of ρ, then the manifold M is topologically the direct product of a straight line I and a 3-dimensional complete manifold $\overline{M}$ of constant scalar curvature $\bar{\kappa}$ and, the metric form of M is given by one of* (2. 5). *If there exists a stationary point of ρ, then the manifold M is a Eucldean space, a hyperbolic space or a sphere.*

3. 4-dimensional Einstein manifolds

Let M be an n-dimensional Einstein manifold admitting a concircular scalar field ρ. Applying Ricci's formula to the equation (1. 1), we have

$$-K_{\nu\mu\lambda}{}^{\kappa}\rho_{\kappa}=\phi_{\nu}g_{\mu\lambda}-\phi_{\mu}g_{\nu\lambda}, \tag{3. 1}$$

and contracting with $g^{\mu\lambda}$,

$$-K_{\nu}^{\kappa}\rho_{n}=(n-1)\phi_{\nu}. \tag{3. 2}$$

Since M is an Einstein manifold, that is, $K_{\nu}^{\kappa}=(n-1)\kappa\phi_{\nu}^{\kappa}$, we have the equation

$$\phi_{\nu}=-\kappa\rho_{\nu}, \quad \text{or} \quad \phi=-\kappa\rho+b \tag{3. 3}$$

and

$$\nabla_{\mu}\nabla_{\lambda}\rho=(-\kappa\rho+b)g_{\mu\lambda}, \tag{3. 4}$$

where b is an integral constant. Hence, in an Einstein manifold, a concircular field is special and the characteristic constant of ρ is equal to the constant scalar curvature κ.

Theorem 2. *If a 4-dimensional Einstein manifold M admits a concircular scalar field, then the manifold is of constant curvature.*

Proof. With respect to an adapted coordinate system (u^{κ}) in a neighborhood U of any ordinary points of M, the equation (3. 4) is reduced to the ordinary equation

$$\rho''=-\kappa\rho+b \quad \text{and} \quad \rho'''=-\kappa\rho'. \tag{3. 5}$$

Substituting these into the third equation of (1. 7) for $n=4$, we have

$$K_{ji}=\overline{K}_{ji}-(2\rho'''+\rho'\rho''')f_{ji}=\overline{K}_{ji}-(2\rho''^2-\kappa\rho'^2)f_{ji}. \tag{3. 6}$$

Since M is an Einstein manifold, we substitute $K_{ji}=3\kappa g_{ji}=3\kappa\rho'^2 f_{ji}$ into (3.6) and obtain the equations

$$\overline{K}_{ji}=(3\kappa\rho'^2+2\rho''^2-\kappa\rho'^2)f_{ji}=2(\kappa\rho'^2+\rho''^2)f_{ji}. \tag{3. 7}$$

This shows that $\overline{M}$ is a 3-dimensional Einstein manifold

$$\overline{K}_{ji}=2\ f_{ji} \tag{3. 8}$$

and constant scalar curvature $\overline{\kappa}$ is equal to

$$\overline{\kappa}=\kappa\rho'^2+\rho''^2. \tag{3. 9}$$

As it is known that a 3-dimensional Einstein manifold is of constant curvature, the manifold $\overline{M}$ is of constant curvature, that is,

$$\overline{K}_{kji}{}^{h}=\overline{\kappa}(\delta_k^h f_{ji}-\delta_j^h f_{ki}). \tag{3. 10}$$

Substituting this and (3. 9) into the third equation of (1. 6), we have

$$\begin{aligned} K_{kji}{}^{h}&=(\overline{\kappa}-\rho''^2)(\delta_k^h f_{ji}-\delta_j^h f_{ki})=\kappa\rho'^2(\delta_k^h f_{ji}-\delta_j^h f_{ki}) \\ &=\kappa(\delta_k^h g_{ji}-\delta_j^h g_{ki}). \end{aligned} \tag{3. 11}$$

The first and second equations of (1. 6) are rewritten as

$$\begin{aligned} K_{1j1}{}^{h}&=-\ .\ \delta_j^h=\kappa(\delta_1^h g_{ji}-\delta_j^h g_{11}) \\ K_{1ji}{}^{h}&=\kappa\rho'^2 f_{ji}=\kappa(\delta_1^1 g_{ji}-\delta_j^1 g_{1i}) \end{aligned} \tag{3. 12}$$

by means of (1. 5). The equations (3. 11) and (3. 12) together make the tensor equation $K_{\nu\mu\lambda}{}^{\kappa}=\kappa(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda})$. Therefore the manifold M is of constant curvature at ordinary points. Since the stationary point of ρ is isolated if there is any, M is of constant curvature.

Q. E. D.

By virtue of this theorem, we give a slight generalization of Kulkarni's theorem [2] in a different way.

Corollary. *Let M and M^* be 4-dimensional Einstein manifolds which are nowhere of constant curvature. Then every conformal map of M into M^* is a homothety.*

Proof. Let f be a conformal map of M into M^*, and denote the metric tensor f^*g by components $g^*_{\mu\lambda}$. Then they are related by the equation $g^*_{\mu\lambda}=\frac{1}{\rho^2}g_{\mu\lambda}$, where ρ is a positive valued scalar field. We indicate by asterisking quantities of $g^*_{\mu\lambda}$ corresponding to those of $g_{\mu\lambda}$. Then we obtain the transformation formulas

$$(3.13)\qquad \left\{{\kappa \atop \mu\lambda}\right\}^*=\left\{{\kappa \atop \mu\lambda}\right\}-\frac{1}{\rho}(\delta_\mu^\kappa\rho_\lambda+\delta_\lambda^\kappa\rho_\mu-g_{\mu\lambda}\rho^\kappa),$$

$$(3.14)\qquad K^*_{\nu\mu\lambda}{}^\kappa=K_{\nu\mu\lambda}{}^\kappa+\frac{1}{\rho}(\delta_\nu^\kappa\nabla_\mu\rho_\lambda-\delta_\mu^\kappa\nabla_\nu\rho_\lambda+g_{\mu\lambda}\nabla_\nu\rho^\kappa-g_{\nu\lambda}\nabla_\mu\rho^\kappa)$$
$$-\frac{1}{\rho^2}\rho_\omega\rho^\omega(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda}),$$

$$(3.15)\qquad K^*_{\mu\lambda}=K_{\mu\lambda}+\frac{1}{\rho}2\nabla_\mu\rho_\lambda+\frac{1}{\rho}g_{\mu\lambda}\nabla_\kappa\rho^\kappa-\frac{1}{\rho^2}3\rho_\kappa\rho^\kappa g_{\mu\lambda}.$$

Since M and M^* are Einstein manifolds, we substitute $K^*_{\kappa\lambda}=3\kappa^*g^*_{\mu\lambda}=3\kappa^*\rho^{-2}g_{\mu\lambda}$ and $K_{\mu\lambda}=3\kappa g_{\mu\lambda}$ into (3.15), and obtain the following equation

$$(3.16)\qquad \nabla_\kappa\rho_\lambda=\frac{\rho}{2}(3\rho^{-2}\kappa^*-3\kappa-\frac{1}{\rho}\nabla_\kappa\rho^\kappa+\frac{3}{\rho^2}\rho^\kappa)g_{\mu\lambda}.$$

This equation means that ρ is a concircular scalar field if ρ would not be constant, and M would be a manifold of constant curvature by Theorem 2. This is a contradiction. Therefore ρ must be a constant, that is, f is a homothety.

Q. E. D.

We notice that κ^* need not be equal to κ.

4. Manifolds of $H_{\sigma\omega\nu\mu\lambda}{}^\kappa=0$

E. Cartan defined the tensor $H_{\sigma\omega\nu\mu\lambda}{}^\kappa$ by the equation

$$(4.1)\qquad H_{\sigma\omega\nu\mu\lambda}{}^\kappa=K_{\sigma\omega\nu}{}^\tau K_{\tau\mu\lambda}{}^\kappa+K_{\sigma\omega\mu}{}^\tau K_{\nu\tau\lambda}{}^\kappa+K_{\sigma\omega\lambda}{}^\tau K_{\nu\mu\tau}{}^\kappa-K_{\sigma\omega\tau}{}^\kappa K_{\nu\mu\lambda}{}^\tau.$$

Theorem 3. *If M is a manifold with the property $H_{\sigma\omega\nu\mu\lambda}{}^\kappa=0$ and admits a concircular scalar field ρ such that ϕ is not identically constant, then the manifold is of constant curvature.*

Proof. We refer to an adapted coordinate system (u^κ) in a neighborhood of any ordinary points of ρ and put the indices $\sigma=1, \omega=k, \kappa=h, \lambda=i, \mu=j, \nu=1$ in the equation (4.1). Taking account of the components (1.6) of the curvature tensor, we have

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$$\frac{\rho''^{1}}{\rho'}\{\overline{K}_{kji}{}^{h}-\rho''^{2}(\delta_k^h f_{ji}-\delta_j^h f_{ki})\}-\frac{\rho'''}{\rho'}(\rho'\rho''')(\delta_j^h f_{ki}-\delta_k^h f_{ji})=0$$

or

(4.2) $$\frac{\rho'''}{\rho'}\{\overline{K}_{kji}{}^{h}-(\rho''^{2}-\rho'\rho''')(\delta_k^h f_{ji}-\delta_j^h f_{ki})\}=0.$$

As $\rho'''\neq 0$, we have from (4.2) the equation

(4.3) $$\overline{K}_{kji}{}^{h}=(\rho''^{2}-\rho'\rho''')(\delta_k^h f_{ji}-\delta_j^h f_{ki}),$$

from which follows $\rho''^2-\rho'\rho'''=\bar{\kappa}$. This implies that $\overline{M}$ is a manifold of constant curvature. Substituting (4.3) into the third of (1.6), we have

$$K_{kji}{}^{h}=-\frac{\rho'''}{\rho'}\delta_k^h g_{ji}-\delta_j^h g_{ki}).$$

From the first and second equations of (1.6) and the other components being zero, we can obtain the tensor equation

$$K_{\omega\mu\lambda}{}^{\kappa}=-\frac{\rho'''}{\rho'}(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda}).$$

Since a stationary point is isolated, M is a manifold of constant curvature.

Q. E. D.

We put $J=\{u\,|\,\rho'''(u)=0\}$. If J contains intervals, we have $\rho=Au^2+Bu+C$, where A, B and C are constant. At the points of the complement J^c of J, the equation $\rho''^2-\rho'\rho'''=\bar{\kappa}$ is satisfied under initial conditions $\rho''(0)=2A$, $\rho'(0)=B$ and $\rho(0)=C$ by a suitable choice of arc-length u. Then the solution is given by

$$\rho(u)=\frac{B^2}{(4A^2-\bar{\kappa})}\left(2A\cosh\frac{\sqrt{4A^2-\bar{\kappa}}}{B}u+\sqrt{4A^2-\bar{\kappa}}\sinh\frac{\sqrt{4A^2-\bar{\kappa}}}{B}u-2A\right)+C.$$

So the differentiability is broken at the point of $u=0$. Therefore, J is equal to the whole straight line I or discrete. When J is the straight line, $\rho'''(u)=0$ for every point of I, that is, $\rho''=b$ and $\nabla_\mu\rho_\lambda=bg_{\mu\lambda}$, b being constant. It follows that ρ_λ is concurrent or parallel. If J is discrete, M is of constant curvature at any point.

5. Manifolds of $\overset{*}{H}_{\sigma\omega\mu\lambda}=0$

We put the tensor,

(5.1) $$H_{\sigma\omega\mu\lambda}=H_{\sigma\omega\nu\mu\lambda}{}^{\nu}=K_{\sigma\omega\mu}{}^{\nu}K_{\nu\lambda}+K_{\sigma\omega\lambda}{}^{\nu}K_{\mu\nu}.$$

Theorem 4. *If M is a manifold having the property* $\overset{*}{H}_{\sigma\omega\mu\lambda}=0$ *and admits a concircular scalar field* ρ *such that* ϕ *is not identically constant, then* M *is an Einstein manifold.*

Proof. Referring to an adapted coordinate system (u^{κ}) in a neighborhood of any ordinary point of ρ, and putting the indices $\sigma=1$, $\omega=j$, $\lambda=i$, $\mu=1$ in the equation (5.1), we have

$$\frac{\rho'''}{\rho'}\{\overline{K}_{ji}-(n-2)(\rho''^2-\rho'\rho''')f_{ji}\}=0. \tag{5.2}$$

As $\rho'''\neq 0$, it follows from (5.2) that

$$\overline{K}_{ji}=(n-2)(\rho''^2-\rho'\rho'')f_{ji}. \tag{5.3}$$

This implies that $\overline{M}$ is an Einstein manifold and the scalar curvature is equal to $\bar{\kappa}=\rho''^2-\rho'\rho'''$. Substituting (5.3) into the third of (1.7), we have

$$K_{ji}=-(n-1)\frac{\rho'''}{\rho'}g_{ji}. \tag{5.4}$$

From the first and second equations of (1.7), we have the tensor equation $K_{\mu\lambda}=-(n-1)\frac{\rho'''}{\rho'}g_{\mu\lambda}$ and hence M is an Einstein manifold.

Q. E. D.

When ρ''' vanishes, the same argument as that of 4 is applicable.

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