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Compactification of Topological Spaces

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COMPACTIFICATION OF TOPOLOGICAL SPACES

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One of the writers[1] has given the compactification of a T-space[2] R as follows:

Let us denote by R^* the totality of all ultrafilters in R. Then it is proved that the family

$$\{U^{*\,3)} \mid U \text{ is an open set in } R\}^{4)}$$

can be taken as a basis of open sets in R^* and R^* becomes a compact[5] *T-space containing the set:*

$$\tilde{R} = \{\mathfrak{F}_x{}^{6)} \mid x \in R;\ \mathfrak{F}_x \text{ is the ultrafilter containing } x\}$$

as a dense subset, moreover, $\tilde{R}$ is homeomorphic with R by the mapping φ defined by

$$\varphi(x) = \mathfrak{F}_x.$$

It is the purpose of this note to make clear some relations among our compactification and Wallman's of a T_0-space and Čech's of a completely regular space.

§1. An extension theorem of continuous functions. First of all we shall prove the

Theorem 1. *Let f be a real valued bounded continuous function defined on R. Then there exists a real valued bounded continuous function f^* defined on R^* such that*

$$f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x) \text{ and } f^*(\mathfrak{F}_x) = f(x).$$

1) T. Inagaki: Contribution à la topologie I, Math. Journ. of Okayama Univ., Vol. 1 (1952), pp. 158 - 166.

2) A T-space is a topological space which satisfies the conditions:

$$1)\ \bar{\varphi} = \varphi;\quad 2)\ \bar{M} \supset M;\quad 3)\ \overline{M \cup N} = \bar{M} \cup \bar{N};\quad 4)\ \bar{\bar{M}} = \bar{M},$$

where M and N are subsets of R and φ is the null set as usual.

3) M^* is the totality of all ultrafilters in R, which contains M.

4) The notation $\{A \mid B\}$ means the totality of all sets A satisfying the condition B.

5) A T-space R is called compact if each filter in R has at least one cluster point.

6) In future, we shall denote by $\mathfrak{F}$ a ultrafilter in R and by $\mathfrak{F}_x$ the ultrafilter containing x.

Since φ is the homeomorphism of R on $\widetilde{R}$, if we regard $\widetilde{R}$ as the space R, then the theorem says that f can be extended on R^*.

Proof. Let $\mathfrak{F} \in R^*$ and let

$$t = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) \quad \text{and} \quad t' = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x).$$

Then it is clear that $t \geqslant t'$, but we can show that $t = t'$. In fact, by definition of t, for every positive number ε there exists a set $A_1 \in \mathfrak{F}$ such that $t \leqslant \sup_{x \in A_1} f(x) < t + \varepsilon$, and hence for some point $x_1 \in A_1$, $t - \varepsilon < f(x_1) < t + \varepsilon$. Therefore, if we denote by B the set $\{x \mid t - \varepsilon < f(x) < t + \varepsilon\}$, then $B \neq \phi$. We now show that $B \in \mathfrak{F}$. Since $\mathfrak{F}$ is an ultrafilter, if $B \overline{\in} \mathfrak{F}$, then CB[1] $\in \mathfrak{F}$ and $A_1 \cap CB \in \mathfrak{F}$. Let x be a point of $A_1 \cap CB$, then $f(x) < t + \varepsilon$ by definition of A_1 and $f(x) \leqslant t - \varepsilon$ or $t + \varepsilon \leqslant f(x)$ by definition of B. Therefore we can say that if $x \in A_1 \cap CB$ then $f(x) \leqslant t - \varepsilon$, and hence $\sup_{x \in A_1 \cap CB} f(x) \leqslant t - \varepsilon$. This contradicts with the definition of t. Thus we have $B \in \mathfrak{F}$ and $\inf_{x \in B} f(x) \geqslant t - \varepsilon$. Therefore, by definition of t', we have $t - \varepsilon \leqslant t'$, and this shows that $t \leqslant t'$; hence $t = t'$.

From what we have just proved above, we can define a function f^* defined on R^* by the equality

$$f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x).$$

It follows evidently that f^* is bounded and $f^*(\mathfrak{F}_x) = f(x)$.

To prove that f^* is continuous, let $t = f^*(\mathfrak{F})$. In proving that $t \leqslant t'$ above, we have shown that for every positive number ε, the set $G = \{x \mid t - \frac{\varepsilon}{2} < f(x) < t + \frac{\varepsilon}{2}\}$ belongs to $\mathfrak{F}$. Since f is continuous, G is open in R and so G^* is open in R^*. Now, it follows from the definition of f^* that $f^*(G) \subset [t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}] \subset (t - \varepsilon, t + \varepsilon)$. This shows that f^* is continuous, Q.E.D.

Remark. As it is easily seen, if f is continuous, then for any set A, $\sup_{x \in A} f(x) = \sup_{x \in \bar{A}} f(x)$ and $\inf_{x \in A} f(x) = \inf_{x \in \bar{A}} f(x)$; therefore the function f^* can be defined by the equality

$$(1) \qquad f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in \bar{A}} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in \bar{A}} f(x).$$

1) CB denote the complement of B, that is, $CB = R - B$.

§2. **Compactification of a T_0-space**[1]. In this place, we suppose that R is a T_0-space. For a point $\mathfrak{F} \in R^*$ we define $\overline{\mathfrak{F}}$ by

$$\overline{\mathfrak{F}} = \{F \mid F \in \mathfrak{F} \text{ and } F \text{ is closed in } R\}.$$

For two points $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of R^*, we write

$$\mathfrak{F}_1 \sim \mathfrak{F}_2, \quad \text{if} \quad \overline{\mathfrak{F}}_1 = \overline{\mathfrak{F}}_2.$$

Obviously the relation $\sim$ is an equivalence relation, hence the relation $\sim$ divides R^* into disjoint classes of equivalent points.

We introduce the notations:

R° = the totality of all classes of equivalent points;

$[\mathfrak{F}]$ = the class of equivalent points, which contains $\mathfrak{F}$.

It is important to remark that if $x \neq y$ then $[\mathfrak{F}_x] \neq [\mathfrak{F}_y]$. For, since $x \neq y$ and R is a T_0-space, at least one of $x \overline{\in} \bar{y}$ and $y \overline{\in} \bar{x}$ holds, from which $\bar{x} \neq \bar{y}$; hence it is not hard to see that $\overline{\mathfrak{F}}_x \neq \overline{\mathfrak{F}}_y$.

We define the mapping ϕ_1 of the set R^* on the set R° such that

$$[\mathfrak{F}] = \phi_1(\mathfrak{F}).$$

Then it is evident that ϕ_1 is one-to-one mapping between $\tilde{R}$ and $\tilde{R}^\circ$, by setting

$$\tilde{R}^\circ = \phi_1(\tilde{R}).$$

Now it is not difficult to see that the family

$$\Gamma = \{F^\circ \mid \phi_1^{-1}(F^\circ) = F^*, \text{ where } F \text{ is closed in } R\}$$

can be taken as a basis of closed sets in R°, and, moreover, R° becomes a T-space and ϕ_1 is continuous.

From this definition, we can prove the

Theorem 2. *R° is a compact T_0-space and contains a dense subset $\tilde{R}^\circ$ which is homeomorphic with R.*

Proof. R° is compact, because R^* is compact and ϕ_1 is continuous in the topology introduced in R°.

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1) A T_0-space is a T-space such that, for any two different points x and y, at least one of $x \overline{\in} \bar{y}$ or $y \overline{\in} \bar{x}$ holds.

To show that R° is a T_0-space, we shall prove first that

$$\phi_1^{-1}(\phi_1(U^*)) = U^* \quad \text{for any open set } U \text{ in } R.$$

In fact, if $\mathfrak{F} \in \phi_1^{-1}(\phi_1(U^*))$, there exists a point $\mathfrak{F}_1 \in U^*$ such that $\bar{\mathfrak{F}} = \bar{\mathfrak{F}}_1$. Hence $U \in \mathfrak{F}_1$ and $CU \bar{\in} \mathfrak{F}_1$. Since CU is closed, we have $CU \bar{\in} \mathfrak{F}$; and so $U \in \mathfrak{F}$. This shows that $\phi_1^{-1}(\phi_1(U^*)) \subset U^*$. Since it is evident that $\phi_1^{-1}(\phi_1(U^*)) \supset U^*$, we have $\phi_1^{-1}(\phi_1(U^*)) = U^*$.

Under this remark, we shall show that R° is a T_0-space. Let $[\mathfrak{F}_1]$ and $[\mathfrak{F}_2]$ be two different points of R°, then $\bar{\mathfrak{F}}_1 \neq \bar{\mathfrak{F}}_2$. Hence, at least one of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, say $\mathfrak{F}_1$, contains a closed set F such that $F \bar{\in} \mathfrak{F}_2$, and so $CF \bar{\in} \mathfrak{F}_1$ and $CF \in \mathfrak{F}_2$. Since CF is open, we get $\phi_1^{-1}(\phi_1((CF)^*)) = (CF)^*$, and clearly $[\mathfrak{F}_1] \bar{\in} \phi_1((CF)^*)$ and $[\mathfrak{F}_2] \in \phi_1((CF)^*)$. Hence it follows that R° is a T_0-space.

In order to show that R and $\tilde{R}^{\circ}$ are homeomorphic, as it is readily seen, it is sufficient to prove that the mapping $\phi_1\varphi$ sends an open set in R to an open set in $\tilde{R}^{\circ}$. Now let G be an open set in R, then

$$\phi_1\varphi(G) = \phi_1(G^* \cap \tilde{R}) = \phi_1(\bigcup_{x \in G} \mathfrak{F}_x) = \bigcup_{x \in G} \phi_1(\mathfrak{F}_x) = \tilde{R}^{\circ} \cap \phi_1(G^*),$$

from which $\phi_1\varphi(G)$ is open in $\tilde{R}^{\circ}$, because $\phi_1(G^*)$ is open in R°, Q.E.D.

Theorem 3. *Let f be a real valued bounded continuous function defined on R. Then there exists a real valued bounded continuous function f° defined on R° such that*

$$f^{\circ}([\mathfrak{F}]) = f^*(\mathfrak{F}).$$

Proof. Since f is continuous, the function f^* defined by the equation (1) in §1 takes the same value at each point which belongs to an equivalence class in R^*. Thus we can define a function f° such that

$$f^{\circ}([\mathfrak{F}]) = f^*(\mathfrak{F}).$$

To prove that f° is continuous, let $[\mathfrak{F}]$ be a point of R° and $t = f^{\circ}([\mathfrak{F}]) = f^*(\mathfrak{F})$. Since f^* is continuous, for every neighborhood $U(t)$ of t, there exists an open set G in R such that $G \in \mathfrak{F}$ and $U(t) \supset f^*(G^*)$. On the other hand, as we proved in the proof of Theorem 2, $G \in \mathfrak{F}$ implies $[\mathfrak{F}] \subset G^*$ and $\phi_1(G^*)$ is open in R°. Hence

$U(t) \supset f^*(G^*) = f^\circ(\phi_1(G^*))$, this shows that f° is continuous, Q.E.D.

Let us denote by $\alpha(R)$ the totality of all dual prime ideals in the lattice $\mathfrak{L}$ composed of all closed sets in R, then we have the

Lemma 1. *There is an one-to-one mapping of the set R° on $\alpha(R)$.*

Proof. To a point $[\mathfrak{F}] \in R^\circ$ we correspond the set $\overline{\mathfrak{F}}$, and we write

$$\overline{\mathfrak{F}} = \psi([\mathfrak{F}]).$$

First of all we prove that $\overline{\mathfrak{F}}$ is a dual prime ideal. It is evident that $\overline{\mathfrak{F}}$ is a dual ideal. In order to show that $\overline{\mathfrak{F}}$ is prime, let F_1 and F_2 be closed sets in R such that $F_1 \cup F_2 \in \overline{\mathfrak{F}}$. If we suppose that $F_1 \bar{\in} \overline{\mathfrak{F}}$, then, since F_1 is closed and $\mathfrak{F}$ is an ultrafilter in R, we have $F_1 \bar{\in} \mathfrak{F}$ and so $CF_1 \in \mathfrak{F}$. Hence it follows that $CF_1 \cap (F_1 \cup F_2) \in \mathfrak{F}$, from which $F_2 \in \mathfrak{F}$ and $F_2 \in \overline{\mathfrak{F}}$, since $CF_1 \cap (F_1 \cup F_2) \subset F_2$ and F_2 is closed. This shows that $\overline{\mathfrak{F}}$ is a dual prime ideal, and hence $\overline{\mathfrak{F}} \in \alpha(R)$. Moreover, it is evident that if $[\mathfrak{F}_1] \neq [\mathfrak{F}_2]$, then $\psi([\mathfrak{F}_1]) \neq \psi([\mathfrak{F}_2])$.

We shall now prove that $\psi(R^\circ) = \alpha(R)$. In fact, let $\mathfrak{M}$ be a dual prime ideal in $\mathfrak{L}$ and let

$$\mathfrak{N} = \{G \mid G \text{ is open in } R \text{ and } CG \bar{\in} \mathfrak{M}\}.$$

Since $R \in \mathfrak{M}$, we have $\phi \bar{\in} \mathfrak{N}$. Next, in order to show that $\mathfrak{N}$ has the finite intersection property, take two sets G_1 and G_2 of $\mathfrak{N}$. Then $CG_1 \bar{\in} \mathfrak{M}$, $CG_2 \bar{\in} \mathfrak{M}$ and $\mathfrak{M} \not\ni CG_1 \cup CG_2 = C(G_1 \cap G_2)$, since $\mathfrak{M}$ is prime. Therefore, $G_1 \cap G_2 \in \mathfrak{N}$ and $G_1 \cap G_2 \neq \phi$, from which we say that $\mathfrak{N}$ has the finite intersection property. Now let $F \in \mathfrak{M}$ and $G \in \mathfrak{N}$. If we suppose that $F \cap G = \phi$, then $F \subset CG$ and so $CG \in \mathfrak{M}$. This contradicts with $CG \bar{\in} \mathfrak{M}$, and hence $F \cap G \neq \phi$. From what we have proved above, we can say that the totality of all sets $F \cap G$, where $F \in \mathfrak{M}$ and $G \in \mathfrak{N}$, forms a basis of a filter. Hence there exists an ultrafilter $\mathfrak{F}$ which contains the above basis: $\mathfrak{M} \cup \mathfrak{N} \subset \mathfrak{F}$. For this ultrafilter $\mathfrak{F}$, we can show that $\overline{\mathfrak{F}} = \mathfrak{M}$. In fact, if $F \in \overline{\mathfrak{F}}$, then $F \in \mathfrak{F}$ and $CF \bar{\in} \mathfrak{F}$, from which $CF \bar{\in} \mathfrak{N}$. Hence $F \in \mathfrak{M}$ by definition of $\mathfrak{N}$, and so $\overline{\mathfrak{F}} \subset \mathfrak{M}$.

From what we have just proved, it follows that $\psi(R^\circ) = \alpha(R)$ and ψ is an one-to-one mapping between R° and $\alpha(R)$, Q.E.D.

We note here that a set $F^\circ \subset R^\circ$ belongs to the closed basis Γ of R°, if and only if there exists a closed set F in R such that $\psi(F^\circ) = \alpha(F)$ by setting

$$\alpha(F) = \{\mathfrak{M} \mid \mathfrak{M} \text{ is a dual prime ideal in } \mathfrak{L} \text{ and } F \in \mathfrak{M}\}.$$

In fact, if F° belongs to Γ, there exists a closed set F in R such that $\phi_1^{-1}(F^\circ) = F^*$. Hence $\psi(F^\circ) = \psi(\phi_1(F^*)) = \psi(\{[\mathfrak{F}] \mid F \in \mathfrak{F}\}) = \alpha(F)$.

Conversely, let F be a closed set in R and $\psi(F^\circ) = \alpha(F)$. Then, since ψ is one-to-one, we have $\phi_1^{-1}(F^\circ) = \phi_1^{-1}(\psi^{-1}(\psi(F^\circ))) = \phi_1^{-1}(\psi^{-1}(\alpha(F))) = \phi_1^{-1}(\{[\mathfrak{F}] \mid F \in \mathfrak{F}\}) = F^*$, and hence F° is contained in the closed basis Γ of R°.

Thus we have the

Lemma 2. *If we introduce the topology in $\alpha(R)$ such that the family $\{\alpha(F) \mid F$ is closed in $R\}$ is a closed basis of $\alpha(R)$, then the mapping $\psi([\mathfrak{F}]) = \overline{\mathfrak{F}}$ is a homeomorphism of the space R° on the space $\alpha(R)$.*

§3. Compactification of a T_1-space. In this section, we suppose that R is a T_1-space. Now let

$$\beta^\circ(R) = \{[\mathfrak{F}] \mid \overline{\mathfrak{F}} \text{ is a maximal dual ideal in } \mathfrak{L}\},$$
$$\beta(R) = \psi(\beta^\circ(R)).$$

Then, it is evident that $\beta(R)$ is a subset of $\alpha(R)$ and consists of all maximal dual prime ideals in $\mathfrak{L}$. Moreover, since $\beta^\circ(R)$ and $\beta(R)$ are regarded as the subspaces of R° and $\alpha(R)$ respectively, $\beta^\circ(R)$ and $\beta(R)$ are homeomorphic with each other.

Since R is a T_1-space, it is important to remark that $[\mathfrak{F}_x]$ is $\mathfrak{F}_x$ itself and hence $\tilde{R} = \tilde{R}^\circ \subset \beta^\circ(R)$.

Under these remarks, we have the well known

Wallman's Theorem. *The space $\beta(R)$ is a compact T_1-space and contains a dense subset $\tilde{R}^\circ$ which is homeomorphic with R.*

But we give a proof of this theorem for the purpose to make clear the relation among the spaces considered in this note.

Proof. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be any two distinct points of $\beta(R)$. Then, since they are maximal ideals in $\mathfrak{L}$, any one of them, say $\mathfrak{M}_1$, contains a closed set F in R such that $F \bar{\in} \mathfrak{M}_2$. However, $\beta(R) - \alpha(F)$ is open in $\beta(R)$ and contains $\mathfrak{M}_2$ and not $\mathfrak{M}_1$, hence $\beta(R)$ is a T_1-space.

To show that $\beta(R)$ is compact, we take an ultrafilter F in $\beta(R)$. Obviously, since F is a filter in $\alpha(R)$ which is compact, there is a cluster point $\mathfrak{F}$ of F. As we know, there is an ultrafilter $\mathfrak{F}_1$ in R such that $\mathfrak{F}_1 \in \beta^\circ(R)$ and $\overline{\mathfrak{F}} \subset \overline{\mathfrak{F}}_1$. Since $\overline{\mathfrak{F}} \subset \overline{\mathfrak{F}}_1$, for any closed set F in R, $\alpha(R) - \alpha(F) \ni \mathfrak{F}_1$ implies $\alpha(R) - \alpha(F) \ni \mathfrak{F}$, and hence we can say that in the space $\alpha(R)$ each neighborhood of $\mathfrak{F}_1$ is also a neighborhood of $\mathfrak{F}$. Hence $\mathfrak{F}_1$ is a cluster point of F, and therefore $\beta(R)$ is compact.

It is almost evident that the subset $\widetilde{R}^c = \widetilde{R}$ of $\beta(R)$ is dense and homeomorphic with R, Q.E.D.

By using Theorem 3, we can prove the

Theorem 4. *A real valued bounded continuous function f defined on R is extendable to a real valued bounded continuous function f_β defined on $\beta(R)$ such that*

$$f_\beta(\overline{\mathfrak{F}}) = f^*(\mathfrak{F}).$$

Finally we give the

Theorem 5. *In order that $\beta(R)$ be normal, it is necessary and sufficient that R be normal.*

Proof. Suppose that $\beta(R)$ is normal, and let F_1 and F_2 be two disjoint closed sets in R. Then, the sets $F_\beta^1 = \beta(R) \cap \alpha(F_1)$ and $F_\beta^2 = \beta(R) \cap \alpha(F_2)$ are disjoint closed sets in $\beta(R)$. Hence there exists a continuous function f_β defined on $\beta(R)$ such that $f_\beta = 0$ on $F_\beta^1, f_\beta = 1$ on F_β^2 and $0 \leqslant f_\beta \leqslant 1$ on $\beta(R)$. If we define a function f by the equality $f(x) = f_\beta(\overline{\mathfrak{F}}_x)$, it is clear that f is continuous and $0 \leqslant f \leqslant 1$ on R. Moreover, if $x \in F_1$, then $\overline{\mathfrak{F}}_x \in \beta(R) \cap \alpha(F_1)$ and hence $f(x) = f_\beta(\overline{\mathfrak{F}}_x) = 0$. Similarly, if $x \in F_2$, then $f(x) = 1$. This shows that R is normal.

Conversely, let R be normal and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two distinct points in $\beta(R)$. Then, as $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are maximal dual ideals in $\mathfrak{L}$, there exist disjoint closed sets F_1 and F_2 such that $F_1 \in \mathfrak{M}_1$ and $F_2 \in \mathfrak{M}_2$. Since R is normal, there exists a continuous function f such that $f(x) = 0$ on F_1, $f(x) = 1$ on F_2 and $0 \leqslant f \leqslant 1$ on R. Let f_β be the function extended from f by Theorems 1 and 4, then it is clear that $f_\beta(\mathfrak{M}_1) = 0$ since $F_1 \in \mathfrak{M}_1$, and similarly $f_\beta(\mathfrak{M}_2) = 1$. This shows that $\beta(R)$ is a Hausdorff space, and hence, as $\beta(R)$ is compact, $\beta(R)$ is normal, Q.E.D.

§4. Compactification of a completely regular space. In this section, we suppose first that R is a complete Hausdorff space.[1] By considering the remark in §1, it is easily seen that, for two points

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1) We means by a complete Hausdorff space the Hausdorff space such that, for any two distinct points x and y, there exists a real valued bounded continuous function taking different values at x and y.

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$\mathfrak{F}_1$ and $\mathfrak{F}_2$, the following propositions are equivalent:

(α). *There is no real valued continuous function f^* defined on R^* such that $f^*(\mathfrak{F}_1)=0$, $f^*(\mathfrak{F}_2)=1$ and $0 \leqslant f^* \leqslant 1$ on R^*.*

(β). *Any real valued bounded continuous function f^* defined on R^* takes the same value at $\mathfrak{F}_1$ and $\mathfrak{F}_2$.*

If two points $\mathfrak{F}_1$ and $\mathfrak{F}_2$ satisfies the proposition, we write $\mathfrak{F}_1 \approx \mathfrak{F}_2$. Evidently the relation $\approx$ is an equivalence relation, hence the relation divides R^* into disjoint classes of equivalent points.

We introduce the notations:

$$\gamma(R) = \text{the totality of all classes of equivalent points;}$$
$$\{\mathfrak{F}\} = \text{the class which contains } \mathfrak{F}.$$

Moreover, we define the mapping ϕ_2 of R^* on $\gamma(R)$ such that

$$\phi_2(\mathfrak{F}) = \{\mathfrak{F}\}.$$

We shall give the

Lemma 3. *If $\overline{\mathfrak{F}}_1 \subset \overline{\mathfrak{F}}_2$, then $f^*(\mathfrak{F}_1)=f^*(\mathfrak{F}_2)$, for every real valued bounded continuous function f^* on R^*.*

Proof. Let $f(x)=f^*(\mathfrak{F}_x)$, then f is a real valued bounded continuous function and equality (1) in §1 holds. On the other hand, since $\overline{\mathfrak{F}}_1 \subset \overline{\mathfrak{F}}_2$, it follows that $\inf_{A \in \mathfrak{F}_1} \sup_{x \in \bar{A}} f(x) \geqslant \inf_{A \in \mathfrak{F}_2} \sup_{x \in \bar{A}} f(x)$ and $\sup_{A \in \mathfrak{F}_1} \inf_{x \in \bar{A}} f(x) \leqslant \sup_{A \in \mathfrak{F}_2} \inf_{x \in \bar{A}} f(x)$. Therefore, it is clear that $f^*(\mathfrak{F}_1) = f^*(\mathfrak{F}_2)$, Q.E.D.

From the lemma, it is not difficult to see that:

(a). $[\mathfrak{F}] \subset \{\mathfrak{F}\}$;

(b). $\{\mathfrak{F}\}$ contains a point $\mathfrak{F}_1$ such that $[\mathfrak{F}_1] \in \beta^\circ(R)$.

Since R is a complete Hausdorff space, for two distinct points x and y, there exists a continuous function f such that $f(x)=0$, $f(y)=1$ and $0 \leqslant f \leqslant 1$ on R. Then, by the equality (1) in §1, $f^*(\mathfrak{F}_x)=0$, $f^*(\mathfrak{F}_y)=1$ and $0 \leqslant f^* \leqslant 1$. Thus we have

(c). If $x \neq y$, then $\phi_2(\mathfrak{F}_x) \neq \phi_2(\mathfrak{F}_y)$.

Hence, if we put

$$\tilde{\gamma}(R) = \phi_2(\tilde{R}),$$

then ϕ_2 gives an one-to-one correspondence between $\tilde{R}$ and $\tilde{\gamma}(R)$.

Moreover, if we take the set:

$$\{F_\gamma \mid \phi_2^{-1}(F_\gamma) \text{ is closed in } R^*\},$$

as the totality of all closed sets in $\gamma(R)$, then $\gamma(R)$ is a T-space and ϕ_2 is a continuous mapping.

Thus we can prove more precisely the

Thorem 6. *The space $\gamma(R)$ is a compact Hausdorff space and contains a dense subset $\tilde{\gamma}(R)$. Moreover, a real valued bounded continuous function f defined on R can be extended to the function f_γ defined on $\gamma(R)$ such that*

$$f_\gamma(\{\mathfrak{F}\}) = f^*(\mathfrak{F}).$$

Proof. Since ϕ_2 is continuous, it follows that $\gamma(R)$ is compact.

Let $\{\mathfrak{F}_1\}$ and $\{\mathfrak{F}_2\}$ be two distinct points in $\gamma(R)$. Then $\mathfrak{F}_1 \not\approx \mathfrak{F}_2$ and, therefore, there exists a continuous function f^* defined on R^* such that $f^*(\mathfrak{F}_1) = 0$, $f^*(\mathfrak{F}_2) = 1$ and $0 \leqq f^* \leqq 1$ on R^*. Hence, if we put $U_1^* = f^{*-1}([0, \frac{1}{2}))$ and $U_2^* = f^{*-1}((\frac{1}{2}, 1])$, then U_1^* and U_2^* are disjoint open sets in R^*. Moreover, it is evident, from the definition of the relation $\approx$, that if $\mathfrak{F} \in U_i^*$ then $\{\mathfrak{F}\} \in U_i^*$, from which it is easily seen that $U_i^* = \phi_2^{-1}(\phi_2(U_i^*))$, $(i = 1, 2)$. Hence, it follows that $\phi_2(U_1^*)$ and $\phi_2(U_2^*)$ are disjoint open sets in $\gamma(R)$ such that $\{\mathfrak{F}_1\} \in \phi_2(U_1^*)$ and $\{\mathfrak{F}_2\} \in \phi_2(U_2^*)$. Thus $\gamma(R)$ is a Hausdorff space.

To prove that f_γ is the function extended from f, it is sufficient to verify that f_γ is continuous. Now, let $\{\mathfrak{F}\}$ be a point of $\gamma(R)$, $t = f_\gamma(\{\mathfrak{F}\})$, and let $U(t)$ be an open set containing t. Then, since $f_\gamma(\{\mathfrak{F}\}) = f^*(\mathfrak{F})$, the set $U^* = f^{*-1}(U(t))$ is open in R^* which contains $\mathfrak{F}$. As it is easily seen, from $U^* = f^{*-1}(U(t))$ and the definition of f_γ, that $f_\gamma^{-1}(U(t)) = \phi_2(f^{*-1}(U(t))) = \phi_2(U^*)$. As we proved above, $\phi_2(U^*)$ is open in $\gamma(R)$, which contains $\{\mathfrak{F}\}$, thus f_γ is continuous, Q.E.D.

Remark. In the same manner as we used above, we can define two spaces $\gamma_1(R)$ and $\gamma^\circ(R)$ from R° and $\beta^\circ(R)$ respectively. That is, if ϕ_3 and ϕ_4 are the mapping of R° on $\gamma_1(R)$ and that of $\beta^\circ(R)$ on $\gamma^\circ(R)$ respectively, then

$$\phi_3([\mathfrak{F}]) = \phi_1(\phi_2(\phi_1^{-1}([\mathfrak{F}]))), \qquad [\mathfrak{F}] \in R^\circ;$$
$$\phi_4([\mathfrak{F}]) = \phi_1(\phi_2(\phi_1^{-1}([\mathfrak{F}]))), \qquad [\mathfrak{F}] \in \beta^\circ(\mathfrak{F}).$$

Therefore, if we denote by ψ_3 and ψ_4 the mapping of $\gamma(R)$ on $\gamma_1(R)$ and that of $\gamma(R)$ on $\gamma^\circ(R)$ respectively, such that $\psi_3(\{\mathfrak{F}\}) = \phi_1(\{\mathfrak{F}\})$,

$\psi_4(\{\mathfrak{F}\}) = \phi_1(\{\mathfrak{F}\})$, then by considering the properties (a) and (b), we can prove that $\gamma(R)$, $\gamma_1(R)$ and $\gamma^\circ(R)$ are homeomorphic with each other.

The space $\tilde{\gamma}(R)$ in the Theorem 6 is a continuous image of $\tilde{R}$, but not necessarily homeomorphic with $\tilde{R}$. As the condition of that $\tilde{\gamma}(R)$ and $\tilde{R}$ be homeomorphic with each other, we have the

Lemma 4. *In order that $\tilde{\gamma}(R)$ and $\tilde{R}$ be homeomorphic, it is necessary and sufficient that R be a completely regular space.*

Proof. The necessity is evident.

Conversely, suppose that R be completely regular and we shall show that the mapping ϕ_2^{-1} of $\tilde{\gamma}(R)$ on $\tilde{R}$ is continuous.

Let F be a closed set of R and $\{\mathfrak{F}_x\}$ be a point of $\tilde{\gamma}(R) - \phi_2(F^* \cap \tilde{R})$. Then x does not belong to F, and, since R is completely regular, there exists a real valued continuous function defined on R such that $f(x) = 0$, $f(y) = 1$ for every point $y \in F$ and $0 \leqslant f \leqslant 1$. Let f_γ be the function extended from f by the Theorems 1 and 6, then it is clear that $f_\gamma(\{\mathfrak{F}_x\}) = 0$ and $f_\gamma(\{\mathfrak{F}\}) = 1$ for every point $\{\mathfrak{F}\} \in \phi_2(F^*)$. This implies that the open set $f_\gamma^{-1}([0, \frac{1}{2}))$ of $\gamma(R)$ contains $\{\mathfrak{F}_x\}$ and does not intersect with $\phi_2(F^*)$, and hence the open set $f_\gamma^{-1}([0, \frac{1}{2})) \cap \tilde{\gamma}(R)$ of $\tilde{\gamma}(R)$ contains $\{\mathfrak{F}_x\}$ and is contained in $\tilde{\gamma}(R) - \phi_2(F^* \cap \tilde{R})$. This shows that $\tilde{\gamma}(R) - \phi_2(F^* \cap \tilde{R})$ is open in $\tilde{\gamma}(R)$ and so $\phi_2(F^* \cap \tilde{R})$ is closed in $\tilde{\gamma}(R)$, Q.E.D.

Thus, as we know, there is the well known

Čech's Theorem. *For a completely regular space R, there is a space W satisfying the following conditions*:

(1) *W is a compact Hausdorff space*;

(2) *$R \subset W$ and $\bar{R} = W$*;

(3) *Any real valued bounded continuous function defined on R can be extended on W.*

Moreover, the spaces which satisfies the three conditions given above are homeomorphic with each other.

In this place, we will give a proof of this theorem for the purpose to make clear the structure of the space W.

Proof. The space $\gamma(R)$ is certainly a space satisfying the conditions (1), (2) and (3), thinking $\tilde{\gamma}(R) = \tilde{R}$ be R. Hence the existence is true.

Let W be a space satisfying the conditions (1), (2) and (3), and we will prove that W and $\gamma(R)$ are homeomorphic, by dividing the proof into seven parts.

(a). Let M be a subset of W. Then, by using the condition (1), it is not difficult to see that an ultrafilter $\mathfrak{F}(M)$ in M converges to only one point w which belongs to $\overline{M}$, and conversely, if $w \in \overline{M}$, then there exists an ultrafilter $\mathfrak{F}(M)$ in M such that $\mathfrak{F}(M)$ converges to w.

(b). Let $g(w)$ be a real valued bounded continuous function defined on W. Let $\mathfrak{F}$ be an ultrafilter in R such that $\mathfrak{F}$ converges to a point w and let $g(w)=t$. Since $g(w)$ is continuous, for every neighborhood $U(t)$ of t there exists a neighborhood $V(w)$ such that $U(t) \supset g(V(w))$, and from which we have:

$$g(w) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} g(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} g(x).$$

If we define a function f such that

$$f(x) = g(x), \quad x \in R \subset W,$$

then $f(x)$ is continuous and the function f^* defined by

$$f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x)$$

is the function obtained in the Theorem 1. Hence it is evident that if $\mathfrak{F}$ converges to w, then

$$g(w) = f^*(\mathfrak{F}).$$

Conversely, let f^* be a continuous function defined on R^*. Then the function f defined by $f(x) = f^*(\mathfrak{F}_x)$ is continuous on R. On the other hand, by the condition (3), there exists a continuons function $g(w)$ defined on W such that $f(x) = g(x)$ for $x \in R$. Since $g(w)$ is continuous, from what we have proved above, we have

$$g(w) = f^*(\mathfrak{F}),$$

for any ultrafilter $\mathfrak{F}$ in R, which converges to w.

(c). For a point $w \in W$, if we denote by $\{\mathfrak{F}\}_w$ the family of all ultrafilters $\mathfrak{F}$ in R which converges to w in W, then R^* is divided into disjoint classes $\{\mathfrak{F}\}_w$. From (b), it follows that $\{\mathfrak{F}\}_w \subset \{\mathfrak{F}\}$.

(d). Let w_1 and w_2 be two distinct points of W, then there exists a real valued bounded continuous function $g(w)$ defined on W such that $g(w_1) \neq g(w_2)$. On the other hand, by (a) and the condition (2), there exist two ultrafilters $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in R such that $\mathfrak{F}_1$ converges to w_1 and $\mathfrak{F}_2$ converges to w_2. Therefore, by (b), it follows that $f^*(\mathfrak{F}_1) \neq f^*(\mathfrak{F}_2)$, from which $\mathfrak{F}_1 \not\approx \mathfrak{F}_2$. This shows that $\{\mathfrak{F}\} \subset \{\mathfrak{F}\}_w$ for every point $w \in W$. Then we have, for $\{\mathfrak{F}\}$ there exists a point $w \in W$ such that $\{\mathfrak{F}\} = \{\mathfrak{F}\}_w$.

(e). By (d), we can define the function ϕ_5 of W on $\gamma(R)$ such that

$$\phi_5(w) = \{\mathfrak{F}\}_w.$$

It is evident that ϕ_5 is one-to-one.

(f). We shall prove that ϕ_5 is continuous. Let $\phi_5(w_0) = \{\mathfrak{F}_0\}$ and U_γ be an open set of $\gamma(R)$ containing $\{\mathfrak{F}_0\}$. By the normality of $\gamma(R)$, there exists a real valued continuous function f_γ defined on $\gamma(R)$ such that $f_\gamma(\{\mathfrak{F}_0\}) = 0$, $f_\gamma(\{\mathfrak{F}\}) = 1$ on $\gamma(R) - U_\gamma$ and $0 \leqslant f_\gamma \leqslant 1$ on $\gamma(R)$. If we define the function f^* on R^* such that $f^*(\mathfrak{F}) = f_\gamma(\{\mathfrak{F}\})$, then f^* is a real valued bounded continuous function on R^*, and, by (b), we get a continuous function g defined on W.

We shall show that the open set $V = g^{-1}([0, \frac{1}{2}))$ of W contains w_0 and $\phi_5(V) \subset U_\gamma$. By (b), $g(w_0) = f^*(\mathfrak{F}_0) = f_\gamma(\{\mathfrak{F}_0\}) = 0$ and so V contains w_0. Let w be a point of V, then $f_\gamma(\phi_5(w)) = f^*(\phi_2^{-1}(\phi_5(w)))$ $= g(w) \in [0, \frac{1}{2})$ and hence $\phi_5(w)$ does not belong to $\gamma(R) - U_\gamma$ and this shows that $\phi_5(w) \in U_\gamma$. Thus the proof of the continuity of ϕ_5 is established.

(g). Since W is compact and $\gamma(R)$ is a Hausdorff space, from (e) and (f), the mapping ϕ_5 is a homeomorphism. Thus the theorem is completely proved, Q.E.D.

Remark. Finally, if $\gamma(R)$ is homeomorphic with $\beta(R)$, then $\beta(R)$ is normal. Then, by Theorem 5, the space R is normal.

Conversely, let R be normal and let $\overline{\mathfrak{F}}_1$ and $\overline{\mathfrak{F}}_2$ be two distinct points of $\beta(R)$. Since $\overline{\mathfrak{F}}_1$ and $\overline{\mathfrak{F}}_2$ are maximal dual ideals in $\mathfrak{L}$, there exists two distinct closed sets F_1 and F_2 in R such that $F_1 \in \overline{\mathfrak{F}}_1$, $F_1 \bar{\in} \overline{\mathfrak{F}}_2$, $F_2 \bar{\in} \overline{\mathfrak{F}}_1$ and $F_2 \in \overline{\mathfrak{F}}_2$. Hence it is evident that $\beta(R) - \alpha(F_1)$ and $\beta(R) - \alpha(F_2)$ are disjoint open sets in $\beta(R)$ and the former contains $\mathfrak{F}_2$, the latter $\mathfrak{F}_1$. Then $\beta(R)$ is a Hausdorff space. Hence $\beta(R)$

satisfies the three conditions (1), (2) and (3) given in the Čech's Theorem, and, therefore, $\gamma(R)$ is homeomorphic with $\beta(R)$.

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