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Abstract

Let n be a positive integer, R a prime ring, U a nonzero right ideal, and d a derivation on R. Under appropriate additional hypotheses, we prove that if $d^n(U)$ is finite, then either R is finite or d is nilpotent. We also provide an extension to semiprime rings.

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ABSTRACT. Let n be a positive integer, R a prime ring, U a nonzero right ideal, and d a derivation on R. Under appropriate additional hypotheses, we prove that if $d^n(U)$ is finite, then either R is finite or d is nilpotent. We also provide an extension to semiprime rings.

In [2] it is proved that if R is a prime ring and d is a derivation on R such that $d(R)$ is finite, then either R is finite or $d = 0$. This result invites an investigation of prime rings with derivation such that $d^n(U)$ is finite for some derivation d, some $n \geq 1$, and some ideal (or right ideal) U. If U is a nonzero ideal, or if U is a nonzero right ideal and R is suitably-restricted, we can show that either R is finite or d is nilpotent on R.

1. Preliminaries

Let R be a ring and S a nonempty subset of R, and let f be a mapping from R to R. We say that f is nilpotent on S if $f^n(S) = \{0\}$ for some positive integer n; more generally, we call f periodic on S if there exist distinct positive integers m, n such that $f^n(x) = f^m(x)$ for all $x \in S$. We denote the right annihilator of S by $A_r(S)$.

We begin by stating and proving a lemma from [1].

Lemma 1.1. *An infinite prime ring contains no nonzero finite right ideal.*

Proof. Let R be infinite and prime, and suppose H is a nonzero finite right ideal. Let $H \setminus \{0\} = \{x_1, x_2, ..., x_n\}$. For each $i = 1, 2, ..., n$, define $f_i : R \to H$ by $f_i(r) = x_i r$ for all $r \in R$. Then $f_i(R)$ is finite, hence $\ker f_i = A_r(x_i)$ is a right ideal of R having finite index in R. Thus $K = \bigcap_{i=1}^{n} \ker f_i$ is a right ideal of finite index, necessarily nonzero, such that $HK = \{0\}$. But this cannot happen in a prime ring. □

It is well-known that if R is a ring of prime characteristic p and d is a derivation on R, then d^p is also a derivation. This observation is the key to the following lemma, which we shall use several times.

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Lemma 1.2. *Let n be a fixed positive integer, and let $\mathcal{R}$ be a class of prime rings with the following property:*

(*) If $R \in \mathcal{R}$ admits a nonzero derivation d such that $d(U)$ is finite for some nonzero ideal (resp. right ideal) U, then R is finite.

Then for any $R \in \mathcal{R}$ and any derivation d such that $d^n(U)$ is finite for some nonzero ideal (resp. right ideal) U, either R is finite or d is nilpotent on R.

Proof. It will suffice to prove the right ideal version. Let $R \in \mathcal{R}$ and U a nonzero right ideal of R, and let d be a derivation on R such that $d^n(U)$ is finite. If $char R = 0$, then $d^n(U) = \{0\}$; and by a result of Chung and Luh [4], d is nilpotent on R. Thus, we assume that R has prime characteristic p. Let P be the smallest power of p which is at least n, and let $\delta = d^P$. Since δ is a derivation and $\delta(U)$ is finite, it follows from $(*)$ that either R is finite or $\delta = 0$; and the latter possibility implies that d is nilpotent on R. □

2. The case of U an ideal

If U is assumed to be an ideal, then we can show that $d^n(U)$ can be finite only in the obvious ways.

Theorem 2.1. *Let n be a fixed positive integer. Let R be a prime ring and d a derivation on R such that $d^n(U)$ is finite for some nonzero ideal U. Then either R is finite or d is nilpotent on R.*

Proof. Let R be any prime ring, U any nonzero ideal and d a derivation on R such that $d(U)$ is finite. Consider the map $\Phi : U \to d(U)$ given by $\Phi(x) = d(x)$ for all $x \in U$. Then $\ker \Phi = \{x \in U \mid d(x) = 0\}$ is a subring of U of finite index in U, so by a result of Lewin[5], $\ker \Phi$ contains an ideal H of U which has finite index in U. If $H = \{0\}$, then U is finite; and by Lemma 1.1, R is finite. Suppose, then, that $H \neq \{0\}$. For all $x \in U$ and $y \in H$, we have $0 = d(yx) = yd(x) + d(y)x = yd(x)$; and therefore $yUd(U) = \{0\}$. But for $y \in H \setminus \{0\}, yU$ is a nonzero right ideal of R, hence $A_r(yU) = \{0\}$. Thus $d(U) = \{0\}$, and it follows easily that $d = 0$. Our result now follows by Lemma 1.2. □

3. The case of U a right ideal

Most of the proof of Theorem 2.1 works if U is assumed to be only a right ideal; the hypothesis that U is a two-sided ideal is used only in showing that $y \in H \setminus \{0\}$ implies $yU \neq \{0\}$. Of course, if R is a domain, the same implication holds; hence, we have

Theorem 3.1. *Let R be a ring with no nonzero divisors of zero, and U a nonzero right ideal of R. If d is a derivation on R and $d^n(U)$ is finite for some positive integer n, then either R is finite or d is nilpotent on R.*

By combining Theorem 2.1 and a result in [3], we obtain

Theorem 3.2. *Let R be a prime ring and U a nonzero right ideal of R. If d is a nonzero derivation and there exists a positive integer n for which $d^n(U)$ is finite and central, then d is nilpotent on R.*

Proof. Assume d is not nilpotent. Then by the final result in [3], R is commutative and hence U is an ideal. By Theorem 2.1, R is finite, hence a finite commutative domain - i.e. a finite field. But it is known that finite fields admit no nonzero derivations. □

Whether we can always replace U in Theorem 2.1 by a right ideal is an open question; however, we do have an affirmative answer for PI-rings.

Theorem 3.3. *Let R be a prime PI-ring, and let d be a derivation on R such that $d^n(U)$ is finite for some nonzero right ideal U and some positive integer n. Then either R is finite or d is nilpotent on R.*

Proof. In view of Lemma 1.2 and its proof, we may assume that $d(U)$ is finite and R has prime characteristic p. It is well known that a prime PI-ring has nonzero center Z; and if $z \in Z \setminus \{0\}$, then $d(z^p) = pz^{p-1}d(z) = 0$, so R has nonzero central constants.

Suppose that $d(U) \neq \{0\}$, and let $|d(U)| = k$. Then for any non-constant $u \in U$ and nonzero central constant z, there exist distinct m, $n \in \{1, 2, ..., k+1\}$ such that $d(z^m u) = d(z^n u)$ - i.e. $(z^m - z^n)d(u) = 0$; and since Z has no elements which are zero divisors in R, we get $z^m = z^n$. It follows easily that there exist distinct integers M, N such that $z^M = z^N$ for all central constants z, hence Z satisfies the identity $z^{Mp} = z^{Np}$ and therefore Z is a finite field.

Since R is a prime PI-ring, its central localization R_Z is a primitive PI-ring [6, Theorem 6.1.30]. Moreover, since Z is a field, $R \cong R_Z$ and hence R is primitive. By a classical result of Kaplansky, R is therefore finite-dimensional over Z; hence R is finite. □

In the proof of this theorem, the right ideal property of U is used only twice: in the proof of Lemma 1.2, to show that d nilpotent on U implies d nilpotent on R, and in the argument above to guarantee that $ZU \subseteq U$. Thus, our methods yield

Theorem 3.4. *Let R be a prime PI-ring and S an additive subgroup such that $ZS \subseteq S$. If R admits a derivation d such that $d^n(S)$ is finite, then either R is finite or d is nilpotent on S.*

4. A theorem on semiprime rings

We conclude the paper with a theorem which replaces "nilpotent" by "periodic", and which is available in the setting of semiprime rings.

Theorem 4.1. *Let R be a semiprime ring having no nonzero finite right ideals. If U is a nonzero right ideal of R and d is a derivation on R such that $d^n(U)$ is finite for some positive integer n, then U contains a nonzero right ideal U_1 of R such that d is periodic on U_1.*

The proof uses a rather general lemma.

Lemma 4.2. *Let R be an arbitrary ring and S a nonempty subset of R. If $f : R \to R$ is a mapping such that $f(S) \subseteq S$ and $f^n(S)$ is finite for some positive integer n, then f is periodic on S.*

Proof. Since $f(S) \subseteq S$, for each positive integer k we have $f^{k+1}(S) = f^k(f(S)) \subseteq f^k(S)$. Thus, if $f^n(S)$ is finite, the chain $f^n(S) \supseteq f^{n+1}(S) \supseteq f^{n+2}(S) \supseteq \ldots$ must become stationary at some point, say at $f^N(S) = \{x_1, x_2, \ldots, x_m\}$. Then for each $u \geq 1$, the ordered m-tuple $(f^u(x_1), f^u(x_2), ..., f^u(x_m))$ is a permutation of $(x_1, x_2, ..., x_m)$. Therefore there exist distinct u, $v \geq 1$ such that $f^u(x_i) = f^v(x_i)$ for all $i = 1, 2, ..., m$. Now for each $x \in S$, $f^N(x) = x_i$ for some $i = 1, 2, ..., m$; therefore $f^{N+u}(x) = f^{N+v}(x)$ for all $x \in S$. □

Proof of Theorem 4.1. Let U be a nonzero right ideal with $d^n(U)$ finite. Let T be the torsion ideal of R; and for each prime p, let T_p be the p-primary component of T. If $T = \{0\}$, then $d^n(U) = 0$, so clearly d is periodic on U. If $T \neq \{0\}$ and $U \cap T = \{0\}$, then $UT = \{0\}$; and it follows easily by semiprimeness that $TU = 0$ as well. It follows that $Ud^m(U) = \{0\} = d^m(U)U$ for all $m \geq n$. By applying d to these equations repeatedly, we see that $d^i(U)d^j(U) = \{0\}$ for all nonnegative i, j with $i \geq n$ or $j \geq n$. By Leibniz' formula, we obtain $d^{2n-1}(U^2) = \{0\}$, hence d is periodic on U^2.

The remaining case is that of $U \cap T \neq \{0\}$, in which case $U \cap T_p \neq \{0\}$ for some prime p. Now by semiprimeness of R, $pT_p = \{0\}$; thus, $V = U \cap T_p$ is a nonzero right ideal of R with $pV = \{0\}$. Moreover, $d^P(V)$ is finite, where P is the smallest power of p which is at least n.

It remains only to prove that if V is any nonzero right ideal with $pV = \{0\}$ and $d^{p^\alpha}(V)$ finite for some α, then d is periodic on some nonzero right ideal contained in V. We use induction on $\left|d^{p^\alpha}(V)\right|$. A crucial observation is that, by Leibniz' formula,

(1)
$$d^{p^\alpha}(xy) = d^{p^\alpha}(x)y + xd^{p^\alpha}(y) \text{ for } x, y \in R \text{ with at least one of } x, y \text{ in } V.$$

If $\left|d^{p^\alpha}(V)\right| = 1$, then $d^{p^\alpha}(V) = \{0\} = d^{p^{\alpha+1}}(V)$, so d is obviously periodic on V. Now assume the result holds for nonzero right ideals $\widehat{V}$ with $p\widehat{V} = \{0\}$ and $\left|d^{p^\alpha}(\widehat{V})\right| < k$, and let V be a nonzero right ideal with $pV = \{0\}$ and $\left|d^{p^\alpha}(V)\right| = k$. If V contains a nonzero right ideal I of R with $\left|d^{p^\alpha}(I)\right| < k$, the desired conclusion is immediate from the inductive hypothesis; hence we assume that for every nonzero right ideal I contained in V, $d^{p^\alpha}(I) = d^{p^\alpha}(V)$. Now since V is infinite and $d^{p^\alpha}(V)$ is finite, V contains a nonzero subset S such that $d^{p^\alpha}(S) = \{0\}$; and since R is semiprime, for $s \in S \setminus \{0\}$, sR is a nonzero right ideal contained in V. Therefore, by (1) we get $d^{p^\alpha}(V) = d^{p^\alpha}(sR) = sd^{p^\alpha}(R) \subseteq V$; hence d^{p^α} is periodic on V by Lemma 4.2. Thus, d is periodic on V. □

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