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A note on commutative separable algebras. II

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A NOTE ON COMMUTATIVE SEPARABLE ALGEBRAS. II

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In this note, we indicate how to employ results concerning descent of projectivity in order to obtain a new proof of the main result in [3, Theorem], namely, that separability for commutative algebras descends by faithful flatness. Following the proof, we comment on the noncommutative case.

Throughout, rings and algebras have identity elements. As usual, if A is a commutative ring and B is an A-algebra, then B is said to be a separable A-algebra if and only if the multiplication map from $B \otimes_A B^o$ to B induces a projective left $B \otimes_A B^o$-module structure on B [2, p. 40].

Theorem. *Let B be a commutative A-algebra and C a commutative faithfully flat A-algebra. If the C-algebra $C \otimes_A B$ is separable, then B is separable over A.*

Proof. We set $X = B \otimes_A B^o$, $Y = C \otimes_A (B \otimes_A B^o)$, and $Z = (C \otimes_A B) \otimes_C (C \otimes_A B^o)$. Let p and p' be the multiplication maps $X \to B$ and $Z \to C \otimes_A B$ respectively. Moreover, let g and h be the canonical isomorphisms

$$Z \to Y \quad \text{and} \quad C \otimes_A B \to Y \otimes_X B$$

respectively. Then, the module $C \otimes_A B$ is a left Z-module (under the p'-structure), and is also a left Y-module (under the $1 \otimes p$-structure). Since $p' = (1 \otimes p)g$ and h is Y-linear, the left Z-module structure on $C \otimes_A B$ may be identified with the left Y-module structure on $Y \otimes_X B$. By hypothesis, $C \otimes_A B$ is a projective left Z-module; moreover, it is cyclic and, a fortiori, finitely generated. Hence $Y \otimes_X B$ is a fintely generated projective left Y-module. Note that Y is a faithfully flat right X-module since faithful flatness is preserved under change of base ring [1, Ch. I, Prop. 5, p. 31]. Therefore, B is a (finitely generated) projective left X-module by virtue of the descent result [1, Ch. I, Prop. 12, p. 35].

Remark. The proof of the theorem was phrased in a way that suggests an attack on the more general context in which B is assumed to be noncommutative. One then needs only to show that B is a flat left

X-module, given that $Y \otimes_X B$ is a flat (indeed, finitely generated projective) left Y-module and Y is faithfully flat over X (on the left and the right). In this generality the problem remains open.

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