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1. Introduction.

We call a (continuous) map $p:(E, F)\to(B, C)$, between two pairs of topological spaces $E\supset F$ and $B\supset C$, a *weak homotopy equivalence of pairs*, if p induces isomorphisms p_* of all the relative homotopy groups $\pi_n(E, F)$ and $\pi_n(B, C)$, i.e.

$$p_*:\pi_n(E, F)\approx\pi_n(B, C), \qquad \text{for any integer } n>0.$$

The purpose of this note is to prove the equivalences of the weak homotopy equivalence of pairs and the conditions (A_i), $i=1, 2, 3$, some sorts of the homotopically lifting homotopy conditions, (cf. §2 and Theorem 3 of §3); and also, by making use of these equivalences, to prove the following theorem, which gives a necessary and sufficient condition that a space is an H-space (a space admitting a map of type (1, 1)).

Theorem 1. *Let F be a CW-complex such that the weak topology of the product complex $F\times F$ is the ordinary product topology of the product space $F\times F$*[1]*. Under these conditions, F is an H-space if, and only if, there exist topological spaces E and B and a map p of E into B, satisfying the following properties:*

(1) *E contains F, and F is contractible in E to a vertex $\varepsilon\in F$ leaving ε fixed throughout the contraction, and*

(2) *$p(F)=b$, a point of B, and the map $p:(E, F)\to(B, b)$ is a weak homotopy equivalence of the two pairs.*

Also we have

Theorem 2. *Let $p:(E, F)\to(B, b)$ is a given map, where E is a CW-complex, F is its locally finite subcomplex, and B is a space containing a point b. If*

(1) *E is contractible in itself to a vertex $\varepsilon\in F$ being ε stationary throughout the contraction, and*

(2) *p is a weak homotopy equivalence of pairs (E, F) and (B, b), then F is a homotopy-associative H-space having an inversion.*

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1) For examples, if F is a countable CW-complex, F has this property.

2. The conditions (A_i), $i=1, 2, 3$.

Let $E \supset F$ and $B \supset C$ be topological spaces and $p: (E, F) \to (B, C)$ a map of pairs. We shall consider the following conditions (A_i), concerning such a map p, which may be considered as generalizations of the lifting homotopy conditions.

(A_1) *Let K be any CW-complex, L its subcomplex, and M a subcomplex of the product complex $K\times I$*[1]. *Let*

$$\xi' : (K\times 0)\cup(L\times I) \to E, \qquad \eta : K\times I \to B$$

be given maps such that $\xi'(M') \subset F$, $(M' = ((K\times 0)\cup(L\times I))\cap M)$, *and* $\eta(M) \subset C$, *and the two maps* $p\circ\xi'$ *and* $\eta \mid (K\times 0)\cup(L\times I)$ *are homotopic each other by a homotopy of pairs*

$$Y'_t : ((K\times 0)\cup(L\times I),\ M') \to (B, C), \quad 0\leqslant t\leqslant 1,$$

with $Y'_0 = p\circ\xi'$ *and* $Y'_1 = \eta \mid (K\times 0)\cup(L\times I)$.

From these assumptions, it follows that ξ' *has an extension*

$$\xi : K\times I \to E, \quad being \quad \xi(M) \subset F,$$

and the two maps $p\circ\xi$ *and* η *are homotopic each other by a homotopy*

$$Y_t : (K\times I,\ M) \to (B, C),\ 0\leqslant t\leqslant 1,\ with\ \ Y_0 = p\circ\xi,\ \ Y_1 = \eta,$$

and also this homotopy Y_t *is taken as an extension of the given homotopy* Y'_t, *i.e.* $Y_t \mid (K\times 0)\cup(L\times I) = Y'_t$ *for* $0\leqslant t\leqslant 1$.

(A_2) *In addition to the assumptions of* (A_1), *we assume that* $K = I^n$ $(= I\times \cdots \times I$ *(n-times)) and its n-cell is* $I^n - \dot{I}^n$ *(=the interior of* I^n*) only, and* $L = \dot{I}^n$ *(=the boundary of* I^n*)*[2]. *Then the conclusions of* (A_1) *follow.*

(A_3) *Moreover, we add the following assumptions to those of* (A_2) : $p\circ\xi' = \eta \mid (I^n\times 0)\cup(\dot{I}^n\times I)$. *Then we have the conclusions of* (A_1), *i.e., there is an extension* ξ *of* ξ' *such that* $\xi(M)\subset F$ *and* $p\circ\xi$ *and* η *are homotopic by a homotopy* $Y_t : (I^n\times I,\ M) \to (B, C)$ *being stationary on* $(I^n\times 0)\cup(\dot{I}^n\times I)$, *i.e.* $Y_t \mid (I^n\times 0)\cup(\dot{I}^n\times I) = p\circ\xi'$ *for* $0\leqslant t\leqslant 1$.

1) $I=[0, 1]$, the closed interval, is considered as a CW-complex whose 1-cell is $(0, 1)$, the open interval, and 0-cells are the two points 0 and 1.

2) We assume that the boundary $\dot{I}^n$ is subdivided arbitrarily into finite cells forming a finite CW-complex.

It follows immediately from the above definitions that the condition (A_{i+1}) is weaker than (A_i) for $i=1, 2$, and we shall prove the equivalences of these conditions in this section.

Before these proofs, we notice about the homotopy extension theorem.

Lemma 1. *Let K be a CW-complex and L, N and M_k $(k=1, 2, \cdots)$ be its subcomplexes such that $M_k \cap M_{k'} = \emptyset$ (the empty set) if $k \neq k'$. Let T be any space and T_k $(k=1, 2, \cdots)$ its subsets; and let a map $f_0: K \to T$ and a homotopy $g_t: L \to T$, $0 \leqq t \leqq 1$, be so given that*

$$g_0 = f_0 \mid L,\ f_0(M_k) \subset T_k;\ g_t \mid L \cap N = f_0 \mid L \cap N,\ g_t(L \cap M_k) \subset T_k,$$

for $0 \leqq t \leqq 1$ and $k=1, 2, \cdots$.

Then there is a homotopy $f_t: K \to T$, $0 \leqq t \leqq 1$, of f_0, such that

$$g_t = f_t \mid L,\quad f_t \mid N = f_0 \mid N,\quad f_t(M_k) \subset T_k,$$

for $0 \leqq t \leqq 1$ and $k=1, 2, \cdots$.

Proof. We define a homotopy $f_t \mid L \cup N: L \cup N \to T$, by setting $f_t \mid N = f_0 \mid N$ and $f_t \mid L = g_t$ for $0 \leqq t \leqq 1$. Since $f_0(M_k) \subset T_k$ and $g_t(L \cap M_k) \subset T_k$, the map $f_0 \mid M_k$ and the homotopy $f_t \mid (L \cup N) \cap M_k$ are considered as mapping into T_k. Hence, by making use of the ordinary homotopy extension theorem for CW-complexes, there are homotopies, of $f_0 \mid M_k$:

$$f_t \mid M_k: M_k \to T_k, \text{ such that } \begin{array}{l} f_t \mid L \cap M_k = g_t \mid L \cap M_k, \\ f_t \mid N \cap M_k = f_0 \mid N \cap M_k, \end{array}$$

for $0 \leqq t \leqq 1$ and every $k=1, 2, \cdots$. These homotopies and the above $f_t \mid L \cup N$ define immediately a homotopy $f_t \mid L \cup N \cup (\bigcup_k M_k): L \cup N \cup (\bigcup_k M_k) \to T$, since $M_k \cap M_{k'} = \emptyset$ for $k \neq k'$. Using again the homotopy extension theorem to f_0 and the last homotopy $f_t \mid L \cup N \cup (\bigcup_k M_k)$, we obtain a homotopy $f_t: K \to T$, $0 \leqq t \leqq 1$, as desired.

Proofs of the equivalences of (A_i), $i=1, 2, 3$, are divided into the following two lemmas.

Lemma 2. *If $p: (E, F) \to (B, C)$ satisfies (A_3), then it also satisfies (A_2).*

Proof. Let maps

$$\xi': (I^n \times 0) \cup (\dot{I}^n \times I)\ (=J^n) \to E, \quad \eta: I^n \times I\ (=I^{n+1}) \to B,$$

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and a homotopy

$Y'_t: (J^n, J^n \cap M) \to (B, C)$ $(0 \leqq t \leqq 1)$ with $Y'_0 = p \circ \xi'$, $Y'_1 = \eta \mid J^n$,
be given by the assumptions of (A_2). Applying Lemma 1 to η and Y'_t by taking $M_1 = M$ and $T_1 = C$, we have a homotopy $Y''_t : I^{n+1} \to B$, $0 \leqq t \leqq 1$, such that

$$Y''_1 = \eta, \quad Y''_t \mid J^n = Y'_t, \quad \text{and} \quad Y''_t(M) \subset C \quad \text{for} \quad 0 \leqq t \leqq 1.$$

We set $\bar{\eta} = Y''_0$. Then $\bar{\eta}(M) \subset C$ and $p \circ \xi' = \bar{\eta} \mid J^n$, and hence maps ξ' and $\bar{\eta}$ satisfy the assumptions of (A_3). It follows from (A_3) that there is an extension $\xi : I^{n+1} \to E$ of ξ', being $\xi(M) \subset F$, and a homotopy

$$\bar{Y}_t : (I^{n+1}, M) \to (B, C), \text{ with } \bar{Y}_0 = p \circ \xi, \ \bar{Y}_1 = \bar{\eta}, \ \bar{Y}_t \mid J^n = p \circ \xi'.$$

Let $\bar{\bar{Y}}_t : (I^{n+1}, M) \to (B, C)$ be a homotopy defined by

$$\bar{\bar{Y}}_t = \bar{Y}_{2t} \text{ for } 0 \leqq t \leqq 1/2, \ \bar{\bar{Y}}_t = Y''_{2t-1} \text{ for } 1/2 \leqq t \leqq 1.$$

Then $\bar{\bar{Y}}_0 = p \circ \xi$, $\bar{\bar{Y}}_1 = \eta$; and also, since $\bar{Y}_t$ is stationary on J^n, $\bar{\bar{Y}}_t \mid J^n$ is homotopic to $Y''_t \mid J^n$ considering as the maps of $J^n \times I$ into B, and this homotopy is taken to be stationary on $J^n \times \dot{I}$ and to be mapping $(J^n \cap M) \times I$ into C. Applying Lemma 1 to the map $\bar{\bar{Y}}_t$ and the last homotopy by taking $N = I^{n+1} \times \dot{I}$, $M_1 = M \times I$ and $T_1 = C$, we have a homotopy of pairs

$$Y_t : (I^{n+1}, M) \to (B, C) \ (0 \leqq t \leqq 1) \text{ with } Y_0 = p \circ \xi, \ Y_1 = \eta,$$

and also $Y_t \mid J^n = Y''_t \mid J^n = Y'_t$. Therefore the map ξ and the homotopy Y_t satisfy the conclusions of (A_2), and we have the above lemma.

Lemma 3. *If* $p : (E, F) \to (B, C)$ *satisfies* (A_2), *then also* (A_1)

Proof. For this lemma, we can apply the same principles of the proofs of Theorem (5. 1) of [1], and we follow proofs briefly.

Let CW-complex K, L and M and maps ξ' and η and a homotopy Y'_t $(0 \leqq t \leqq 1)$ be so given as to satisfy the assumptions of (A_1) for the map $p : (E, F) \to (B, C)$, and let $\bar{K}^q = K^q \cup L$ $(q \geqq -1)$[1] and $P_q = (K \times 0) \cup (\bar{K}^q \times I) \subset K \times I$.

Let $n \geqq 0$, and assume inductively that ξ' has an extension $\xi_{n-1} : P_{n-1} \to E$ such that $\xi_{n-1}(P_{n-1} \cap M) \subset F$, and also that Y'_t has an ex-

1) K^q is the q-section of K.

tension $Y_t^{n-1}:(P_{n-1}, P_{n-1}\cap M)\to(B, C)$, which is a homotopy between $Y_0^{n-1}=p\circ\xi_{n-1}$ and $Y_1^{n-1}=\eta \mid P_{n-1}$. Let $\{e_r^n \mid r\in R\}$ be the set of all n-cells of $K-L$. For each $r\in R$, let $\phi_r: I^n\to K$ be a map such that $\phi_r(\dot{I}^n)\subset K^{n-1}$ and $\phi_r \mid I^n-\dot{I}^n$ is a homeomorphism onto e_r^n. Let ψ_r: $I^n\times I\to P_n$ be defined by

$$\psi_r(z, t)=(\phi_r(z), t), \qquad \text{for } z\in I^n,\ t\in I.$$

Then $\psi_r(J^n)=P_{n-1}\cap\psi_r(I^{n+1})$ $(J^n=(I^n\times 0)\cup(\dot{I}^n\times I),\ I^{n+1}=I^n\times I)$. Also, as easily seen, there is a subcomplex M_r of the product complex $I^n\times I$ such that $\psi_r(M_r)=\psi_r(I^{n+1})\cap M$, since M is a subcomplex of the product complex $K\times I$, for each $r\in R$.

It follows immediately from the above hypotheses that the maps

$$\xi_{n-1}\circ\psi_r \mid J^n: J^n\to E \text{ and } \eta\circ\psi_r: I^{n+1}\to B$$

and the homotopy of pairs

$$Y_t^{n-1}\circ\psi_r \mid J^n:(J^n, J^n\cap M)\to(B, C) \quad (0\leqslant t\leqslant 1)$$

satisfy the assumptions of (A_2) by taking M_r instead of M. Since the given map $p:(E, F)\to(B, C)$ satisfies the condition (A_2), we have a map $\lambda_r:(I^{n+1}, M_r)\to(E, F)$ and a homotopy $Z_t^r:(I^{n+1}, M_r)\to(B, C)$ $(0\leqslant t\leqslant 1)$ such that

$$\lambda_r \mid J^n=\xi_{n-1}\circ\psi_r \mid J^n;\ Z_0^r=p\circ\lambda_r,\ Z_1^r=\eta\circ\psi_r, \text{ and}$$
$$Z_t^r \mid J^n=Y_t^{n-1}\circ\psi_r \mid J^n, \text{ for } 0\leqslant t\leqslant 1.$$

Therefore, it follows from the property $\psi_r(J^n)=P_{n-1}\cap\psi_r(I^{n+1})$ that a map $\xi_n: P_n\to E$ and a homotopy $Y_t^n: P_n\to B$ $(0\leqslant t\leqslant 1)$ are defined by

$$\left.\begin{array}{l}\xi_n \mid P_{n-1}=\xi_{n-1},\quad \xi_n\circ\psi_r(z)=\lambda_r(z);\\ Y_t^n \mid P_{n-1}=Y_t^{n-1},\ Y_t^n\circ\psi_r(z)=Z_t^r(z);\end{array}\right\} \text{ for } z\in I^{n+1}.$$

It is easy to see that the map ξ_n and the homotopy Y_t^n satisfy the above hypotheses of the induction. Therefore, starting with $\xi_{-1}=\xi'$ and $Y_t^{-1}=Y_t'$, we can construct ξ_n and Y_t^n of above sorts for every $n\geqslant 0$. Since $K\times I=\bigcup_n P_n$ and $K\times I$ has the weak topology, a map $\xi: K\times I\to E$ and a homotopy $Y_t: K\times I\to B$ $(0\leqslant t\leqslant 1)$ are defined by $\xi \mid P_n=\xi_n$ and $Y_t \mid P_n=Y_t^n$. Clearly ξ and Y_t satisfy the conclusions of the condition (A_1) and Lemma 2 is proved.

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As a consequence of these two lemmas, we have the equivalences of the conditions (A_i), $i = 1, 2, 3$.

3. The weak homotopy equivalence and the conditions (A_i). We shall prove the following two lemmas.

Lemma 4. *If* $p : (E, F) \to (B, C)$ *is a weak homotopy equivalence, then it satisfies the codition* (A_3).

Proof. Let $\xi' : (I^n \times 0) \cup (\dot{I}^n \times I) \ (= J^n) \to E$ and $\eta : I^n \times I \ (= I^{n+1}) \to B$ be the given maps such that $p \circ \xi' = \eta \mid J^n$. We consider two cases separately by the situation of the subcomplex M, which satisfies $\eta(M) \subset C$, of the product complex $I^n \times I$.

(a) *The case either* $M \cap ((I^n - \dot{I}^n) \times 1) = \emptyset$ *or* $M = I^{n+1}$. Let $\theta : I^{n+1} \to J^n$ be a strong deformation retraction, i. e. $\theta \mid J^n =$ the identity map and $\theta \sim$ the identity map : $I^{n+1} \to I^{n+1}$, relative J^n. We consider the map $\xi : I^{n+1} \to E$, defined by $\xi = \xi' \circ \theta$. ξ, thus defined, is clearly an extension of ξ'. In the first case, $M \subset J^n$ and so $\xi(M) \subset F$, and also $p \circ \xi = p \circ \xi' \circ \theta = \eta \circ \theta \sim \eta$, relative J^n. In the second case, $\xi(I^{n+1}) \subset F$ and the conclusions of (A_3) are satisfied evidently.

(b) *The case* $I^n \times 1 \subset M \subset I^{n+1}$. Let $y = \xi'(*)$, $b = p(y)$, $(* = (0, \cdots, 0, 1) \in J^n)$, and let $\alpha \in \pi_n(E, F, y)$ and $\beta \in \pi_n(B, C, b)$ be the elements determined by the maps

$$\xi' : (J^n, \dot{J}^n, *) \to (E, F, y) \text{ and } \eta \mid J^n : (J^n, \dot{J}^n, *) \to (B, C, b),$$

respectively, $(\dot{J}^n = \dot{I}^n \times 1)$. Since η is defined on I^{n+1} and $\eta(I^n \times 1) \subset \eta(M) \subset C$, the map $\eta \mid J^n$ is homotopic, relative $\dot{J}^n$, to the map whose image is contained in C, and hence $\beta = 0$. Since $p \circ \xi' = \eta \mid J^n$, $p_*(\alpha) = \beta$ and so $p_*(\alpha) = 0$, and we have $\alpha = 0$ because $p_* : \pi_n(E, F, y) \to \pi_n(B, C, b)$ is an isomorphism by the weak homotopy equivalence of p.

Therefore there exists a map $\xi_1 : (J^n \times I, \dot{J}^n \times I, * \times I) \to (E, F, y)$ such that $\xi_1(z, 0) = \xi'(z)$ $(z \in J^n)$ and $\xi_1(J^n \times 1) = y$. Since $p \circ \xi_1(z, 0) = p \circ \xi'(z) = \eta(z)$ for $z \in J^n$, $p \circ \xi_1 : (J^n \times I, \dot{J}^n \times I, * \times I) \to (B, C, b)$ is a homotopy of $\eta \mid J^n$. Since $p \circ \xi_1((J^n \cap (I^n \times 1)) \times I) = p \circ \xi_1(\dot{J}^n \times I) \subset C$, we can apply Lemma 1 of §2 to η and $p \circ \xi_1$ by taking $M_1 = I^n \times 1$ and $T_1 = C$, and hence we have a map $\eta_1 : I^{n+1} \times I \to B$ such that

$$\eta_1(z, 0) = \eta(z) \text{ for } z \in I^{n+1}; \ \eta_1((I^n \times 1) \times I) \subset C;$$
$$\eta_1(z, t) = p \circ \xi_1(z, t) \text{ for } z \in J^n \text{ and } t \in I.$$

Since $\eta_1(J^n \times 1) = p \circ \xi_1(J^n \times 1) = b$ and $\eta_1((I^n \times 1) \times 1) \subset C$, the map

$\eta_1 \mid I^{n+1}\times 1 : (I^{n+1}\times 1,\ \dot{I}^{n+1}\times 1,\ J^n\times 1) \to (B,\ C,\ b)$ determines an element of $\pi_{n+1}(B,\ C,\ b)$. Therefore there is a map $\xi_1' : (I^{n+1}\times 1,\ \dot{I}^{n+1}\times 1, J^n\times 1) \to (E,\ F,\ y)$ such that

$$p\circ \xi_1' \sim \eta_1 \mid I^{n+1}\times 1 : (I^{n+1}\times 1,\ \dot{I}^{n+1}\times 1, J^n\times 1) \to (B,\ C,\ b),$$

because the induced homomorphism $p_* : \pi_{n+1}(E,\ F,\ y) \to \pi_{n+1}(B,\ C,\ b)$ is onto by the weak homotopy equivalence of p. We denote this homotopy by $\zeta_t : (I^{n+1},\ \dot{I}^{n+1},\ J^n) \to (B,\ C,\ b)$, $0 \leqslant t \leqslant 1$, with $\zeta_0 = p\circ\xi_1'$ and $\zeta_1 = \eta_1 \mid I^{n+1}\times 1$.

The map $\xi_1 : J^n\times I \to E$, defined previously, gives clearly an homotopy of $\xi_1' \mid J^n\times 1 =$ the constant map. If we apply Lemma 1 to ξ_1', ξ_1 and $M_1 = I^n\times 1$ and $T_1 = F$, we have a map $\xi_1 : I^{n+1}\times I \to E$ such that

$$\xi_1 \mid I^{n+1}\times 1 = \xi_1', \quad \xi_1 \mid J^n\times 0 = \xi', \quad \xi_1((I^n\times 1)\times I) \subset F.$$

We now show that the map $\xi : I^{n+1} \to E$, defined by $\xi(z) = \xi_1(z,\ 0)$ for $z \in I^{n+1}$, satisfies the conclusions of (A_3). It is an extension of ξ', and $\xi(M) \subset \xi(M \cap J^n) \cup \xi(I^n\times 1) \subset F$, since $I^n\times 1 \subset M \subset \dot{I}^{n+1} = (I^n\times 1) \cup J^n$. We define a map $\bar{Y} : I^{n+1}\times I \to B$ and a homotopy $\bar{Y}_s : J^n\times I \to B$, $0 \leqslant s \leqslant 1$, as follows:

$$\begin{aligned} \bar{Y}(z,\ t) &= p\circ\xi_1(z,\ 4t), && \text{for } 0 \leqslant t \leqslant 1/4,\\ &= \zeta_{(4t-1)/2}(z), && \text{for } 1/4 \leqslant t \leqslant 3/4,\\ &= \eta_1(z,\ 4(1-t)), && \text{for } 3/4 \leqslant t \leqslant 1, \end{aligned}$$

where $z \in I^{n+1}$; and

$$\begin{aligned} \bar{Y}_s(z, t) &= p\circ\xi_1(z, 4t-2s) && \text{for } 0\leqslant s\leqslant 1,\ s/2 \leqslant t \leqslant \min((2s+1)/4, 1/2),\\ &= b, && \text{for } 0\leqslant s \leqslant 1/2,\ (2s+1)/4 \leqslant t \leqslant (3-2s)/4,\\ &= \eta_1(z, 4-4t-2s), && \text{for } 0\leqslant s\leqslant 1,\ \max((3-2s)/4, 1/2) \leqslant t \leqslant (2-s)/2,\\ &= p\circ\xi(z) = \eta(z), && \text{for otherwise,} \end{aligned}$$

where $z \in J^n$. The map $\bar{Y}$ is well defined and it gives a homotopy of $p\circ\xi$ and η. The homotopy $\bar{Y}_s$ is well defined, since $p\circ\xi_1 \mid J^n\times I = \eta_1 \mid J^n\times I$ and $\zeta_t(J^n) = b$. Also $\bar{Y}_0 = \bar{Y} \mid J^n\times I$, $\bar{Y}((I^n\times 1)\times I) \subset C$, $\bar{Y}_s(\dot{J}^n\times I) \subset C$, and $\bar{Y}_s \mid J^n\times \dot{I}$ is stationary. Therefore, by applying Lemma 1 to $\bar{Y}$, $\bar{Y}_s$ and $N = I^{n+1}\times \dot{I}$, $M_1 = (I^n\times 1)\times I$, and $T_1 = C$, we have a map $Y : I^{n+1}\times I \to B$ being homotopic to $\bar{Y}$; and hence a homotopy $Y_t : I^{n+1} \to B$, $0 \leqslant t \leqslant 1$, defined by $Y_t(z) = \bar{Y}(z,\ t)$ for $z \in I^{n+1}$. The homotopy Y_t, thus defined, has the following properties: for $z \in I^{n+1}$,

$$Y_0(z) = \bar{Y}(z,\ 0) = p\circ\xi(z),\ \ Y_1(z) = \bar{Y}(z,\ 1) = \eta(z);$$

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and, for $z \in J^n$ and $0 \leqslant t \leqslant 1$, $Y_t(z) = \overline{Y}_1(z, t) = p \circ \xi(z) = p \circ \xi'(z)$. Also $Y_t(I^n \times 1) \subset C$, and hence we have $Y_t(M) \subset C$, since $M \subset \dot{I}^{n+1} = J^n \cup (I^n \times 1)$.

Therefore we have the map ξ and the homotopy Y_t satisfying the conclusions of (A_3), and Lemma 4 is proved completely.

Lemma 5. *If $p:(E, F) \to (B, C)$ satisfies the condition* (A_1), *then it is a weak homotopy equivalence between two pairs* (E, F) *and* (B, C).

Proof. Let y be any point of F, $b = p(y)$, and n be any positive integer.

(a) We show first that the induced homomorphism $p_* : \pi_n(E, F, y) \to \pi_n(B, C, b)$ is onto. Let α be any element of $\pi_n(B, C, b)$ and $\eta : (I^n, \dot{I}^n, J^{n-1}) \to (B, C, b)$ be a map which determines α. Further, let $\xi' : J^{n-1} \to E$ be the constant map, defined by $\xi'(z) = y$ for $z \in J^{n-1}$. Then the maps ξ' and η satisfy the assumptions of (A_1) by taking $K = I^{n-1}$, $L = \dot{I}^{n-1}$, and $M = \dot{I}^n$, and $Y_t' = p \circ \xi' = b$. Hence it follows from (A_1) that there exists an extension $\xi : I^n \to E$ of ξ' such that

$$\xi(J^{n-1}) = y, \quad \xi(\dot{I}^n) \subset F, \text{ and } p \circ \xi \sim \eta : (I^n, \dot{I}^n, J^{n-1}) \to (B, C, b).$$

Therefore the element β of $\pi_n(E, F, y)$ determined by the map $\xi : (I^n, \dot{I}^n, J^{n-1}) \to (E, F, y)$ is mapped to α by p_*, and the onto-ness is proved.

(b) Let β be a element $\pi_n(E, F, y)$, and $\xi_0 : (I^n, \dot{I}^n, J^{n-1}) \to (E, F, y)$ be a map of the homotopy class β. We assume that $p_*(\beta) = 0$, i.e. the map $p \circ \xi_0 : (I^n, \dot{I}^n, J^{n-1}) \to (B, C, b)$ is homotopic, relative J^{n-1}, to the constant map, remaining the image of $\dot{I}^n$ in C. We denote this homotopy by $\eta : (I^n \times I, \dot{I}^n \times I, J^{n-1} \times I) \to (B, C, b)$ with $\eta(z, 0) = p \circ \xi_0(z)$ for $z \in I^n$ and $\eta(I^n \times 1) = b$. Let $\xi' : (I^n \times 0) \cup (J^{n-1} \times I) \to E$ be the map defined by $\xi'(z, 0) = \xi_0(z)$ for $z \in I^n$ and $\xi'(J^{n-1} \times I) = y$. Then the maps ξ' and η satisfy the assumptions of (A_1) by taking $K = I^n$, $L = J^{n-1}$, $M = (\dot{I}^n \times I) \cup (I^n \times 1)$ and the homotopy $Y_t' = p \circ \xi'$.

Therefore, it follows from (A_1) that there is a map $\xi : I^n \times I \to E$ such that $\xi(z, 0) = \xi'(z, 0) = \xi_0(z)$ for $z \in I^n$, $\xi(J^{n-1} \times I) = y$, and $\xi((\dot{I}^n \times I) \cup (I^n \times 1)) \subset F$. Let $\xi_1 : I^n \to E$ be the map defined by $\xi_1(z) = \xi(z, 1)$ for $z \in I^n$. Then, ξ gives a homotopy $\xi_0 \sim \xi_1 : (I^n, \dot{I}^n, J^{n-1}) \to (E, F, y)$, and so ξ_0 and ξ_1 determine the same element β of $\pi_n(E, F, y)$. Also, by the property of ξ, we have $\xi_1(I^n) \subset F$, and this shows that $\beta = 0$. These complete the proofs of the fact that p_* is isomorphic and hence that p is a weak homotopy equivalence of the pairs (E, F) and (B, C). Thus we have Lemma 5.

By the above four lemmas, we have

Theorem 3. *A map $p:(E, F)\to(B, C)$ between two pairs of spaces $E\supset F$ and $B\supset C$ is a weak homotopy equivalence, i.e. the induced homomorphism $p_*:\pi_n(E, F)\to\pi_n(B, C)$ is an isomorphism onto for any positive integer n, if and only if the map p satisfies the condition* (A_i) $(i=1, 2, 3)$.

Remark. For the case that $p: E\to B$ is a fibre map (in the sense of Serre) and $F=p^{-1}(b)$ the fibre over a point $b\in B$, the map $p:(E, F)\to(B, b)$ has the ordinary lifting homotopy property; and, for the case of a quasi-fibre space (introduced by A. Told and R. Thom), the projection p has the homotopically lifting homotopy property which is stronger than (A_1), (cf. [7], §1). Therefore it may be considered as a generalization of the notion of the (quasi)-fibre space that a map $p:(E, F)\to(B, b)$ is a weak homotopy equivalence of pairs.

4. Some properties of H-spaces.

We say that a space F is an H-space (has an H-structure), if there is a multiplication μ in F, i.e. a map $\mu: F\times F\to F$, such that $\mu(\varepsilon, x)=\mu(x, \varepsilon)=x$ for some point ε (called an unit) of F and every $x\in F$[1]. (We often write xy or $x\cdot y$ instead of $\mu(x, y)$.)

We consider the following condition (B) for an H-space F.

(B) *Both of the two maps l_1 and l_2 of $F\times F$ into itself, defined by*

$$l_1(x, y)=(x\cdot y, x),\quad l_2(x, y)=(x\cdot y, y),$$

for $x, y\in F$, are homotopy equivalences of $(F\times F, (\varepsilon, \varepsilon))$ into itself.

If (B) is satisfied, we denote a homotopy inverse of l_i by m_i, and a homotopy of $m_i\circ l_i$ and the identity map by $L_t^i:(F\times F, (\varepsilon, \varepsilon))\to(F\times F, (\varepsilon, \varepsilon))$ $(0\leqslant t\leqslant 1)$ and that of $l_i\circ m_i$ and the identity map by $M_t^i:(F\times F, (\varepsilon, \varepsilon))\to(F\times F, (\varepsilon, \varepsilon))$ $(0\leqslant t\leqslant 1)$, respectively, for $i=1, 2$.

Remark. It is easy to see that a homotopy-associative H-space having an inversion satisfies the above condition (B); and (B) implies

1) More generally, H-spaces are defined by the weaker condition that there is a homotopy-unit ε, i.e. two maps $x\to\varepsilon\cdot x$ and $x\to x\cdot\varepsilon$ of F into itself are both homotopic, relative ε, to the identity map $x\to x$. But, when F is a CW-complex such that the weak topology of the product complex $F\times F$ is the ordinary product topology, the conditions of the above definition are satisfied by H-spaces of generally defined, cf. Lemma (6.4) of [2].

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the existence of right and left inversions, (more precisely, $q_2 \circ m_1(\varepsilon, x)$ and $q_1 \circ m_2(\varepsilon, x)$ are right and left inversions respectively, where q_i is the natural projections from $F \times F$ onto F of the i-th factor for $i = 1, 2$).[1]

We now notice the following property.

Lemma 6. *Suppose that F is a CW-complex and the weak topology of the product complex $F \times F$ is the ordinary product topology. Then, if F has an H-structure, it satisfies the property* (B).

Proof. The map l_1 of $F \times F$ into $F \times F$ induces the homomorphisms l_{1*} of the homotopy groups:

$$l_{1*} : \pi_n(F \times F) \to \pi_n(F \times F),$$

for all positive integers n. We shall prove that l_{1*} are isomorphisms of $\pi_n(F \times F)$ onto itself.

Let q_i be the natural projections as in the above remark, and r_1 and r_2 be the natural imbedding homeomorphisms of F onto the subsets $F \times \varepsilon$ and $\varepsilon \times F$ of $F \times F$ respectively. Then we have the following two isomorphisms between $\pi_n(F \times F)$ and $\pi_n(F) + \pi_n(F)$ (the direct sum of two groups):

$$(q_{1*}, q_{2*}) : \pi_n(F \times F) \approx \pi_n(F) + \pi_n(F),$$
$$r_{1*} + r_{2*} : \pi_n(F) + \pi_n(F) \approx \pi_n(F \times F).$$

From the definition of l_1: $F \times F \to F \times F$, it follows immediately

$$\begin{array}{l} (q_{1*}, q_{2*}) \circ l_{1*} \circ r_{1*}(\alpha) = (\alpha, \alpha), \\ (q_{1*}, q_{2*}) \circ l_{1*} \circ r_{2*}(\beta) = (\beta, 0), \end{array} \quad \text{for } \alpha, \beta \in \pi_n(F).$$

Hence, $(q_{1*}, q_{2*}) \circ l_{1*} \circ (r_{1*} + r_{2*})(\beta, \alpha - \beta) = (\alpha, \beta)$; and, if $(q_{1*}, q_{2*}) \circ l_{1*} \circ (r_{1*} + r_{2*})(\alpha, \beta) = (\alpha + \beta, \alpha) = (0, 0) =$ the zero element of $\pi_n(F) + \pi_n(F)$, then $\alpha = 0$ and $\beta = 0$. Therefore l_{1*} is an isomorphism of $\pi_n(F \times F)$ onto itself, and it follows from Theorem of J. H. C. Whitehead that l_1 is an homotopy equivalence since $F \times F$ is a CW-complex by assumptions. Moreover, since $l_1(\varepsilon, \varepsilon) = (\varepsilon, \varepsilon)$, l_1 is also an homotopy equivalence of the pair $(F \times F, (\varepsilon, \varepsilon))$ to itself[2].

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1) If two maps $(x, y, z) \to (xy)z$ and $(x, y, z) \to x(yz)$ of $F \times F \times F$ into F are homotopic each other, rel. $(\varepsilon, \varepsilon, \varepsilon)$, we say that F is homotopy-associative. F has an inversion, if there exists a map $\sigma : F \to F$ such that the two maps $x \to \sigma(x) \cdot x$ and $x \to x \cdot \sigma(x)$ of F into F are both homotopic, rel. ε, to the constant map $x \to \varepsilon$. If only one of these two maps has this property, we say σ is an one-sided (left or right) inversion.

2) This is an immediate consequence of Theorem (3.1) of [1].

By the same way, Lemma 6 is proved for the map l_2.

5. Constructions and some properties of the map $p: F\circ F \to \hat{F}$. In this section, let F be an H-space. The constructions of the map $p: F\circ F \to \hat{F}$ are the analogy of the constructions of n-universal bundle having a topological group as its structure group [3], and also generalizations of the Hopf fibering $S^{2k+1}\to S^{k+1}$ for $k=1, 3, 7$.

Let $F\circ F$ be the join of two copies of F, i. e. the identification space obtained from $F\times F\times I$ by identifying each set of the form $x\times F\times 0$ with $(x, 0)\in F\times 0$ and each set of the form $F\times x\times 1$ with $(x, 1)\in F\times 1$. The point of $F\circ F$, being the image of $(x_1, x_2, t_2)\in F\times F\times I$, will be denoted by the symbol $t_1x_1\oplus t_2x_2$ where $t_1+t_2=1$ and the element x_i may be chosen arbitrary or omitted whenever $t_i=0$.

Let $\hat{F}$ be the suspension of F, i. e., the identification space obtained from $F\times I$ by shrinking each of the subspaces $F\times 0$ and $F\times 1$ to different points respectively. A point of $\hat{F}$ will be denoted by the symbol (x, t) $(x\in F,\ t\in I)$, where the element x may be chosen arbitrary or omitted whenever $t=0$ or 1.

We also define notations as follows:

$$F\circ F\supset F_i=\{t_1x_1\oplus t_2x_2 \mid t_i=1\},$$
$$F\circ F\supset U_i=\{t_1x_1\oplus t_2x_2 \mid t_i>0\}\supset F_i,\ U_3=U_1\cap U_2,$$
$$F_i\ni \varepsilon_i=(t_1x_1\oplus t_2x_2 \mid t_i=1 \text{ and } x_i=\varepsilon),\quad \text{for } i=1,\ 2;$$
$$\hat{F}\supset V_1=\{(x, t)\mid t>0\},\ V_2=\{(x, t)\mid t<1\},\ V_3=V_1\cap V_2,$$
$$V_1\ni\bar{\varepsilon}_1=(x, 1),\quad V_2\ni\bar{\varepsilon}_2=(x, 0).$$

Then U_i and V_i are open sets of $F\circ F$ and $\hat{F}$ respectively for $i=1$, 2, 3, and F_i is the homeomorphic image of F under the natural map $x\to 1x\oplus 0$ or $x\to 0\oplus 1x$. We shall identify F_1 with F by this natural homeomorphism.

Let p be the (continuous) map of $F\circ F$ into $\hat{F}$, defined by

$$\begin{aligned} p(t_1x_1\oplus t_2x_2) &= (x_1x_2, t_1), && \text{for } t_1, t_2\neq 1,\\ &= \bar{\varepsilon}_i, && \text{for } t_i=1,\ i=1,\ 2. \end{aligned}$$

This map p is clearly continuous by the fact that the map $t_1x_1\oplus t_2x_2\to x_i$ of $F\circ F$ onto F is continuous whenever $t_i\neq 0$. Also $p^{-1}(V_i)=U_i$ and $p^{-1}(\bar{\varepsilon}_i)=F_i$.

About these spaces and maps, we have

Theorem 4. *If the H-space F satisfies the condition* (B) *of* §4, *the map* $p : (F \circ F, F) \to (\hat{F}, \bar{\varepsilon}_1)$, *defined above, is a weak homotopy equivalence between two pairs, i. e.* p *induces isomorphisms* $p_* : \pi_n(F \circ F, F) \to \pi_n(\hat{F}, \bar{\varepsilon}_1)$ *for all positive integers* n.

Before proving this theorem, we consider some properties of $p : F \circ F \to \hat{F}$, where F is an H-space satisfying (B).

Define the maps $p_i : U_i \to F$ and $\phi_i : V_i \times F \to U_i$, for $i = 1, 2$, as follows:

$$p_i(t_1x_1 \oplus t_2x_2) = x_i, \qquad \text{for } t_1x_1 \oplus t_2x_2 \in U_i;$$

$$\left.\begin{array}{l}\phi_i((x, t), y) = tx_1 \oplus (1-t)x_2, \text{ with} \\ x_i = y,\ x_j = q_j \circ m_i(x, y),\ \{i, j\} = \{1, 2\},\end{array}\right. \qquad \text{for } (x, t) \in V_i,\ y \in F,$$

where m_i is a homotopy inverse of l_i of (B). These maps p_i and ϕ_i are well defined and continuous, and have the following properties: $p_i \mid F_i$ is the natural homeomorphism; $\phi_i(V_3 \times F) \subset U_3$, and $\phi_i \mid \bar{\varepsilon}_i \times F$ is a homeomorphism onto F_i. Also, it holds the following lemma among these maps:

Lemma 7. *For* $i = 1, 2$, *the two maps* $(p, p_i) : (U_i, U_3) \to (V_i \times F, V_3 \times F)$[1] *and* $\phi_i : (V_i \times F, V_3 \times F) \to (U_i, U_3)$ *are homotopy equivalences of pairs and they are homotopy inverses of the other, relative* F_i *and* $\bar{\varepsilon}_i \times F$ *respectively. More precisely speaking, there are homotopies* $\Phi_t^i : (U_i, U_3) \to (U_i, U_3)$ *and* $\Psi_t^i : (V_i \times F, V_3 \times F) \to (V_i \times F, V_3 \times F)$, $0 \leqq t \leqq 1$, *such that*

$$\Phi_0^i = \phi_i \circ (p, p_i), \quad \Psi_0^i = (p, p_i) \circ \phi_i,$$

Φ_1^i, $\Phi_t^i \mid F_i$, Ψ_1^i, $\Psi_t^i \mid \bar{\varepsilon}_i \times F$ *are the identity maps of* U_i, F_i, $V_i \times F$, $\bar{\varepsilon}_i \times F$ *respectively, for* $0 \leqq t \leqq 1$.

Proof. We define a homotopy $\Phi_t^i : U_i \to U_i$, $0 \leqq t \leqq 1$, as follows, for $i = 1, 2$:

$$\Phi_t^i(t_1x_1 \oplus t_2x_2) = t_1\,{}^1\Phi_t^i(x_1, x_2) \oplus t_2\,{}^2\Phi_t^i(x_1, x_2), \quad \text{with}$$
$${}^i\Phi_t^i(x_1, x_2) = x_i, \quad {}^j\Phi_t^i(x_1, x_2) = q_j \circ L_t^i(x_1, x_2), \quad \{i, j\} = \{1, 2\},$$

for $t_1x_1 \oplus t_2x_2 \in U_i$, where L_t^i is a homotopy between $m_i \circ l_i$ and the identity map mentioned in §4. This homotopy is well defined and continuous; and, for the special case $t_i = 1$, these definitions are read as follows: $\Phi_t^1(1x \oplus 0) = 1x \oplus 0$, $\Phi_t^2(0 \oplus 1x) = 0 \oplus 1x$. $\Phi_t^i(U_3) \subset U_3$ is evident.

By definitions, for $t_1x_1 \oplus t_2x_2 \in U_i$, $i = 1, 2$,

1) It is defined by $(p, p_i)(u) = (p(u), p_i(u)) \in V_i \times F$ for $u \in U_i$.

$$\phi_i \circ (p, p_i)(t_1x_1 \oplus t_2x_2) = \phi_i((x_1x_2, t_1), x_i) = t_1y_1 \oplus t_2y_2,$$

with

$$\begin{aligned} y_i &= x_i = {}^i\Phi_t^i(x_1, x_2),\\ y_j &= q_j \circ m_i(x_1x_2, x_i) = q_j \circ m_i \circ l_i(x_1, x_2)\\ &= q_j \circ L_0^i(x_1, x_2) = {}^j\Phi_0^i(x_1, x_2), \end{aligned}$$

where $\{i, j\} = \{1, 2\}$. Also ${}^j\Phi_1^i(x_1, x_2) = q_j \circ L_1^i(x_1, x_2) = q_j(x_1, x_2) = x_j$. From these equations, it follows immediately that Φ_t^i satisfies the properties of Lemma 7.

We also define a homotopy $\Psi_t^i : V_i \times F \to V_i \times F$, $0 \leqq t \leqq 1$, as follows, for $i = 1, 2$:

$$\begin{aligned} &\Psi_t^i((x, t), y) = ((\overline{\Psi}_t^i(x, y), t), y), \text{ with}\\ &\overline{\Psi}_t^i(x, y) = (q_2 \circ M_{1-2t}^1(x, y)) \cdot (q_2 \circ m_1(x, y)), \text{ for } i = 1,\ 0 \leqq t \leqq 1/2,\\ &\qquad = (q_1 \circ m_2(x, y)) \cdot (q_1 \circ M_{1-2t}^2(x, y)), \text{ for } i = 2,\ 0 \leqq t \leqq 1/2,\\ &\qquad = q_i \circ M_{2t-1}^i(x, y), \qquad \text{for } 1/2 \leqq t \leqq 1, \end{aligned}$$

for $(x, t) \in V_i$, $y \in F$, where M_t^i is a homotopy between $l_i \circ m_i$ and the identity map mentioned in §4. This homotopy is well defined and continuous; and, for the special case $(x, t) = \bar{\varepsilon}_i$, these definitions are read as follows: $\Psi_t^i(\bar{\varepsilon}_i, y) = (\bar{\varepsilon}_i, y)$ for $i = 1, 2$. $\Psi_t^i(V_3 \times F) \subset V_3 \times F$ is evident.

By definitions, for $(x, t) \in V_1, y \in F$,

$$\begin{aligned} (p, p_1) \circ \phi_1((x, t), y) &= (p, p_1)(ty \oplus (1-t)q_2 \circ m_1(x, y))\\ &= ((y \cdot (q_2 \circ m_1(x, y)), t), y)\\ &= ((\overline{\Psi}_0^1(x, y), t), y) = \Psi_0^1((x, t), y), \end{aligned}$$

since $q_2 \circ M_1^1(x, y) = y$. Similarly, we have $(p, p_2) \circ \phi_2 = \Psi_0^2$. Also, $\Psi_1^i((x, t), y) = ((q_i \circ M_1^i(x, y), t), y) = ((x, t), y)$. These show that Ψ_t^i satisfy the properties of Lemma 7, and proofs are completed.

6. Proof of Theorem 4 of § 5.

We shall prove that the map $p : (F \circ F, F) \to (\hat{F}, \bar{\varepsilon}_1)$ satisfies the condition (A_3).

Let $\xi' : (I^n \times 0) \cup (\dot{I}^n \times I) (= J^n) \to F \circ F$ and $\eta : I^n \times I \to \hat{F}$ be given maps such that $p \circ \xi' = \eta \mid J^n$ and $\xi'(J^n \cap M) \subset F$, $\eta(M) = \bar{\varepsilon}_1$ for a given subcomplex M of the product complex $I^n \times I$. Assume that I^n has been so finely subdivided, by $(n-1)$-planes perpendicular to the axes, into finite numbers of n-cubes $\{I_r^n\}$, $r = 1, 2, \ldots, N_1$, and also the unit interval I has been so finely divided at $0 = t_1, t_2, \ldots, t_{N_2+1} = 1$, in such a way that $\eta(I^n_r \times [t_s, t_{s+1}])$ is contained in either the open set V_1 or V_2, for each $r=1, \dots, N_1$ and $s=1, \dots, N_2$.

Thus we have a sequence of finite numbers of $(n+1)$-cubes $\{I_k \mid k = 1, 2, \dots, N_1N_2\}$ such that $\bigcup_k I_k = I^{n+1}$ $(= I^n \times I)$ and $\eta(I_k)$ is contained in either V_1 or V_2 for each $1 \leqslant k \leqslant N_1N_2$, by setting $I_k = I^n_r \times [t_s, t_{s+1}]$, $k = (r-1)N_2 + s$, $1 \leqslant r \leqslant N_1$, $1 \leqslant s \leqslant N_2$.

I_k has $2(n+1)$ n-cubes on its boundary $\dot{I}_k$ for each k, and the total of these n-cubes will be denoted by $\{I^n_\nu\}$. For each $i=1, 2, 3$, we denote by W_i the point-set union of I^n_ν such that $\eta(I^n_\nu) \subset V_i$. Then, we have immediately the following relations:

$$\eta(W_i) \subset V_i, \text{ for } i=1, 2, 3; \quad W_1 \cap W_2 \supset W_3, \quad M \cap W_3 \text{ is empty.}$$

Let $Q_k = J^n \cup (\bigcup_{k'=1}^{k} I_{k'})$ and $Q_0 = J^n$. Let k be $1 \leqslant k \leqslant N_1N_2$, and we assume that ξ' is extended to a map $\xi_{k-1} : Q_{k-1} \to F \circ F$ and also there is a homotopy $Y^{k-1}_t : Q_{k-1} \to \hat{F}$, $0 \leqslant t \leqslant 1$, with the following properties:

(1_{k-1}) $\xi_{k-1}(Q_{k-1} \cap M) \subset F$, $\quad \xi_{k-1}(Q_{k-1} \cap W_i) \subset U_i$, $(i=1, 2, 3)$,

(2_{k-1}) $Y^{k-1}_0 = p \circ \xi_{k-1}$, $\quad Y^{k-1}_1 = \eta \mid Q_{k-1}$, $\quad Y^{k-1}_t \mid J^n = p \circ \xi'$,

(3_{k-1}) $Y^{k-1}_t(Q_{k-1} \cap M) = \bar{\varepsilon}_1$, $\ Y^{k-1}_t(Q_{k-1} \cap W_i) \subset V_i$, $(i=1, 2, 3)$.

Then we have the following

Lemma 8. *From these hypotheses, it follows that ξ_{k-1} and Y^{k-1}_t have extensions $\xi_k : Q_k \to E$ and $Y^k_t : Q_k \to B$ $(0 \leqslant t \leqslant 1)$ satisfying* (1_k), (2_k) and (3_k).

It follows from this lemma and the induction on k, starting with $\xi_0 = \xi'$ and $Y^0_t = p \circ \xi'$, that there is a map $\xi : I^{n+1} \to E$ and a homotopy $Y_t : I^{n+1} \to B$ $(0 \leqslant t \leqslant 1)$ satisfying the conclusions of the condition (A_3), since $Q_{N_1N_2} = I^{n+1}$. Therefore, to prove Theorem 4 of §5, it is sufficient to prove the above lemma, by Theorem 3 of §3.

Proof of Lemma 8. By the definition of $\{I^k\}$, $\eta(I_k)$ is contained in either V_1 or V_2. Let $i_k = 1$ or 2 be such that $\eta(I_k) \subset V_{i_k}$.

We set $J_k = I_k \cap Q_{k-1}$. Then J_k is a union of n-cubes of $\{I^n_\nu\}$ and is a strong deformation retract of I_k, as be easily seen. This retraction will be denoted by $\theta_k : I_k \to J_k$. Also, $\xi_{k-1}(J_k) \subset U_{i_k}$ and $Y^{k-1}_t(J_k) \subset V_{i_k}$, from $J_k \subset W_{i_k}$ and (1_{k-1}), (3_{k-1}).

We now define a map $\zeta' : I_k \to U_{i_k} \subset F \circ F$ and a homotopy $X'_t : J_k \to U_{i_k} \subset F \circ F$, $0 \leqslant t \leqslant 1$, as follows:

$$\zeta'(z)=\phi_{i_k}(\eta(z),\ p_{i_k}\circ\xi_{k-1}\circ\theta_k(z)),\qquad \text{for } z\in I_k;$$
$$X'_t(z)=\phi_{i_k}(Y^{k-1}_{1-2t}(z),\ p_{i_k}\circ\xi_{k-1}(z)),\qquad \text{for } 0\leqq t\leqq 1/2,\ z\in J_k,$$
$$=\Phi^{i_k}_{2t-1}\circ\xi_{k-1}(z),\qquad \text{for } 1/2\leqq t\leqq 1,\ z\in J_k;$$

where p_i, ϕ_i and Φ^i_t are maps and homotopies, mentioned in Lemma 7. X'_t is well defined, since, for $z\in J_k$,

$$\phi_{i_k}(Y^{k-1}_0(z),\ p_{i_k}\circ\xi_{k-1}(z))=\phi_{i_k}\circ(p,\ p_{i_k})\circ\xi_{k-1}(z)=\Phi^{i_k}_0\circ\xi_{k-1}(z).$$

Also, for $z\in J_k$, $\zeta'(z)=\phi_{i_k}(\eta(z),\ p_{i_k}\circ\xi_{k-1}(z))=X'_0(z)$; and hence X'_t is a homotopy of $\zeta'\mid J_k$. Further ζ' and X'_t have properties:

$$\zeta'(I_k\cap M)\subset F,\ \ \zeta'(I_k\cap W_3)\subset U_3;\ X'_t(J_k\cap M)\subset F,\ \ X'_t(J_k\cap W_3)\subset U_3;$$

which are shown immediately from Lemma 7 and (1_{k-1}), (3_{k-1}) of above. Hence, by applying Lemma 1 of § 2 to ζ' and X'_t, and $M_1=I_k\cap M$, $T_1=F$, $M_2=I_k\cap W_3$, and $T_2=U_3$, we have a homotopy $X_t:\ I_k\rightarrow U_{i_k}\subset F\circ F$ such that

$$X_0=\zeta',\quad X_t\mid J_k=X'_t,\quad X_t(I_k\cap M)\subset F,\quad X_t(I_k\cap W_3)\subset U_3,$$

for $0\leqq t\leqq 1$. The second equation shows $X_1\mid J_k=X'_1=\xi_{k-1}\mid J_k$.

From the last property, we can define a map $\xi_k:Q_k\rightarrow F\circ F$ by

$$\xi_k\mid Q_{k-1}=\xi_{k-1},\quad \xi_k\mid I_k=X_1.$$

This map ξ_k has the property (1_k), as be easily seen from the above constructions and (1_{k-1}).

We now consider the map $p\circ\xi_k$. We denote by $q:\ V_i\times F\rightarrow V_i$ the natural projection. Let $Z:I_k\times I\rightarrow V_{i_k}\subset\hat{F}$ be a map defined by, for $z\in I_k$,

$$Z(z,t)=p\circ X_{(2-3t)/2}(z),\qquad \text{for } 0\leqq t\leqq 2/3,$$
$$=q\circ\Psi^{i_k}_{3t-2}(\eta(z),\ p_{i_k}\circ\xi_k\circ\theta_k(z)),\qquad \text{for } 2/3\leqq t\leqq 1,$$

where Ψ^i_t is a homotopy of $(p,\ p_i)\circ\phi_i$ and the identity map, mentioned in Lemma 7. Z is well defined, since $X_0=\zeta'=\phi_{i_k}\circ(\eta,\ p_{i_k}\circ\xi_k\circ\theta_k)$ and $q\circ\Psi^i_0=p\circ\phi_i$. Also,

$$Z(z,0)=p\circ X_1(z)=p\circ\xi_k(z),\ Z(z,1)=\eta(z),\ \text{for } z\in I_k;$$

and $Z((I_k\cap M)\times I)=\bar{e}_1$, $Z((I_k\cap W_3)\times I)\subset V_3$, by making use of Lemma 7. By definitions, the map $Z\mid J_k\times I$ is read as follows, for $z\in J_k$,

$$Z(z,t)=p\circ\Phi^{i_k}_{1-3t}\circ\xi_k(z),\qquad \text{for } 0\leqq t\leqq 1/3,$$
$$=p\circ\phi_{i_k}(Y^{k-1}_{3t-1}(z),\ p_{i_k}\circ\xi_k(z)),\qquad \text{for } 1/3\leqq t\leqq 2/3,$$
$$=q\circ\Psi^{i_k}_{3t-2}(\eta(z),\ p_{i_k}\circ\xi_k(z)),\qquad \text{for } 2/3\leqq t\leqq 1.$$

Let $Z'_s:(J_k\times I)\cup(I_k\times\dot{I})\rightarrow V_{i_k}\subset\hat{F}$, $0\leqq t\leqq 1$, be a homotopy defined by, for $z \in J_k$,

$$
\begin{aligned}
Z_s^t(z, t) &= p \circ \Phi_{1-3t-3s}^{i_k} \circ \xi_k(z), && \text{for } 0 \leqq t \leqq 1/3,\ 0 \leqq s \leqq (1-3t)/3, \\
&= q \circ \Psi_{(3t+3s-1)/2}^{i_k} \circ (p, p_{i_k}) \circ \xi_k(z), && \\
& && \text{for } 0 \leqq t \leqq 1/3, (1-3t)/3 \leqq s \leqq 1-t, \\
&= q \circ \Psi_{3s/2}^{i_k}(Y_{3t-1}^{k-1}(z),\ p_{i_k} \circ \xi_k(z)), && \text{for } 1/3 \leqq t \leqq 2/3,\ 0 \leqq s \leqq 2/3, \\
&= q \circ \Psi_{(6t+3s-4)/2}^{i_k}(\gamma_j(z), p_{i_k} \circ \xi_k(z)), && \text{for } 2/3 \leqq t \leqq 1,\ 0 \leqq s \leqq 2(1-t), \\
&= \gamma_j(z), && \text{for } 2/3 \leqq t \leqq 1, 2(1-t) \leqq s \leqq t, \\
&= Y_{(t+s-1)/(2s-1)}^{k-1}(z), && \text{for } 2/3 \leqq s \leqq 1,\ 1-s \leqq t \leqq s;
\end{aligned}
$$

and, for $z \in I_k$,

$$
\begin{aligned}
Z_s^t(z, 0) &= p \circ \Phi_{1-3s}^{i_k} \circ \xi_k(z), && \text{for } 0 \leqq s \leqq 1/3, \\
&= q \circ \Psi_{(3s-1)/2}^{i_k} \circ (p, p_{i_k}) \circ \xi_k(z), && \text{for } 1/3 \leqq s \leqq 1\ , \\
Z_s^t(z, 1) &= \gamma_j(z), && \text{for } 0 \leqq s \leqq 1\ .
\end{aligned}
$$

From the properties concerning Φ_t^i, Ψ_t^i and Y_t^{k-1} for $t=0,\ 1$, simple calculations show that this homotopy is well defined; and also $Z_0^t = Z \mid (J_k \times I) \cup (I_k \times \dot{I})$ and $Z_1^t(z, 0) = p \circ \xi_k(z)$, $Z_1^t(z, 1) = \gamma_j(z)$, for $z \in I_k$; and

$$Z_s^t(z,\ t) = \bar{e}_1 \quad \text{if } z \in M, \quad Z_s^t(z, t) \in V_3 \quad \text{if } z \in W_3.$$

We extend Z_t^t on $I_k \times I$, by applying Lemma 1 of §2 to Z and Z_t^t, and $M_1 = (I_k \cap M) \times I$, $T_1 = \bar{e}_1$, $M_2 = (I_k \cap W_3) \times I$, and $T_2 = V_3$. Therefore, we have a map $Z_1 : I_k \times I \to V_{i_k} \subset \hat{F}$, being homotopic to Z and having the following properties:

$$
\begin{gathered}
Z_1(z, 0) = Z_1^t(z, 0) = p \circ \xi_k(z),\ \ Z_1(z, 1) = Z_1^t(z, 1) = \gamma_j(z), \ \text{ for } z \in I_k; \\
Z_1(z, t) = Z_1^t(z, t) = Y_t^{k-1}(z), \ \text{ for } z \in J_k \text{ and } 0 \leqq t \leqq 1; \\
Z_1((I_k \cap M) \times I) = \bar{e}_1,\ \ Z_1((I_k \cap W_3) \times I) \subset V_3.
\end{gathered}
$$

From these properties, we can define a homotopy $Y_t^k : Q_k \to \hat{F}$, $0 \leqq t \leqq 1$, by

$$Y_t^k \mid Q_{k-1} = Y_t^{k-1},\ \ Y_t^k(z) = Z_1(z, t) \qquad \text{for } z \in I_k.$$

It follows immediately, from the above constructions and (2_{k-1}), (3_{k-1}), that this homotopy Y_t^k has the desired properties (2_k) and (3_k).

Therefore we have Lemma 8, and Theorem 4 of §5 is proved completely.

Remark. In the above proofs, we use only Lemma 7. Therefore, if there are open sets $U_i \subset E$, $V_i \subset B$ and maps p_i and q_i, $i=1,\ 2$, such that $\{V_i\}$ is a covering of B and they satisfy Lemma 7, then we can prove that $p : (E, F) \to (B, b)$ satisfies the condition (A_3), and hence, that p is a weak homotopy equivalence.

We also notice that the number of the index set $\{i\}$ of the covering $\{V_i\}$ of B may be infinite, if homotopies Φ_t^i and Ψ_t^i of Lemma 7 can be taken as $\Phi_t^i(U_i\cap U_{i_1}\cap\cdots\cap U_{i_n})\subset U_i\cap U_{i_1}\cap\cdots\cap U_{i_n}$ and $\Psi_t^i((V_i\cap V_{i_1}\cap\cdots\cap V_{i_n})\times F)\subset(V_i\cap V_{i_1}\cap\cdots\cap V_{i_n})\times F$ for $0\leqq t\leqq 1$ and for all n and $i_1, \cdots, i_n$.

7. Proof of Theorem 1 of § 1.

From the fact that $F=F_1$ is contractible to a point ε in $F\circ F$ leaving $\varepsilon\in F$ fixed, and from Lemma 6 and Theorem 4, it follows that $F\circ F$, $\hat{F}$ and $p:(F\circ F, F)\to(\hat{F}, \bar{\varepsilon}_1)$, constructed in §5, satisfy (1), (2) of Theorem 1. Therefore the existence of E, B, b and p in Theorem 1 is proved.

To prove the sufficiency of Theorem 1, and also for the later purpose, we prove the follwing lemma.

Lemma 9. *Let $E\supset\bar{F}\supset F$ and $B\ni b$ be given spaces such that $\bar{F}$ is a CW-complex, F its subcomplex and also the weak topology of the product complex $\bar{F}\times F$ is the ordinary product topology of $\bar{F}\times F$; and let $p:(E, F)\to(B, b)$ be a weak homotopy equivalence between two pairs. Further, we assume that $\bar{F}$ is contractible to a vertex $\varepsilon\in F$ in E with ε stationary. Then there is a map $\bar{\mu}:\bar{F}\times F\to E$ such that*

(1) *$\bar{\mu}(F\times F)\subset F$ and $\bar{\mu}(u, \varepsilon)=u$, $\bar{\mu}(\varepsilon, x)=x$, for $u\in F$, $x\in F$, and*

(2) *the map $p\circ\bar{\mu}:\bar{F}\times F\to B$ is homotopic, relative $F\times F$, to the map $\bar{p}:\bar{F}\times F\to B$ defined by $\bar{p}(u, x)=p(u)$ for $u\in\bar{F}$, $x\in F$.*

Proof. Since $\bar{F}\times F$ is a CW-complex and $\bar{F}\vee F=(\bar{F}\times\varepsilon)\cup(\varepsilon\times F)$ is its subcomplex by assumptions, we can apply the same processes of the proof of Theorem 2 of [6].

Let $k_t:(\bar{F}, \varepsilon)\to(E, \varepsilon)$ $(0\leqq t\leqq 1)$ be the contraction of $\bar{F}$ into ε, i.e. $k_1(\bar{F})=\varepsilon$ and $k_0=$the identity map of $\bar{F}$. We define a map $g_0:\bar{F}\times F\to E$ by $g_0(u, x)=x$, and a homotopy $g_t':\bar{F}\vee F\to F$ $(0\leqq t\leqq 1)$ by

$$g'(u, \varepsilon)=k_{1-t}(u),\quad g_t'(\varepsilon, x)=x, \qquad \text{for } u\in\bar{F}, x\in F.$$

Then g_t' is a homotopy of $g_0\mid\bar{F}\vee F$, and hence, by extending this homotopy, we have a homotopy $g_t:\bar{F}\times F\to E$, $0\leqq t\leqq 1$. The map g_1 satisfies

$$g_1(u, \varepsilon)=u,\ g_1(\varepsilon, x)=x,\ p\circ g_1(u, x)=p(u),\ \text{for } (u, x)\in\bar{F}\vee F.$$

By using this homotopy, we also define a map $h':\bar{F}\times F\times I\to B$ as follows:

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$$h'(u, x, t)=p\circ g_{1-2t}(u, x), \qquad \text{for } 0\leqq t\leqq 1/2,$$
$$=p\circ k_{2-2t}(u), \qquad \text{for } 1/2\leqq t\leqq 1.$$

Then $h'(u, x, 0)=p\circ g_1(u, x)$, $h'(\varepsilon\times F\times I)=b$ and $h'(u, x, 1)=p(u)$. Also, $h'\mid(\bar{F}\vee F)\times I$ is homotopic, relative $(\bar{F}\times\varepsilon\times\dot{I})\cup(\varepsilon\times F\times I)$, to the map $h:(\bar{F}\vee F)\times I\to B$ such that $h(u, x, t)=p(u)$. We can extend this homotopy on $\bar{F}\times F\times I$ so that it is stationary on $\bar{F}\times F\times\dot{I}$. Therefore, we have a map $h:\bar{F}\times F\times I\to B$, being homotopic to h' and satisfying the following properties:

$$h(u, x, 0)=p\circ g_1(u, x), \qquad \text{for } (u, x)\in\bar{F}\times F,$$

$$h(u, x, t)=p(u), \qquad \text{for } \begin{cases} t=1, \text{ and } (u, x)\in\bar{F}\times F, \\ 0\leqq t\leqq 1, \text{ and } (u, x)\in\bar{F}\vee F. \end{cases}$$

Let $g':(\bar{F}\times F\times 0)\cup((\bar{F}\vee F)\times I)\to E$ be the map defined by, for $u\in\bar{F}$, $x\in F$,

$$g'(u, x, 0)=g_1(u, x), \quad g'(u, \varepsilon, t)=u, \quad g'(\varepsilon, x, t)=x.$$

Then, as be easily seen, the maps g' and h satisfy the assumptions of (A_1) by taking $K=\bar{F}\times F$, $L=\bar{F}\vee F$, $M=(F\times F\times 1)\cup((F\vee F)\times I)$, and Y'_t is stationary. Since $p:(E, F)\to(B, b)$ is a weak homotopy equivalence and hence it satisfies (A_1), it follows that there is a map $g:\bar{F}\times F\times I\to E$ such that $g\mid(\bar{F}\times F\times 0)\cup((\bar{F}\vee F)\times I)=g'$, $g(F\times F\times 1)\subset F$, and $p\circ g\sim h$: $\bar{F}\times F\times I\to B$, relative $(\bar{F}\times F\times 0)\cup((\bar{F}\vee F)\times I)\cup(F\times F\times I)$. We define $\bar{\mu}:\bar{F}\times F\to E$ by $\bar{\mu}(u, x)=g(u, x, 1)$ for $u\in\bar{F}$, $x\in F$. It follows immediately from the above properties that the map $\bar{\mu}$ satisfies (1), (2) of Lemma 9.

Proof of the sufficiency of Theorem 1. By the conditions (1), (2) of Theorem 1, Lemma 9 is able to be applied by taking $\bar{F}=F$. Therefore the sufficiency is an immediate consequence of Lemma 9.

Remark. The sufficiency is a generalization of Theorem (1. 1) of [5] and the above proofs are similar to it.

8. Proof of Theorem 2 of § 1.

By the assumptions of Theorem 2, we can apply Lemma 9 by taking $\bar{F}=E$. Therefore Theorem 2 follows immediately from the following theorem:

Theorem 5. *Suppose that $p:(E,F)\to(B,b)$ is a weak homotopy equivalence and there is a map $\bar{\mu}:E\times F\to E$ satisfying* (1), (2) *of Lemma* 9 *by taking $\bar{F}=E$. Further we assume that E is contractible in itself to $\varepsilon(=unit)$ with ε stationary.*

Then there is an H-homomorphism[1] *f, which is also a weak homotopy equivalence, of the H-space F, having the multiplication $\mu=\bar{\mu}\mid F\times F$, into the H-space $\Lambda(B)$ of loops in B with the base point b, having the natural multiplication (composition of loops).*

Further, if F is a locally finite CW-complex, the H-structure $\mu=\bar{\mu}\mid F\times F$ of F is homotopy-associative and also has a (two-sided) inversion.

This is a generalization of Theorem 1 of [4] and Theorem 3 of [6], and is proved by the essentially same manner, and we follow several lemmas.

Lemma 10. *Under the assumptions of Theorem* 5, *the map f: $F\to\Lambda(B)$, defined by*

$$f(x)(t)=p\circ k_t(x), \qquad \text{for } x\in F,\ 0\leqq t\leqq 1,$$

where $k_t:(E,\varepsilon)\to(E,\varepsilon)$ is a homotopy between $k_0=$the identity map and $k_1(E)=\varepsilon$, is a weak homotopy equivalence, i. e. f induces isomorphisms f_* of all the homotopy groups of F and $\Lambda(B)$.

Proof. This lemma is an immediate consequence of the commutativity of the following diagram:

$$\begin{array}{ccc} \pi_{n+1}(E,F) & \xrightarrow{\partial} & \pi_n(F) \\ \downarrow p_* & T & \downarrow f_* \\ \pi_{n+1}(B) & \longrightarrow & \pi_n(\Lambda(B)), \end{array}$$

where ∂ is the homotopy boundary homomorphism, which is an isomorphism since $\pi_m(E)=0$, and T is the natural isomorphism.

The commutativity is proved as follows. If a map $\varphi:(I^n,\dot{I}^n)\to(F,\varepsilon)$ represents an element $\alpha\in\pi_n(F)$, the map $\bar{\varphi}:(I^{n+1},\dot{I}^{n+1},J_1^n)\to(E,F,\varepsilon)$, defined by $\bar{\varphi}(x,t)=k_t\circ\varphi(x)$ for $(x,t)\in I^n\times I=I^{n+1}$, $(J_1^n=(I^n\times 1)\cup(\dot{I}^n\times I))$, represents $\beta\in\pi_{n+1}(E,F)$ being $\partial(\beta)=\alpha$. Since $T(p\circ\bar{\varphi}(x))(t)=p\circ\bar{\varphi}(x,t)=p\circ k_t\circ\varphi(x)=(f\circ\varphi(x))(t)$, we have $T\circ p_*(\beta)=f_*(\alpha)=f_*\circ\partial(\beta)$.

---

1) For *H*-spaces X and Y with multiplications μ and μ' respectively, a map $f:X\to Y$ is called an *H*-homomorphism, if two maps $(x_1,x_2)\to f\circ\mu(x_1,x_2)$ and $(x_1,x_2)\to\mu'(f(x_1),f(x_2))$ of $X\times X$ into Y are homotopic each other.

Lemma 11. *The map f, defined above, is an H-homomorphism.*

Proof. As the same to § 4 of [4], we define a map $\Phi : F\times F\times I^2 \to E$, first on $F\times F\times \dot{I}^2$ by, for $x, y\in F$,

$$\begin{aligned}\Phi(x, y, t, s) &= \varepsilon, && \text{for } t=1, \ 0 \leqslant s \leqslant 1,\\ &= \mu(x, y), && \text{for } t=0, \ 0 \leqslant s \leqslant 1,\\ &= k_t(\mu(x, y)), && \text{for } s=0, \ 0 \leqslant t \leqslant 1,\\ &= \bar{\mu}(k_{2t}(x), y), && \text{for } s=1, \ 0 \leqslant t \leqslant 1/2,\\ &= k_{2t-1}(y), && \text{for } s=1, \ 1/2 \leqslant t \leqslant 1,\end{aligned}$$

and then on $F\times F\times I^2$, by mapping the segment from $(t, s)\in \dot{I}^2$ to (1/2, 1/2) on the path, described by the point $\Phi(x, y, t, s)$ under the contraction $k_t : E\to E$. Then the homotopy $\Psi_s : F\times F\to \Lambda(B)$, $0\leqslant s\leqslant 1$, defined by $\Psi_s(x, y)(t) = p\circ\Phi(x, y, t, s)$, is a homotopy of $\Psi_0 = f\circ\mu$ and Ψ_1. The map $p\circ\Phi \mid F\times F\times [0, 1/2]\times 1$ is the map $(x, y, t)\to p\circ\bar{\mu}(k_{2t}(x), y)$, and hence, is homotopic, relative $((F\times F\times 0)\cup(F\times F\times 1/2))\times 1$, to the map $(x, y, t) \to p\circ k_{2t}(x)$, since $\bar{\mu}$ has the property (2) of Lemma 9 of § 7 by taking $\bar{F} = E$. Therefore the map Ψ_1 is homotopic to the map $\mu'\circ(f\times f)$, where μ' is the natural multiplication (composition of loops) on the loop-space $\Lambda(B)$. This shows that two map $f\circ\mu$ and $\mu'\circ(f\times f)$ of $F\times F$ into $\Lambda(B)$ are homotopic, and so, f is an H-homomorphism.

Proof of Theorem 5. The first half is the above two lemmas.

Since f induces isomorphisms between every homotopy groups of F and $\Lambda(B)$, two maps of CW-complex into F are homotopic if, and only if, the two composed maps of these maps and f are homotopic each other. Therefore, the homotopy-associativity of F, i. e. the fact that two maps $(x, y, z)\to\mu(x, \mu(y, z))$ and $(x, y, z)\to\mu(\mu(x, y), z)$, of $F\times F\times F$ into F, are homotopic, is an immediate consequence of the fact that f is an H-homomorphism and that the H-space $\Lambda(B)$ of loops in B with natural multiplication is homotopy-associative.

On the other hand, by Lemma 6 and Remark of §4, μ has a left inversion; and we show the latter is also a right inversion as follows, by using the homotopy-associativity of μ.

Let $\sigma : (F, \varepsilon)\to(F, \varepsilon)$ be a left inversion. As the map $x\to\mu(\sigma(x), x)$ is homotopic, relative ε, to the constant map $x\to\varepsilon$, the map $x\to\sigma\circ\sigma(x) = \mu(\sigma\circ\sigma(x), \varepsilon)$ of F into itself is so to the map $x\to\mu(\sigma\circ\sigma(x), \mu(\sigma(x), x))$, and latter to the map $x\to\mu(\mu(\sigma\circ\sigma(x), \sigma(x)), x)$, and so, to the identity map $x\to x$. Therefore the map $x\to\mu(x, \sigma(x))$ is homotopic, relative ε, to the map $x\to\mu(\sigma\circ\sigma(x), \sigma(x))$, and hence to constant map $x\to\varepsilon$ of F into itself.

This shows that σ is also a right inversion of μ.

Thus we have Theorem 5, and Theorem 2 of § 1 is proved.

Remark. I cannot prove the inverse of Thenrem 2 yet. The inverse may be proved, by generalizing the methods of constructions in [3], if the H-structure μ of F is restricted by additional conditions : $\mu(x, y) = \mu(x', y)$ and $\mu(x, y) = \mu(x, y')$ imply $x = x'$ and $y = y'$, respectively.

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