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# On Rings of Certain Type Associated With Simple Ring-extensions

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# ON RINGS OF CERTAIN TYPE ASSOCIATED WITH SIMPLE RING-EXTENSIONS

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Let $R$ be a Noetherian domain (which is commutative and has a unit), let $R[X]$ be a polynomial ring, let $\alpha$ be an element of an algebraic extension field of the quotient field $K$ of $R$ and let $\pi: R[X] \to R[\alpha]$ be the $R$-algebra homomorphism sending $X$ to $\alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of $\alpha$ over $K$ with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Then $\eta_i \in K$ $(1 \le i \le d)$ are uniquely determined by $\alpha$. Put $d = [K(\alpha) : K]$, $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$. If $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$, we say that $\alpha$ is *anti-integral* over $R$ (*cf.* [3]). Put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Then $J_{[\alpha]} = \mathrm{c}(I_{[\alpha]}\varphi_\alpha(X))$, where $\mathrm{c}(\ )$ denotes the ideal generated by the coefficients of the polynomials in ( ), that is, the content ideal of ( ). If $J_{[\alpha]} \not\subseteq p$ for all $p \in \mathrm{Dp}_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{depth} R_p = 1\}$, the element $\alpha$ is called a *super-primitive element* over $R$. A super-primitive element over $R$ is anti-integral over $R$ (*cf.*[3, Theorem 1.12]). It is also known that any algebraic element over a Krull domain $R$ is super-primitive over $R$ (*cf.*[3, Theorem 1.13]), and hence $\alpha$ is anti-integral over $R$. We also note here that $I_{[\alpha]} = R \Leftrightarrow R[\alpha]$ is integral over $R$ and that $J_{[\alpha]} = R \Leftrightarrow R[\alpha]$ is flat over $R$, provided that $\alpha$ is anti-integral over $R$.

Assume that $\alpha$ is an anti-integral element of degree $d \ge 2$ over $R$. We have seen in [1] that $R[\alpha] \cap R[\alpha^{-1}] = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ as an $R$-module, where $\zeta_i := \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i$ $(1 \le i \le d)$. Note that $\zeta_d = \varphi_\alpha(\alpha) = 0$. Put $\zeta_0 := 1$ and $\eta_0 := 1$ for convenience. It is easy to see that $\alpha\zeta_i = \zeta_{i+1} - \eta_{i+1}$ $(0 \le i \le d-1)$ by definition. Note that $R[\alpha] \cap R[\alpha^{-1}]$ and $R[\alpha]$ are birational and their quotient fields are equal to $K(\alpha)$. Let $H$ be an ideal of $R$ and put $C_H := R + H\zeta_1 + \cdots + H\zeta_{d-1}$, which is an $R$-submodule in $K(\alpha) = K \oplus K\zeta_1 \oplus \cdots \oplus K\zeta_{d-1}$. Our objective of this paper is to investigate when $C_H$ forms a subring of $K(\alpha)$ ($R$-algebra).

Throughout this paper, we use the above notation and conventions unless otherwise specified. Our general reference for unexplained terminology is [2].

We start with the following lemma, which is obviousely seen.

**Lemma 1.** *Assume that $\alpha$ is an anti-integral element of degree $d$*

*over $R$. Let $H$ denote an ideal of $R$. Then $C_H$ forms a subring of $K(\alpha)$ if and only if $H^2\zeta_i\zeta_j \subseteq C_H$ for all $i, j \in \{1, 2, \ldots, d-1\}$ with $j \leq i$.*

**Proposition 2.** *Let $H$ denote an ideal of $R$. Assume that $\alpha$ is an anti-integral element over $R$ of degree $d = 2$. Then $C_H$ forms a subring of $K(\alpha)$ if and only if $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$.*

*Proof.* Note $\zeta_1 = \alpha + \eta_1$ and hence $\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1 = \eta_1\zeta_1 - \eta_2$ (here $\zeta_2 = 0$). Assume that $C_H$ is a subring of $K(\alpha)$. Then $H^2\zeta_1^2 \in C_H$ yields that for any $h \in H$, $h^2\eta_1\zeta_1 - h^2\eta_2 \in R + H\zeta_1 \subseteq K \oplus K\zeta_1$. Consequently, $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$. Conversely, take $a + h_1\zeta_1, b + h_2\zeta_1 \in C_H$ with $h_1, h_2 \in H$ and $a, b \in R$. Then $(a+h_1\zeta_1)(b+h_2\zeta_1) = ab+(ah_2+bh_1)\zeta_1+h_1h_2\zeta_1^2 \in R+H\zeta_1$ because $h_1h_2\zeta_1^2 \in H^2\zeta_1^2 = H^2(\eta_1\zeta_1 - \eta_2) \subseteq R + H\zeta_1 = C_H$.

We consider the case $d = 3$, and compute $\zeta_i\zeta_j$ by a definite calculation.

**Example 3.** Assume that $\alpha$ is an anti-integral element of degree $d = 3$ over $R$. Note $\zeta_1 = \alpha + \eta_1$, $\zeta_2 = \alpha\zeta_1 + \eta_2$ and $\zeta_3 = \alpha\zeta_2 + \eta_3$. Thus

$$\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1,$$

$$\zeta_2\zeta_1 = \zeta_2(\alpha + \eta_1) = \alpha\zeta_2 + \zeta_2\eta_1 = \zeta_3 - \eta_3 + \eta_1\zeta_2 = -\eta_3 + \eta_1\zeta_2 \quad \text{and}$$

$$\zeta_2^2 = \zeta_2(\alpha\zeta_1 + \eta_2) = (\alpha\zeta_2)\zeta_1 + \zeta_2\eta_2 = (\zeta_3 - \eta_3)\zeta_1 + \zeta_2\eta_2 = -\eta_3\zeta_1 + \eta_2\zeta_2$$

because $\zeta_3 = 0$.

Assume that $C_H$ is a subring of $K(\alpha)$. Then we have $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$ and $H^2\eta_3 \subseteq H$ since $\zeta_1^2, \zeta_2\zeta_1$ and $\zeta_2^2$ are in $C_H$.

Conversely, if $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$ and $H^2\eta_3 \subseteq H$, then $\zeta_1^2, \zeta_2\zeta_1$ and $\zeta_2^2$ are in $C_H$. So we conclude that $C_H$ is a subring of $k(\alpha)$ by Lemma 1.

**Lemma 4.** *Assume that $\alpha$ is an anti-integral element of dgree $d(\geq 3)$ over $R$ and that $1 \leq j \leq i < d$.*

(i) *If $i + j < d$, then*

$$(*) \qquad \zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}.$$

(ii) *If $i + j \geq d$, then*

$$(**) \qquad \zeta_i\zeta_j = -\sum_{t=1}^{d-i} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1} \eta_{j-s}\zeta_{i+s}.$$

*(Note that $d - i > 0$ and $d - i - 1 \geq 0$ because $1 \leq j \leq i < d$.)*

*Proof.* Note first that $\eta_0 := 1$, $\zeta_0 := 1$, $\zeta_d = 0$ and $\zeta_{i+1} = \alpha\zeta_i + \eta_{i+1}$. Now we compute $\zeta_i\zeta_j$ as follows :

$$\begin{aligned}
\zeta_i\zeta_j &= \zeta_i(\alpha\zeta_{j-1} + \eta_j) \\
&= \alpha\zeta_i\zeta_{j-1} + \eta_j\zeta_i \\
&= (\zeta_{i+1} - \eta_{i+1})\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+1}\zeta_{j-1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+2}\zeta_{i-2} - \eta_{i+2}\zeta_{j-2} + \eta_{j-1}\zeta_{i+1} - \eta_{i+1}\zeta_{j-1} + \eta_j\zeta_i \\
&= \zeta_{i+2}\zeta_{i-2} - (\eta_{i+2}\zeta_{j-2} + \eta_{i+1}\zeta_{j-1}) + (\eta_j\zeta_i + \eta_{j-1}\zeta_{i+1}) \\
&\quad \cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots .
\end{aligned}$$

(i) Repeat the above process, we have

$$\zeta_i\zeta_j = \zeta_{i+j}\zeta_0 - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s},$$

that is,

$$\zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}.$$

(ii) Put $\ell := i + j - d$. Then $j \le i < d$ yields $\ell < d - 1$ and $j - \ell \ge 1$. Thus continuing the above process yields

$$\zeta_i\zeta_j = \zeta_d\zeta_\ell - \sum_{t=1}^{j-\ell} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-\ell-1} \eta_{j-s}\zeta_{i+s},$$

that is,

$$(**) \qquad \zeta_i\zeta_j = -\sum_{t=1}^{d-i} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1} \eta_{j-s}\zeta_{i+s}.$$

**Remark 5.** Assume that $\alpha$ is an anti-integral element of dgree $d(\ge 3)$ over $R$ and that $1 \le j \le i < d$. For $(i,j)$ with $i + j < d$, let $\Delta_1^{(j)} := \{\ j - t \mid t = 1, \dots, j\ \}$ and $\Delta_1^{(i,j)} := \{\ i + s \mid s = 0, \dots, j-1\ \}$. For $(i,j)$ with $i + j \ge d$, let $\Delta_2^{(j,i)} := \{\ j - t \mid t = 1, \dots, d - i\ \}$ and $\Delta_2^{(i)} := \{\ i + s \mid s = 0, \dots, d - i - 1\ \}$.

(i) If $i+j<d$, then $\Delta_1^{(i,j)} \cap \Delta_1^{(j)} = \emptyset$ because $j-t<i+s$, and $\bigcup_{i+j<d}(\Delta_1^{(i,j)} \cup \Delta_1^{(j)}) = \{0,1,2,\dots,d-1\}$.

(ii) Assume $i+j \geq d$. Then $\Delta_2^{(i)} \cap \Delta_2^{(j,i)} = \emptyset$ because $j-t<i+s$. Put $e_k := -1$ if $k \in \Delta_2^{(j,i)}$ and $e_k := 1$ if $k \in \Delta_2^{(i)}$. Then

$$(**) \qquad \zeta_i\zeta_j = \sum_{k\in\Delta_2^{(j,i)}\cup\Delta_2^{(i)}} e_k\eta_{i+j-k}\zeta_k.$$

Consider $\zeta_{d-1}^2$. Then we have

$$\begin{aligned}\zeta_{d-1}^2 &= \zeta_{d-1}(\alpha\zeta_{d-2}+\eta_{d-1})\\ &= \alpha\zeta_{d-1}\zeta_{d-2}+\eta_{d-1}\zeta_{d-1}\\ &= (\zeta_d-\eta_d)\zeta_{d-2}+\eta_{d-1}\zeta_{d-1}\\ &= -\eta_d\zeta_{d-2}+\eta_{d-1}\zeta_{d-1}.\end{aligned}$$

Hence if $\zeta_{d-1}^2 \in C_H$, then $H^2\eta_d \in H$.

**Theorem 6.** *Assume that $\alpha$ is an anti-integral element of degree $d(\geq 3)$ over $R$. Let $H$ be an ideal of $R$. Then $C_H := R+H\zeta_1+\cdots+H\zeta_{d-1}$ is a subring of $K(\alpha)$ if and only if $H^2\eta_i \subseteq H$ for all $i$ $(1\leq i\leq d)$.*

*Proof.* Our conclusion follows Lemmas 1 and 4 and Remark 5.

**Corollary 7.** *Assume that $\alpha$ is an anti-integral element of degree $d(\geq 3)$ over $R$. Let $H$ be an ideal of $R$. If $C_H$ is a subring of $K(\alpha)$, then every element in $H\eta_i$ is integral over $R$ for all $i$ $(1\leq i\leq d)$.*

*Proof.* We have $H^2\eta_i \subseteq H$ for all $i$ $(1\leq i\leq d)$ by Theorem 6. So $H\eta_i \subseteq H :_K H$. Since $H$ is a finitely generated ideal, any element in $H :_K H$ is integral over $R$. Thus every element of $H\eta_i$ is integral over $R$ for all $i$ $(1\leq i\leq d)$.

**Corollary 8.** *Assume that $R$ is normal and that $\alpha$ is of degree $d(\geq 3)$ over $R$. Let $H$ be an ideal of $R$. If $C_H$ is a subring of $K(\alpha)$, then $H\subseteq I_{[\alpha]}$ and hence $C_H \subseteq R[\alpha]\cap R[\alpha^{-1}]$.*

*Proof.* Since $R$ is normal, the element $\alpha$ is anti-integral over $R$ by [3]. We have $H\eta_i \subseteq H :_K H = R$ for all $i$ $(1\leq i\leq d)$, that is, $H\subseteq I_{[\alpha]}$.

**Theorem 9.** *Let $H$ be an ideal of $R$. Assume that $\alpha$ is an anti-integral element of degree $d(\geq 2)$ over $R$.*

(1) *If $C_H$ is a subring of $K(\alpha)$, then $H^2 \subseteq I_{[\alpha]}$.*

(2) *Assume that* grade$(H) > 1$. *Then $C_H$ is a subring of $K(\alpha)$ if and only if $\eta_i \in R$ for all $i$ if and only if $\alpha$ is integral over $R$.*

(3) *If $I_{[\alpha]} \neq R$ and if $C_H$ is a subring of $K(\alpha)$, then* grade$(H) \leq 1$.

(4) *When $H$ is an invertible ideal and $d \geq 3$, $H \subseteq I_{[\alpha]}$ if and only if $C_H$ is a subring of $K(\alpha)$.*

*Proof.* (1) Assume that $d \geq 3$. Since $C_H$ is a ring, we have $H^2\eta_i \subseteq H$ for all $(1 \leq i \leq d)$ by Theorem 6. Hence $H^2\eta_i \subseteq H \subseteq R$, which means that $H^2 \subseteq I_{\eta_i}$ for all $(1 \leq i \leq d)$. Thus $H^2 \subseteq I_{[\alpha]}$. Next assume that $d = 2$. Then Proposition 2 yields our conclusion.
(2) Assume that $C_H$ is a subring of $K(\alpha)$. Then $H^2 \subseteq I_{[\alpha]}$ by (1). Hence $1 <$ grade$(H) \leq$ grade$(I_{[\alpha]})$, which means that $I_{[\alpha]} = R$ because $I_{[\alpha]}$ is a divisorial ideal of $R$. So we have $I_{\eta_i} = R$ for all $i$ $(1 \leq i \leq d)$, that is, $\eta_i \in R$ for all $i$ $(1 \leq i \leq d)$. Furthermore $\alpha$ is integral over $R$ by [3]. Conversely, assume that $\alpha$ is integral over $R$. Then $I_{[\alpha]} = R$ (*cf.* [3]), and hence $\eta_i \in R$ for all $(1 \leq i \leq d)$. So if $d \geq 3$, $H^2\eta_i \subseteq H$ yields that $C_H$ is a ring by Theorem 6. If $d = 2$, $C_H$ is a ring by Proposition 2.
(3) Our conclusion follows the result (2).
(4) Since $d \geq 3$, $C_H$ is a subring if and only if $H^2\eta_i \subseteq H$ for all $i$ $(1 \leq i \leq d)$. Since $H$ is invertible, $H^2\eta_i \subseteq H$ for all $i$ $(1 \leq i \leq d)$ if and only if $H\eta_i \subseteq R$ for all $i$ $(1 \leq i \leq d)$ if and only if $H \subseteq I_{[\alpha]}$.

**Theorem 10.** *Assume that $\alpha$ is an anti-integral element of degree $d(\geq 2)$ over $R$. Let $H$ be an ideal of $R$. If $H \subseteq I_{[\alpha]}$, then $C_H$ is a subring of $K(\alpha)$.*

*Proof.* The inclusion $H \subseteq I_{[\alpha]}$ yields that $H^2\eta_i \subseteq H(\subseteq R)$ for all $i$ $(1 \leq i \leq d)$. Hence $C_H$ is a ring by Proposition 2 and Theorem 6.

**Corollary 11.** *Assume that $R$ is a locally factorial domain and that $\alpha$ is an element of degree $d(\geq 2)$. The collection $\Delta := \{C_H \mid H$ is an ideal of $R$ and $C_H$ is a subring of $K(\alpha)\}$ has the maximum member $C_{I_{[\alpha]}}(= R[\alpha] \cap R[\alpha^{-1}])$.*

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*Proof.* Note first that $\alpha$ is anti-integral over $R$ because $R$ is a Krull domain (*cf.* [3]). Take $C_H \in \Delta$. Then $H^2 \subseteq I_{[\alpha]}$ by Theorem 9(1). Since $I_{[\alpha]}$ is a divisorial ideal, we have grade$(H)$ = grade$(H^2) = 1$. Since $R$ is locally factorial, every non-zero ideal of grade one is invertible. So $H$ is invertible. Hence $H \subseteq I_{[\alpha]}$ by Theorem 9(4).

Finally we close this paper by the following results concerned with the ring $R[\alpha] \cap R[\alpha^{-1}]$.

**Proposition 12.** *Assume that $\alpha$ is an anti-integral element of degree $d$. If an element $a$ in $R$ is a non-zero-divisor on $R/I_{[\alpha]}$, then $R[a\alpha] \cap R[\alpha^{-1}] = R + I_{[\alpha]}(a\zeta_1) + \cdots + I_{[\alpha]}(a^{d-1}\zeta_{d-1})$.*

*Proof.* We have only to show the inclusion $(\subseteq)$ because $a^i I_{[\alpha]}\zeta_i \subseteq R[a\alpha] \cap R[\alpha^{-1}]$. Take an element $\beta \in R[a\alpha] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a\alpha)^n + \cdots + x_1(a\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots + y_m(\alpha^{-1})^m$ with $x_i, y_j \in R$. Then we have $\alpha^m\beta = x_n a^n \alpha^{n+m} + \cdots + x_0\alpha^m = y_0\alpha^m + y_1\alpha^{m-1} + \cdots + y_m$. Put $f(X) := x_n a^n X^{n+m} + \cdots + x_0 X^m - (y_0 X^m + y_1 X^{m-1} + \cdots + y_m) \in \mathrm{Ker}(\pi)$, where $\pi \colon R[X] \to R[\alpha]$ denotes the canonical $R$-homomorphism. Since $\alpha$ is an anti-integral element over $R$, $\mathrm{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $x_n a^n \in I_{[\alpha]}$. Since $a$ is a non-zero-divisor of $R/I_{[\alpha]}$, we have $x_n \in I_{[\alpha]}$. Put $\beta_*(X) := x_n a^n X^n + \cdots + x_1 a X + x_0$. Since $x_n\varphi_\alpha(X) \in R[X]$, considering $\deg(\beta_*(X) - x_n\varphi_\alpha(X)X^{n-d}) < n$ if $n \geq d$, we can use the induction on $n$ and assume that $n < d$. So considering $\beta - x_n a^n \zeta_n$ instead of $\beta$, we an conclude that $\beta \in R + I_{[\alpha]}(a\zeta_1) + \cdots + I_{[\alpha]}(a^{d-1}\zeta_{d-1})$ by induction on $n$.

**Proposition 13.** *Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. If an element $a$ in $R$ is a non-zero-divisor on $R/I_{[\alpha^{-1}]}$, then $R[a^{-1}\alpha] \cap R[\alpha^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$.*

*Proof.* We have only to show the inclusion $(\subseteq)$. Take an element $\beta \in R[a^{-1}\alpha] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a^{-1}\alpha)^n + \cdots + x_1(a^{-1}\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots + y_m(\alpha^{-1})^m$ with $x_i, y_j \in R$. Then $a^n\beta = x_n\alpha^n + \cdots + a^n x_0 = a^n y_0 + a^n y_1(\alpha^{-1}) + \cdots + a^n y_m(\alpha^{-1})^m$. Since $\alpha$ is anti-integral over $R$, so is $\alpha^{-1}$ (*cf.* [1]). Putting $f(X) := a^n y_m X^{m+n} + \cdots + a^n X^n - (a^n x_0 X^n + \cdots + x_n)$, we have $f(\alpha^{-1}) = 0$ and hence $f(X) \in I_{[\alpha^{-1}]}\varphi_{\alpha^{-1}}(X)$. Thus we have $a^n y_m \in I_{[\alpha^{-1}]}$. Since $a$ is a non-zero-divisor on $R/I_{[\alpha^{-1}]}$, we have $y_m \in I_{[\alpha^{-1}]}$. Hence $y_m\varphi_{\alpha^{-1}}(X) \in R[X]$. Put $\beta_*(X) := y_0 + y_1 X + \cdots + y_m X^m$. Considering $\deg(\beta_*(X) - y_m\varphi_{\alpha^{-1}}(X)) < m$, we can assume that $m < d$ by induction. Let $\varphi_{\alpha^{-1}}(X) := X^d + \eta'_1 X^{d-1} + \cdots + \eta'_d$ be the monic minimal polynomial of $\alpha^{-1}$ over $K$ and let $\zeta'_i := (\alpha^{-1})^i + \eta'_1(\alpha^{-1})^{i-1} + \cdots + \eta'_i$ $(1 \leq i \leq d)$. Considering $\beta - y_m\zeta'_m$ instead of $\beta$, we obtain that $\beta - y_m\zeta'_m \in R + I_{[\alpha^{-1}]}\zeta'_1 + \cdots + I_{[\alpha^{-1}]}\zeta'_{d-1} = R[\alpha^{-1}] \cap R[\alpha]$ (*cf.* [1]) by induction on $m$. Thus $\beta \in R[\alpha^{-1}] \cap R[\alpha]$. Therefore we have $R[a^{-1}\alpha] \cap R[\alpha^{-1}] \subseteq R[\alpha] \cap R[\alpha^{-1}]$.

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