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SUMS AND PARTIAL SUMS OF DOUBLE POWER SERIES ASSOCIATED WITH THE GENERALIZED ZETA FUNCTION AND THEIR N-FRACTIONAL CALCULUS

Maged G. BIN-SAAD

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1. Introduction

An interesting definition of the zeta function, due to Erdélyi [1, p.27 (1)], is recalled here, which is of the form:

$$\Phi(y, z, a) = \sum_{n=0}^{\infty} \frac{y^n}{(a+n)^z}, \quad |y| < 1, \quad a \neq 0, -1, -2, \dots \ . \tag{1.1}$$

This is expressed as the integral form

$$\Phi(y, z, a) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-at} (1 - ye^{-t})^{-1} \, dt, \tag{1.2}$$

for $\operatorname{Re} a > 0$, and either $|y| \leq 1$, $y \neq 1$, $\operatorname{Re} z > 0$ or $y = 1$, $\operatorname{Re} z > 1$, where $\Gamma(z)$ denotes the gamma function. This function is continued to a meromorphic function over the whole z-plane ([1, p.27, 1.11]). The Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s, a)$ are special cases of (1.1) ([1, p.32]):

$$\Phi(1, z, 1) = \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \tag{1.3}$$

$$\Phi(1, z, a) = \zeta(z, a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^z}. \tag{1.4}$$

Erdélyi's function (1.1) has since been extended by Goyal and Laddha [2, p. 100 (1.5)] in the form:

$$\Phi^*_{\mu}(y, z, a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^z} \frac{y^n}{n!}, \quad |y| < 1, \quad \operatorname{Re} a > 0, \quad \mu \geq 1, \tag{1.5}$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ and it is expressed as the integral form

$$\Phi^*_\mu(y,z,a) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-at}(1-ye^{-t})^{-\mu}\,dt. \tag{1.6}$$

Obviously, when $\mu = 1$, (1.5) reduces to (1.1). Recently, Katsurada [4, p. 23] introduced two hypergeometric–type generating functions of the Riemann zeta function as follows:

$$e_z(x) = \sum_{m=0}^\infty \zeta(z+m)\,\frac{x^m}{m!}, \qquad |x| < +\infty, \tag{1.7}$$

$$f_z(\nu;x) = \sum_{m=0}^\infty (\nu)_m\,\zeta(z+m)\,\frac{x^m}{m!}, \qquad |x| < 1, \tag{1.8}$$

where ν and z are arbitrary fixed complex parameter s. Motivated by the results of Erdélyi (1.1) and the work of Katsurada [4], we aim here at presenting two new type of generating functions associated with the zeta function $\Phi(y,z,a)$ and at deriving their various properties and formulas including their integral representations, generating functions, partial sums and N–fractional calculus.

Definition Let a, $a \neq 0,-1,-2,\ldots$, and z, $\operatorname{Re} z > 1$, be complex parameters. We define

$$\zeta(x,y;z,a) = \sum_{m=0}^\infty \Phi(y,z+m,a)\,\frac{x^m}{m!}, \qquad |y| < 1, \tag{1.9}$$

$$\zeta_\nu(x,y;z,a) = \sum_{m=0}^\infty (\nu)_m\,\Phi(y,z+m,a)\,\frac{x^m}{m!}, \quad |y| < 1,\ |x| < |a|, \tag{1.10}$$

where Φ is the generalized zeta function defined by (1.1).

It is important to note that the functions $\zeta(x,y;z,a)$ and $\zeta_\nu(x,y;z,a)$ can be continued meromorphically to the whole z-plane. Clearly, for $y = 1$, definitions (1.9) and (1.10) reduce to

(1.11)

$$\zeta(x,1;z,a) = \sum_{m=0}^\infty \Phi(1,z+m,a)\,\frac{x^m}{m!} = \sum_{m=0}^\infty \zeta(z+m,a)\,\frac{x^m}{m!},$$

(1.12)

$$\zeta_\nu(x,1;z,a) = \sum_{m=0}^\infty (\nu)_m\,\Phi(1,z+m,a)\,\frac{x^m}{m!} = \sum_{m=0}^\infty (\nu)_m\,\zeta(z+m,a)\,\frac{x^m}{m!},$$

respectively, where $\zeta(z,a)$ is the Hurwitz zeta function defined by (1.4). Further, on putting $a = 1$ in equations (1.11) and (1.12), we get the above mentioned results (1.7) and (1.8). In fact, if we let $\nu = z$ in (1.8), this formula reduces to a well-known result of Ramanujan [9]:

$$\zeta(z, 1-x) = \sum_{m=0}^{\infty} (z)_m \, \zeta(z+m) \, \frac{x^m}{m!}, \qquad |x| < 1. \tag{1.13}$$

As an immediate consequence from definitions (1.9) and (1.10), the following Propositions are proved by substituting (1.1) and by changing the order of summation.

Proposition 1.1. *For any complex z, ν and a, $a \notin \{0, -1, -2, \dots\}$, we have*

$$\zeta(x, y; z, a) = \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!(a+n)^{z+m}} = \sum_{n=0}^{\infty} e^{x/(a+n)} \frac{y^n}{(a+n)^z},$$

for $|y| < 1$, and

$$\zeta_\nu(x, y; z, a) = \sum_{m,n=0}^{\infty} \frac{(\nu)_m \, x^m y^n}{m!(a+n)^{z+m}} = \sum_{n=0}^{\infty} \left(1 - \frac{x}{a+n}\right)^{-\nu} \frac{y^n}{(a+n)^z},$$

for $|x| < |a|$ and $|y| < 1$.

The case $y = 0$ of this proposition gives

Proposition 1.2. *Under the same assumptions as in Proposition 1.1, we have*

$$\begin{aligned}
\zeta(x, 0; z, a) &= a^{-z} e^{x/a}, \qquad |x| < +\infty, \\
\zeta_\nu(x, 0; z, a) &= a^{-z} \left(1 - \frac{x}{a}\right)^{-\nu}, \quad |x| < |a|, \\
\zeta(0, y; z, a) &= \zeta_\nu(0, y; z, a) = \Phi(y, z, a).
\end{aligned}$$

2. Integral Representations for the functions $\zeta(x, y; z, a)$ and $\zeta_\nu(x, y; z, a)$

By using Eulerian integral formula of second kind (see e.g. [2]):

$$a^{-z}\Gamma(z) = \int_0^{\infty} e^{-at} t^{z-1} \, dt, \quad \operatorname{Re} z > 0, \quad \operatorname{Re} a > 0, \tag{2.1}$$

it is easy to derive the following integral representations:

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Theorem 2.1. *Let* $\operatorname{Re} a > 0$ *and either*

$$|x| < 1,\ |y| \le 1,\ y \ne 1,\ \operatorname{Re} z > 0 \quad or \quad |x| < 1,\ y = 1,\ \operatorname{Re} z > 1.$$

Then

$$\zeta(x, y; z, a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \left(\frac{e^{-at}}{1 - ye^{-t}} \right) {}_0F_1[-; z; xt]\, dt, \tag{2.2}$$

and

$$\zeta_\nu(x, y; z, a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \left(\frac{e^{-at}}{1 - ye^{-t}} \right) {}_1F_1[\nu; z; xt]\, dt, \tag{2.3}$$

where ${}_0F_1[-; \beta; z] = \sum_{n=0}^{\infty} \frac{x^n}{(\beta)_n\, n!}$ *and*

$${}_1F_1[\alpha; \beta; z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n\, x^n}{(\beta)_n\, n!}, \quad \beta \ne 0, -1, -2, \ldots .$$

Proof. Denote for convenience the right-hand side of equation (2.2) by I. Then, it is easily seen that

$$I = \sum_{m=0}^{\infty} \frac{x^m}{m!\,(z)_m} \frac{1}{\Gamma(z)} \int_0^\infty t^{z+m-1} e^{-at} \left(1 - ye^{-t}\right)^{-1} dt.$$

Now, in view of the integral formula (1.2) and of the definition (1.9), we get the left-hand side of formula (2.2). In the same manner, one can derive the formula (2.3). □

Moreover, by using the contour integral formula [1, p.14 (4)]:

$$2i \sin(\pi z)\Gamma(z) = -\int_\infty^{(0+)} (-t)^{z-1}\, e^{-t}\, dt, \quad |\arg(-t)| \le \pi,$$

one can derive the following contour integral representations.

Theorem 2.2. *Let* $\operatorname{Re} a > 0$ *and* $|\arg(-t)| \le \pi$, *then*

$$\zeta(x, y; z, a) = -\frac{\Gamma(1-z)}{2\pi i} \int_\infty^{(0+)} (-t)^{z-1} \left(\frac{e^{-at}}{1 - ye^{-t}} \right) {}_0F_1[-; z; xt]\, dt, \tag{2.4}$$

and

$$\zeta_\nu(x, y; z, a) = -\frac{\Gamma(1-z)}{2\pi i} \int_\infty^{(0+)} (-t)^{z-1} \left(\frac{e^{-at}}{1 - ye^{-t}} \right) {}_1F_1[\nu; z; xt]\, dt. \tag{2.5}$$

If $x = 0$, equations (2.2) and (2.3) would immediately reduce to (1.2). Whereas for $x = 0$, (2.4) and (2.5) reduce to another known result [1, p.28 (5)].

3. Integrals involving $\zeta(x, y; z, a)$ and $\zeta_\nu(x, y; z, a)$

In this section we evaluate definite integrals involving the functions $\zeta(x, y; z, a)$ and $\zeta_\nu(x, y; z, a)$ in terms of other kinds of zeta and hypergeometric functions. First, we recall the Eulerian integral formula of first kind (see e.g. [10]):

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \operatorname{Re} x > 0,\ \operatorname{Re} y > 0. \tag{3.1}$$

From the term-by-term integration, we can derive the following formulas:

Theorem 3.1. *Let* $\operatorname{Re}(c-b) > 0$ *and* $\operatorname{Re} b > 0$. *Then*

$$\begin{aligned} \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\zeta(xt, y; z, a)\,dt \\ = \sum_{n=0}^{\infty} {}_1F_1[b; c; x/(a+n)]\frac{y^n}{(a+n)^z}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\zeta_\nu(xt, y; z, a)\,dt \\ = \sum_{n=0}^{\infty} {}_2F_1[\nu, b; c; x/(a+n)]\frac{y^n}{(a+n)^z}. \end{aligned} \tag{3.3}$$

Proof. Denote for convenience the left-hand side of equation (3.2) by I. Then in view of Proposition 1, it is easily seen that

$$I = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!\,(a+n)^{z+m}} \int_0^1 t^{b+m-1}(1-t)^{c-b-1}\,dt.$$

Upon using (3.1) and the relation $\Gamma(b+m) = (b)_m\Gamma(b)$, we are finally led to relation (3.2). The derivation of the integral formula (3.3) runs parallel to that of (3.2). □

On putting $y = a = 1$ in (3.2) and (3.3) and noting (1.3), (1.7) and (1.8), the assertions (3.2) and (3.3) reduce to

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}e_z(xt)\,dt = G_z(b, c; x) \tag{3.4}$$

and

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}f_z(\nu;xt)\,dt = G_z(\nu,b,c;x), \tag{3.5}$$

respectively, where

$$G_z(b,\mu;c;x) = \sum_{m=0}^{\infty}\frac{(b)_m(\mu)_m}{(c)_m}\,\zeta(z+m)\,\frac{x^m}{m!}, \qquad |x|<1, \tag{3.6}$$

and

$$G_z(b;c;x) = \sum_{m=0}^{\infty}\frac{(b)_m}{(c)_m}\,\zeta(z+m)\,\frac{x^m}{m!}. \tag{3.7}$$

Note that the integral formulas (3.4) and (3.5) are known results (see [4, p.24 (5.5) and (5.6)]). Now, if we use the integral formula (2.1), other integral formula would occur as follows:

Theorem 3.2. *Let* $\mathrm{Re}|, z>0$, $\mathrm{Re}\,\mu>0$ *and* $\mathrm{Re}\,\lambda<1$. *Then*

$$\begin{aligned}\frac{1}{\Gamma(\mu)\Gamma(z)}\int_0^{\infty}\int_0^{\infty} u^{\mu-1}\nu^{z-1}e^{-u-a\nu}\,\zeta(xue^{-\nu},y;z,a)\,du\,d\nu \\ = \sum_{n=0}^{\infty}\Phi^*_{\mu}[x/(a+n),z,a]\frac{y^n}{(a+n)^z},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\frac{1}{\Gamma(z)}\int_0^{\infty} t^{z-1}e^{-at}\,\zeta_{\nu}(xe^{-t},y;z,a)\,dt \\ = \sum_{n=0}^{\infty}\Phi^*_{\nu}[x/(a+n),z,a]\frac{y^n}{(a+n)^z},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{1}{\Gamma(1-\lambda)}\int_0^{\infty} t^{-\lambda}e^{-at}\,\zeta(x/t,ye^{-t};z,a)\,dt \\ = \Phi(y,z-\lambda+1,a)\,{}_0F_1[-;\lambda;-x],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{1}{\Gamma(1-\lambda)}\int_0^{\infty} t^{-\lambda}e^{-at}\,\zeta_{\nu}(x/t,ye^{-t};z,a)\,dt \\ = \Phi(y,z-\lambda+1,a)\,{}_1F_1[\nu;\lambda;-x],\end{aligned} \tag{3.11}$$

$$\frac{1}{\Gamma(\nu)}\int_0^{\infty} t^{\nu-1}e^{-t}\,\zeta(xt,y;z,a)\,dt = \zeta_{\nu}(x,y;z,a). \tag{3.12}$$

Proof. Denote for convenience the left-hand side of equation (3.8) by I. Then, in view of (1.9), it is easily seen that

$$I = \sum_{m,\ n=0}^{\infty} \frac{x^m y^n}{m!\,(a+n)^{z+m}} \frac{1}{\Gamma(\mu)} \int_0^\infty u^{\mu+m-1} e^{-\mu}\, du;$$
$$\times \frac{1}{\Gamma(z)} \int_0^\infty \nu^{z-1} e^{-\nu(a+m)}\, d\nu.$$

Upon using the integral formula (2.1) and the definition (1.5), we are finally led to right-hand side of formula (3.8). Proceeding in the same manner, it is equally straightforward to derive the formulae (3.9) to (3.12). □

Now some special cases of Theorem 3.2 are of interest. First, if we let $y = 0$ in (3.8), in view of Proposition 1.2, we get an integral representation for the zeta function defined by (1.5) in the form

$$\Phi^*_\mu(x, z, a) = \frac{1}{\Gamma(\mu)\Gamma(z)} \int_0^\infty \int_0^\infty u^{\mu-1} \nu^{z-1} e^{-u-a\nu} e^{(xue^{-\nu})}\, du\, d\nu. \tag{3.13}$$

Secondly, for $y = 0$, equation (3.9) reduces to (1.6) with x and ν replaced by xa and μ respectively. Moreover, for $x = 0$ and $y = 1$, equations (3.10) and (3.11) reduce to an integral relation between the Hurwitz zeta function defined by (1.4) and Erdélyi zeta function defined by (1.1) as follows:

$$\frac{1}{\Gamma(1-\lambda)} \int_0^\infty t^{-\lambda} e^{-at} \Phi(e^{-t}, z, a)\, dt = \zeta(z - \lambda + 1, a). \tag{3.14}$$

4. Sums of series

First we derive the following basic sums of series.

Theorem 4.1. *Let* $z \neq 1, 2, 3, \ldots$. *Then*

$$\sum_{k=0}^{\infty} \zeta(x, y; z-k, a) \frac{w^k}{k!} = e^{aw}\, \zeta(x, ye^w; z, a), \quad |y| < 1, \tag{4.1}$$

$$\sum_{k=0}^{\infty} \zeta_\nu(x, y; z-k, a) \frac{w^k}{k!} = e^{aw}\, \zeta_\nu(x, ye^w; z, a), \quad |x| < |a|,\ |y| < 1. \tag{4.2}$$

Proof. If in formula (1.9) we replace z by $z - k$, $k \notin Z^+ \cup \{0\}$, multiply throughout by $w^k/k!$ and sum up, then we get (4.1). The proof of (4.2) is similar to that of (4.1). □

Again, starting from (1.9) and (1.10), substituting the series expression for $\Phi(y, z, a)$ into (1.9) and (1.10), and changing the order of summations, we get the following theorem.

Theorem 4.2. *Let* $\operatorname{Re} z > 0$ *and* $a \neq 0, -1, -2$*. Then*

$$\sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \zeta(x, y; z+k, a) \frac{w^k}{k!} = \sum_{n=0}^{\infty} e^{(x/(a+n))} {}_1F_1[b; c; w/(a+n)] \frac{y^n}{(a+n)^z}, \tag{4.3}$$

for $|w| < 1$ *and* $|x| < 1$*, and*

$$\sum_{k=0}^{\infty} \frac{(b)_k (f)_k}{(c)_k} \zeta_\nu(x, y; z+k, a) \frac{w^k}{k!} = \sum_{n=0}^{\infty} \left(1 - \frac{x}{a+n}\right)^{-\nu} {}_2F_1[b, f; c; w/(a+n)] \frac{y^n}{(a+n)^z} \tag{4.4}$$

for $|w| < 1$ *and* $|x| < |a|$*.*

If we let $x = 0$, $y = a = 1$ in (4.3) and (4.4) and use (1.3), then we obtain the two Dirichlet series expressions due to Katsurada [4, p.24 (5.3) and (5.4)]:

$$G_z(b; c; w) = \sum_{n=1}^{\infty} {}_1F_1[b; c; w/n] n^{-z}, \tag{4.5}$$

$$G_z(b, f; c; x) = \sum_{n=1}^{\infty} {}_2F_1[b, f; c; w/n] n^{-z}. \tag{4.6}$$

Next, we set $f = c$, $b = z$ and let $x = 0$ in (4.4). Upon noting that

$$ {}_2F_1[z, c; c; w/(a+n)] = (a+n)^z (a+n-w)^{-z}, $$

the assertion (4.4) reduces to

$$\sum_{k=0}^{\infty} (z)_k \Phi(y; z+k, a) \frac{w^k}{k!} = \Phi(y; z, a-w), \tag{4.7}$$

which for $y = 1$ and $b = z$ reduces to a known result [8, p.396 (6)]:

$$\sum_{k=0}^{\infty} (z)_k \zeta(z+k, a) \frac{w^k}{k!} = \zeta(z, a-w). \tag{4.8}$$

Further, from definitions (1.9) and (1.10), we easily have the following interesting series relations.

Theorem 4.3. *Let $|x| < 1$, $|y| < 1$, $|w| < |a|$ and $|t| < |a|$. Then, for any complex number z and $|b|$,*

$$\zeta(x, y; z, a - w) = \sum_{k=0}^{\infty} \zeta_{z+k}(w, y; z + k, a) \frac{x^k}{k!}, \tag{4.9}$$

$$\zeta_b(x, y; b, a - w) = \sum_{k=0}^{\infty} (b)_k \, \zeta_{b+k}(x, y; b + k, a) \frac{w^k}{k!}, \tag{4.10}$$

$$\begin{aligned} &\sum_{k=0}^{\infty} (b)_k \, \zeta(x, y; z + k, a - t) \frac{w^k}{k!} \\ &= \sum_{n=0}^{\infty} {}^3\varphi_G^{(1)} \left[z, 1, b; 1, z; \frac{t}{a+n}, \frac{w}{a+n}, \frac{x}{a+n} \right] \frac{y^n}{(a+n)^z}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\sum_{k=0}^{\infty} \zeta_\nu(x, y; z + k, a - t) \frac{w^k}{k!} \\ &= \sum_{n=0}^{\infty} {}^3\varphi_G^{(1)} \left[z, 1, \nu; 1, z; \frac{t}{a+n}, \frac{w}{a+n}, \frac{x}{a+n} \right] \frac{y^n}{(a+n)^z}, \end{aligned} \tag{4.12}$$

where ${}^3\varphi_G^{(1)}$ is Jain's confluent hypergeometric function of three variables ([3]):

$${}^3\varphi_G^{(1)}(a, b, c; e, f; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b)_m(c)_n x^m y^n z^p}{(e)_m (f)_{n+p} m! n! p!}.$$

Proof. Starting from (1.9), we have

$$\zeta(x, y; z, a - w) = \sum_{k,n=0}^{\infty} \frac{x^k y^n}{k!(a + n - w)^{z+k}}.$$

Now, on using the binomial theorem, the above equation gives us

$$\zeta(x, y; z, a - w) = \sum_{k=0}^{\infty} \sum_{m,n=0}^{\infty} \frac{(z + k)_m w^m y^n x^k}{k!(a + n)^{z+m+k} m!},$$

which, in view of (1.10), gives the left hand side of (4.9). This complete the proof of relation (4.9). In the same manner one can prove relations (4.10) to (4.12). □

Now some special cases of Theorem 4.3 are of interest. First for $w = 0$, equation (4.9) reduces to (1.9). Whereas for $x = 0$, equation (4.9) yields

$$\Phi(y, z, a - w) = \zeta_z(w, y; z, a). \tag{4.13}$$

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If we further put $x=0$, $y=1$ and $b=z$ in (4.10), then (4.10) reduces to a particular case of formula (4.8). Finally from (4.11) and (4.12), we find that

$$\sum_{k=0}^{\infty}(b)_k\,\zeta(x,y;z+k,a-t)\,\frac{w^k}{k!}=\sum_{m=0}^{\infty}\zeta_b(w,y;z+m,a-t)\,\frac{x^m}{m!}. \tag{4.14}$$

5. Partial sums

Let

$$_{[r]}\Phi(y,z,a)=\sum_{n=0}^{r}\frac{y^n}{(a+n)^z}$$

be the truncation of (1.1), and

$$^{[r+1]}\Phi(y,z,a)=\sum_{n=r+1}^{\infty}\frac{y^n}{(a+n)^z},\quad r=0,1,2,\ldots,$$

be its remainder. Motivated by the above mentioned definitions, we here aim at further investigating the functions ζ and ζ_ν in their truncated forms $_{[r]}\zeta$ and $_{[r]}\zeta_\nu$ and reminders $^{[r+1]}\zeta$ and $^{[r+1]}\zeta_\nu$.

Definition. Let a and z be complex parameters such that $a\neq 0,-1,-2,\ldots$, $\operatorname{Re} z>1$ and $r=0,1,2,\ldots$. We define

$$_{[r]}\zeta(x,y;z,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{r}\frac{x^my^n}{m!(a+n)^{z+m}}, \tag{5.1}$$

$$^{[r+1]}\zeta(x,y;z,a)=\sum_{m=0}^{\infty}\sum_{n=r+1}^{\infty}\frac{x^my^n}{m!(a+n)^{z+m}}, \tag{5.2}$$

$$_{[r]}\zeta_\nu(x,y;z,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{r}\frac{(\nu)_mx^my^n}{m!(a+n)^{z+m}}, \tag{5.3}$$

$$^{[r+1]}\zeta_\nu(x,y;z,a)=\sum_{m=0}^{\infty}\sum_{n=r+1}^{\infty}\frac{(\nu)_mx^my^n}{m!(a+n)^{z+m}}. \tag{5.4}$$

Theorem 5.1. *Let* $\operatorname{Re} a>0$ *and either* $|x|<1$, $|y|\le 1$, $y\neq 1$, $\operatorname{Re} z>0$ *or* $|x|<1$, $y=1$, $\operatorname{Re} z>1$. *Then*

(5.5)
$$_{[r]}\zeta(x,y;z,a)=\frac{1}{\Gamma(z)}\int_0^{\infty}t^{z-1}\left(\frac{1-(ye^{-t})^{r+1}}{e^t-y}\right)e^{(1-a)t}\,{}_0F_1[-;z,xt]\,dt,$$

(5.6)

$$ {}^{[r+1]}\zeta(x,y;z,a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \left(\frac{y^{r+1}e^{-(a+r)t}}{e^t - y} \right) {}_0F_1[-;z,xt]\,dt, $$

(5.7)

$$ {}_{[r]}\zeta_\nu(x,y;z,a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \left(\frac{1-(ye^{-t})^{r+1}}{e^t - y} \right) e^{(1-a)t}\,{}_1F_1[\nu;z,xt]\,dt, $$

(5.8)

$$ {}^{[r+1]}\zeta_\nu(x,y;z,a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \left(\frac{y^{r+1}e^{-(a+r)t}}{e^t - y} \right) {}_1F_1[\nu;z,xt]\,dt, $$

(5.9)

$$ {}_{[r]}\zeta(x,y;z,a) + {}^{[r+1]}\zeta(x,y;z,a) = \zeta(x,y;z,a), $$

(5.10)

$$ {}_{[r]}\zeta_\nu(x,y;z,a) + {}^{[r+1]}\zeta_\nu(x,y;z,a) = \zeta_\nu(x,y;z,a). $$

Proof. Using (5.1) and (2.1), we get

$$ \begin{aligned} {}_{[r]}\zeta(x,y;z,a) &= \sum_{m=0}^{\infty}\sum_{n=0}^{r} \frac{x^m y^n}{m!\Gamma(z+m)} \int_0^\infty t^{z+m-1} e^{-(a+n)t}\,dt \\ &= \sum_{m=0}^{\infty} \frac{x^m}{m!\Gamma(z+m)} \int_0^\infty t^{z+m-1} \left(\sum_{n=0}^{r} y^n e^{-(a+n)t} \right) dt \\ &= \frac{1}{\Gamma(z)} \sum_{m=0}^{\infty} \frac{x^m}{m!(z)_m} \int_0^\infty t^{z+m-1} e^{(1-a)t} \left(\frac{1-(ye^{-t})^{r+1}}{e^t - y} \right) dt \\ &= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{(1-a)t} \left(\frac{1-(ye^{-t})^{r+1}}{e^t - y} \right) dt \sum_{m=0}^{\infty} \frac{(xt)^m}{m!(z)_m}\,dt. \end{aligned} $$

Consequently, we get (5.5). The derivation of the equations (5.6) to (5.8) runs parallel to that of (5.5) and we skip the details. The proofs of (5.9) and (5.10) are obtained clearly by the definitions (1.9), (1.10) and (5.1) – (5.4), respectively. □

Theorem 5.2. *Let* $p_j \geq 1$, $\operatorname{Re} z > 1$ *and* $\operatorname{Re} p_j z > 1$, $j = 1, 2, \ldots, n$. *Then*

$$ \prod_{j=1}^{n} {}_{[r_j]}\zeta(x_j, y_j; p_j z, a_j) = \prod_{j=1}^{n} \left\{ \sum_{k_j=0}^{r_j} y_j^{k_j} e^{B_j} \right\} \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-At}\,dt, $$

$$\prod_{j=1}^{n} {}^{[r_j+1]}\zeta(x_j, y_j; p_j z, a_j) = \prod_{j=1}^{n}\left\{\sum_{k_j=r_j+1}^{\infty} y_j^{k_j} e^{B_j}\right\}\frac{1}{\Gamma(z)}\int_0^{\infty} t^{z-1}e^{-At}\,dt, \tag{5.11}$$

$$\prod_{j=1}^{n} {}_{[r_j]}\zeta_\nu(x_j, y_j; p_j z, a_j) = \prod_{j=1}^{n}\left\{\sum_{k_j=0}^{r_j} y_j^{k_j}\left(1 - x_j/(a_j+k_j)\right)^{-\nu}\right\}\frac{1}{\Gamma(z)}\int_0^{\infty} t^{z-1}e^{-At}\,dt, \tag{5.12}$$

$$\prod_{j=1}^{n} {}^{[r_j+1]}\zeta_\nu(x_j, y_j; p_j z, a_j) = \prod_{j=1}^{n}\left\{\sum_{k_j=r_j+1}^{\infty} y_j^{k_j}\left(1 - x_j/(a_j+k_j)\right)^{-\nu}\right\}\frac{1}{\Gamma(z)}\int_0^{\infty} t^{z-1}e^{-At}\,dt, \tag{5.13}$$

where $B_j = x_j/\left(a_j + k_j\right)$ *and* $A = \prod_{j=1}^{n}(a_j + k_j)^{p_j}$.

Proof. To prove (5.11), let I denote the right hand side of (5.11). Then, from (2.1) and (5.11), we find that

(5.14)

$$I = \left\{\sum_{m_1=0}^{\infty}\sum_{k_1=0}^{r_1}\frac{x_1^{m_1}y_1^{k_1}}{m_1!(a_1+k_1)^{m_1}}\cdots\sum_{m_n=0}^{\infty}\sum_{k_n=0}^{r_n}\frac{x_n^{m_n}y_n^{k_n}}{m_n!(a_n+k_n)^{m_n}}\right\}A^{-z}$$
$$= \left\{\sum_{m_1=0}^{\infty}\sum_{k_1=0}^{r_1}\frac{x_1^{m_1}y_1^{k_1}}{m_1!(a_1+k_1)^{m_1+p_1z}}\cdots\sum_{m_n=0}^{\infty}\sum_{k_n=0}^{r_n}\frac{x_n^{m_n}y_n^{k_n}}{m_n!(a_n+k_n)^{m_n+p_nz}}\right\}.$$

Therefore, we have (5.11) from (5.15). To prove formulas (5.12) to (5.14), we refer to the proof of (5.11). □

It is important to note that Theorem 5.2 includes a number of known results due to Nishimoto ([6, Theorems 2–5 and 6] and [7, Theorems 2–5 and 6]) as special cases.

6. N–FRACTIONAL CALCULUS

There are many definitions of a differ-integral of arbitrary order. In this work we use the definition of fractional calculus given by Nishimoto [5].

Definition(by Nishimoto): Let $C = \{C_-, C_+\}$ and $D = \{D_-, D_+\}$, where C_- is a curve surrounding in the positive direction the cut joining two the points z and $-\infty + i\,\mathrm{Im}\, z$, C_+ is a curve surrounding in the negative direction the cut joining the two points z and $\infty + i\,\mathrm{Im}\, z$, and $D_\mp$ is the domain surrounded by $C_\mp$ respectively. (Here D contains the points over the curve C) . Moreover, let $f = f(z)$ be analytic (regular) function in $z \in D$. Then we define

$$(6.1) \qquad f_\nu = (f)_\nu (z) = {}_C (f)_\nu (z) = \frac{\Gamma(\nu+1}{2\pi i} \int_{\boldsymbol{C}} \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\, d\zeta, \quad \nu \notin \boldsymbol{Z}^-,$$

$$(f)_{-n} (z) = \lim_{\nu \to -n} (f)_\nu (z), \quad n \in \boldsymbol{Z}^+,$$

where $-\pi \le \arg(\zeta - z) \le \pi$ for $\zeta \in C_-$, $0 \le \arg(\zeta - z) \le 2\pi$ for $\zeta \in C_+$, $\zeta \ne z$ and $z \in C$. Then $(f)_\nu$ is the fractional differ–integration for arbitrary order ν (derivation of order ν for $\mathrm{Re}\,\nu > 0$ and integral of order -ν for $\mathrm{Re}\,\nu < 0$) with respect to z of the function f, if $|(f)_\nu| < \infty$.

Obeying the above definition of N–fractional calculus, Nishimoto obtained the following result.

Lemma 6.1. *We have*

$$(6.2) \qquad (e^{az})_\alpha (z) = a^\alpha e^{az}, \quad a \ne 0,$$

$$(6.3) \qquad \left(z^\beta\right)_\alpha (z) = e^{-\pi\alpha} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)}\, z^{\beta-\alpha}, \quad \left| \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} \right| < \infty.$$

We shall now use this Lemma in order to derive certain transformation formulas and relationships for the functions ζ and ζ_ν.

Theorem 6.1. *Let* $\alpha,\ \beta \in \boldsymbol{R}$, $a > 0,\ a \ne 1$ *and* $\mathrm{Re}\, z > 1$. *Then*

$$(6.4) \qquad (\zeta(x, y; z, a))_\alpha (z) = e^{i\pi\alpha} \sum_{n=0}^{\infty} (\log(a+n))^\alpha\, e^{x/(a+n)} \frac{y^n}{(a+n)^z},$$

$$(6.5) \qquad (\zeta_\nu(x, y; z, a))_\alpha (z)$$
$$= e^{i\pi\alpha} \sum_{n=0}^{\infty} (\log(a+n))^\alpha\, (1 - x/(a+n))^{-\nu}\, \frac{y^n}{(a+n)^z},$$

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$$(6.6)\qquad \left((\zeta(x,y;z,a))_\alpha(z)\right)_\beta(z) = \Big((\zeta(x,y;z,a))_\beta(z)\Big)_\alpha(z) = (\zeta(x,y;z,a))_{\beta+\alpha}(z),$$

$$(6.7)\qquad \left((\zeta_\nu(x,y;z,a))_\alpha(z)\right)_\beta(z) = \Big((\zeta_\nu(x,y;z,a))_\beta(z)\Big)_\alpha(z) = (\zeta_\nu(x,y;z,a))_{\beta+\alpha}(z).$$

Proof. From (1.9), we have

$$(\zeta(x,y;z,a))_\alpha(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^m y^n}{m!(a+n)^m}\left(e^{zlog(a+n)^{-1}}\right)_\alpha(z),$$

which, on using (6.2), yields

$$(6.8)\qquad (\zeta(x,y;z,a))_\alpha(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^m y^n}{m!(a+n)^m}\left(\log(a+n)^{-1}\right)^\alpha.$$

Therefore, we have (6.4) from (6.8) under the conditions. The proof of the result (6.5) is similar to that of (6.4). To prove (6.6), we start from (6.4) and have

$$(6.9)\qquad \left((\zeta(x,y;z,a))_\alpha(z)\right)_\beta(z) = e^{i\pi\alpha}\sum_{n=0}^{\infty}(\log(a+n))^\alpha e^{x/(a+n)} y^n \left(e^{z\log(a+n)^{-1}}\right)_\beta$$

$$(6.10)\qquad = e^{i\pi(\alpha+\beta)}\sum_{n=0}^{\infty}(\log(a+n))^{\alpha+\beta} e^{x/(a+n)} y^n \frac{y^n}{(a+n)^z}.$$

In the same way, we have

$$(6.11)\qquad \Big((\zeta(x,y;z,a))_\beta(z)\Big)_\alpha(z) = e^{i\pi(\alpha+\beta)}\sum_{n=0}^{\infty}(\log(a+n))^{\alpha+\beta} e^{x/(a+n)} y^n \frac{y^n}{(a+n)^z},$$

and replacing α by $\alpha+\beta$ in (6.4), we get

$$(6.12)\qquad (\zeta(x,y;z,a))_{\beta+\alpha}(z) = e^{i\pi(\alpha+\beta)}\sum_{n=0}^{\infty}(\log(a+n))^{\alpha+\beta} e^{x/(a+n)} y^n \frac{y^n}{(a+n)^z}.$$

Clearly, we have (6.6) from (6.10), (6.11) and (6.12). Similarly, one can prove (6.7). □

Theorem 6.2. *Let* $\alpha,\ \beta \in \boldsymbol{R}$, $\operatorname{Re} a > 0$, $\operatorname{Re} z > 1$ *and* $|\Gamma(1-\beta)/\Gamma(1-\alpha)| < \infty$. *Then*

$$\begin{aligned}(6.13)\quad & \left(x^{\alpha-1}\zeta(x,y;z,a)\right)_{(\alpha-\beta)}(x) \\ &= e^{-i\pi(\alpha-\beta)}\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)}x^{\beta-1}\sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\beta)_m}\Psi(y,z+m,a)\frac{x^m}{m!},\end{aligned}$$

$$\begin{aligned}(6.14)\quad & \left(x^{\alpha-1}\zeta_\nu(x,y;z,a)\right)_{(\alpha-\beta)}(x) \\ &= e^{-i\pi(\alpha-\beta)}\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)}x^{\beta-1}\sum_{m=0}^{\infty}\frac{(\nu)_m(\alpha)_m}{(\beta)_m}\Phi(y,z+m,a)\frac{x^m}{m!}.\end{aligned}$$

Proof. By (1.9), we have

$$\left(x^{\alpha-1}\zeta(x,y;z,a)\right)_{(\alpha-\beta)}(x) = \sum_{m=0}^{\infty}\Phi(y,z+m,a)\frac{\left(x^{\alpha+m-1}\right)_{(\alpha-\beta)}(x)}{m!},$$

which is immediately derived from (1.9) by applying the term-by-term fractionally differential operation. Next, on using (6.3) and (1.1), we are led to the right–hand side of equation (6.13). Similarly, one can prove (6.14). □

An interesting special cases arise form the relations (6.13) and (6.14) when $y = a = 1$ in the forms:

$$(6.15)\quad \left(x^{\alpha-1}f_z(\nu;x)\right)_{(\alpha-\beta)}(x) = e^{-i\pi(\alpha-\beta)}\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)}x^{\beta-1}G_z(\nu,\alpha;\beta;x),$$

and

$$(6.16)\quad \left(x^{\alpha-1}e_z(\nu;x)\right)_{(\alpha-\beta)}(x) = e^{-i\pi(\alpha-\beta)}\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)}x^{\beta-1}G_z(\alpha;\beta;x).$$

Equations (6.15) and (6.16) exhibit the fact that $G_z(\nu,\alpha;\beta;x)$ and $G_z(\alpha;\beta;x)$ are essentially the fractional differ–integral of the functions $f_z(\nu;x)$ and $e_z(x)$, respectively. Finally, when $\nu = \beta$, formula (6.14) gives the elegant result

$$(6.17)\quad \left(x^{\alpha-1}\zeta_\beta(x,y;z,a)\right)_{(\alpha-\beta)}(x) = e^{-i\pi(\alpha-\beta)}\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)}x^{\beta-1}\zeta_\alpha(x,y;z,a).$$

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