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ALGEBRAIC EXTENSIONS OF SIMPLE RINGS I

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In this paper we study the local finiteness of algebraic extensions of simple rings. Our study starts with the preliminary section § 1, which contains some matrix calculations of independent interest. In § 2, we consider simple algebras of finite rank, and show under what conditions they are locally finite, of bounded degree over simple subrings. In § 3, we study the equality of left and right dimensionalities for some types of simple ring extensions. In § 4, we consider QG-1 extensions of simple rings which are generalized Galois extensions, and in § 5, we study the local finiteness of QG-1 division ring extensions of infinite rank. For QG-1 extensions of simple rings, the situations will be studied in a continuation of the present title, II.

Throughout the present paper, $A = \sum_1^n D e_{ij}$ will represent a simple ring where $E = \{e_{ij}\}$ is a system of matrix units and D is the centralizer of E in A which is a division ring, and B a simple subring of A containing the identity 1 of A . For any subset S of A , we denote by $V_A(S)$ the centralizer of S in A . Specially we use the following convention: V means $V_A(B)$. Moreover we write $C = V_A(A)$, $Z = V_B(B)$, $C_0 = V_V(V)$ the centers of A , B , and V respectively. If, for every finite subset F of A , $[B[F] : B]_l < \infty$ (resp. $[B[F] : B]_r < \infty$) then we say that A is left (resp. right) locally finite over B . Given rings R , S , we shall understand by an R - S -module a two-sided module which is treated as a left R - and right S -module. For any simple subring T of A , we denote by $[T|T]$ the uniquely determined number of irreducible direct summands of T -module T . If $T \supset E$ then one can see that $[T|T] = [A|A]$. A subring T of A is said to be regular if T and $V_A(T)$ are both simple. By \mathcal{R} we denote the set of all regular subrings of A containing B , and we shall use the following notations:

$$\mathcal{R}^0 = \{B' \in \mathcal{R} ; [B'|B'] = [A|A]\},$$

$$\mathcal{R}_{l,j} = \{B' \in \mathcal{R} ; [B' : B]_l < \infty\} (\mathcal{R}_{r,j} = \{B' \in \mathcal{R} ; [B' : B]_r < \infty\}),$$

$\mathcal{R}/S = \{B' \in \mathcal{R} ; B' \supset S\}$, where S is a subset of A , and we set $\mathcal{R}_{l,j}^0 = \mathcal{R}^0 \cap \mathcal{R}_{l,j}$, $\mathcal{R}^0/S = \mathcal{R}^0 \cap \mathcal{R}/S$, $\mathcal{R}_{l,j}/S = \mathcal{R}_{l,j} \cap \mathcal{R}/S$, $\mathcal{R}_{l,j}^0/S = \mathcal{R}_{l,j}^0 \cap \mathcal{R}/S$. As to other notations and terminologies used in this paper, we follow [3—11].

1. Preliminaries

We consider the following

$$u(E, d) = de_{21} + \sum_3^n e_{ii-1}$$

with non-zero $d \in D$, which plays an important role in our subsequent consideration. First we shall prove the following

Lemma 1. *Let $n > 1$, and T a left Artinian unital subring of A . If T contains $u(E, d)$ and $a = \sum x_{ij}e_{ij}$ ($x_{ij} \in D$) with $x_{1n} = 1$ and $x_{in} = 0$ for every $i > 1$, then T contains E, d , and x_{ij} 's.*

Proof. Since A is T - A -irreducible by [7, Lemma 8], T is in \mathcal{R} (cf. [11, p. 69]). We set $u = u(E, d)$. Then $u^{k-1}au^{n-1} = d^2e_{k1}$ is a non-zero element of $T \cap e_{kk}A$ ($k = 1, \dots, n$), and so by [7, Lemma 5] T contains $e_{11}, e_{22}, \dots, e_{nn}$. It follows therefore $e_{1n} = e_{11}ae_{nn} \in T$ and $d = (u + e_{1n})^n \in T$, so that $e_{21} = d^{-2}(d^2e_{21}) \in T$. It follows then $\sum_3^n e_{ii-1} = (1-d)e_{21} + u \in T$, and therefore our assertion is a direct consequence of [7, Lemma 6].

The next contains evidently [1, Satz].

Corollary 1. *Let $\phi(\subset C)$ be a perfect field. If A/ϕ is 3-algebraic then A/ϕ is locally finite. In case $n > 1$, if A/ϕ is 2-algebraic then A/ϕ is locally finite.*

Proof. Firstly, assume that A is a division ring and 3-algebraic over ϕ . If a_1, a_2, a_3 are in A then $\phi[a_1, a_2, a_3] = \phi[a'_1, a'_2]$ with some $a'_1, a'_2 \in A$ ([7, Th. 1]). Hence, an easy induction will prove that A/ϕ is locally finite. Next, assume that $n > 1$ and A/ϕ is 2-algebraic. Let d_1, d_2, d_3 be non-zero elements of D . If $a_1 = \sum x_{ij}e_{ij}$ where $x_{11} = d_1, x_{21} = d_2, x_{1n} = 1$ and each of other x_{ij} 's is 0, and $a_2 = u(E, d_3)$, then $\phi[a_1, a_2]$ is finite over ϕ and contains $\phi[d_1, d_2, d_3, E]$ (Lemma 1), which means that D/ϕ is 3-algebraic. Hence, A/ϕ is locally finite by the first assertion proved above.

Proposition 1. *Let A be left algebraic (resp. left algebraic and of bounded degree) over B . If $B \not\subset C$ then there exists some $B' = \sum_1^n D'e'_{ij} \in \mathcal{R}_{i,j}^0$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic (resp. left algebraic and of bounded degree).*

Proof. It suffices to prove the proposition for the case $n > 1$. By [7, Lemma 7], there exists an element $r \in A^*$ such that $B\bar{r}$ contains an element $a = \sum x_{ij}e_{ij}$ ($x_{ij} \in D$) with $x_{1n} = 1$ and $x_{in} = 0$ for every $i > 1$. Given an arbitrary non-zero element $d \in D$, we set $u = u(E, d)$. Then, $(B\bar{r})[u]$ is left finite over $B\bar{r}$ and contains $\{E, d\}$ (Lemma 1). If we set $B^* = (B\bar{r})[E] = \sum D^*e_{ij}$ with the division ring $D^* = V_{B^*}(E)$, then D/D^* is left

algebraic. Hence, $B^* \tilde{r}^{-1}$ can be taken as B' requested.

Proposition 2. *Let $[B : Z] < \infty$ (or $[A : C] < \infty$). If $[A : B]_l < \infty$ then $[A : B]_r < \infty$.*

Proof. By [9, Lemma], we obtain $[A : C] < \infty : A = \sum_i^n a_i C$. Since $C \cdot B = \sum_i^n B c_j$ ($c_j \in C$), it follows $\sum_{i,j} a_i c_j B = \sum_i a_i C \cdot B = A$, which means $[A : B]_r < \infty$.

Corollary 2. *Let $[B : Z] < \infty$. If A/B is left locally finite then it is locally finite.*

2. Algebraic extensions of a simple algebra of finite rank

The present section is devoted exclusively to the treaty of algebraic extensions of a simple algebra of finite rank.

Proposition 3. *Let A be left algebraic over B .*

(a) *In order that $[A : C] < \infty$, it is necessary and sufficient that $[V : C] < \infty$ and $[B : Z] < \infty$.*

(b) *If $[A : C] < \infty$ then A/B is locally finite.*

Proof. (a) By the validity of [10, Lemma], it suffices to prove the sufficiency. In case B is contained in C , our assertion is trivial. If B is not contained in C then there exists some $B' = \sum_i^n D' e'_{ij} \in \mathcal{R}_{B'}^0$ such that $V_A(\{e'_{ij}\})$ is left algebraic over D' (Prop. 1). Since $[B' : V_{B'}(B')] < \infty$ again by [10, Lemma], we may assume from the beginning that A is a division ring. Then, noting that $B \cdot C = B \otimes_Z Z \cdot C$, there holds $[B \cdot C : C] = [B \cdot C : Z \cdot C] \cdot [Z \cdot C : C] \leq [B : Z] \cdot [V : C] < \infty$. Hence, $B \cdot C = V_A^2(B \cdot C) = V_A(V)$, and so $[A : C] = [A : B \cdot C] \cdot [B \cdot C : C] = [V : C] \cdot [B \cdot C : C] < \infty$.

(b) In case B is contained in C , our assertion is clear by [2, Prop. 10.12.3] or by a direct computation. In what follows, we shall assume that $B \not\subset C$. Then, by [7, Th. 1], there exists an element $a \in A$ such that $A = (B \cdot C)[a]$. Now, let F be an arbitrary finite subset of A . Then, C contains a finite subset F' such that $B[F] \subset B[F', a] = B[a] \cdot B[F']$. Since we can easily see that $[B[F'] : B]_l < \infty$, our assertion is a consequence of (a) and Cor. 2.

The next theorem contains obviously Prop. 3 (b).

Theorem 1. *Let $[B : Z] < \infty$, and let A be left algebraic over B . In order that A/B be locally finite, it is necessary and sufficient that $A/B \cdot C$ be left locally finite.*

Proof. It suffices to prove the sufficiency. At first we shall prove

our assertion for the case that B is a regular subring. Since $B \cdot C = B \otimes_Z C \cdot Z$, $Z \cdot C$ is a subfield of C_0 and $B \cdot C$ is a simple ring with $[B \cdot C : Z \cdot C] = [B : Z] < \infty$. Now, let F be an arbitrary finite subset of A . Then, $A' = (B \cdot C)[E, F]$ is a simple ring with $[A' : B \cdot C]_r < \infty$. Recalling here that $[B \cdot C : Z \cdot C] < \infty$, we have then $[A' : V_{A'}(A')] < \infty$ by [10, Lemma]. Hence, A'/B is locally finite by Prop. 3 (b), which means obviously that A/B is locally finite. Next, we shall proceed into the general case. By the first step, we may restrict our attention to the case that $B \not\subset C$. Then, by Props. 1 and 2 (a), there exists some $B' = \sum_{i=1}^n D' e'_{ij} \in \mathcal{R}_{i,j} = \mathcal{R}_{i,j}^0$ such that $V_A(\{e'_{ij}\})$ is left algebraic over D' . One may remark here $[D' : V_{B'}(D')] \leq [B' : V_{B'}(B')] < \infty$ ([10, Lemma]). Since $B' \cdot C = \sum_{i=1}^n (D' \cdot C) e'_{ij}$ is left finite over $B \cdot C$, $A/B' \cdot C$ is left locally finite, or what is the same, $V_A(\{e'_{ij}\})/D' \cdot C$ is left locally finite. Hence, by the first step, $V_A(\{e'_{ij}\})/D'$ is locally finite, or equivalently A/B' is locally finite. Consequently, it follows that A/B is locally finite.

Theorem 2. *Let A be left algebraic over a simple ring B . Then, the following conditions are equivalent :*

- (1) $[B : Z] < \infty$ and A/B is of bounded degree.
- (2) $[A : C] < \infty$ and $B \cdot C/B$ is of bounded degree.
- (3) $[A : C] < \infty$ and $Z \cdot C/Z$ is of bounded degree.

Proof. Evidently, $B \cdot C = B \otimes_Z Z \cdot C$ and (2) \Leftrightarrow (3).

(3) \Rightarrow (1). We set $A = \sum_{i=1}^s Ca_i$, where $a_i = 1$. There exists a positive integer k such that every subring of the form $B[c]$ ($c \in C$) possesses a linearly independent B -basis consisting of at most k elements of C . Accordingly, if c_1, \dots, c_t are in C then $B[c_1, \dots, c_t]$ possesses a linearly independent B -basis consisting of at most k^t elements of C . In case $B \subset C$, the last yields $[B[x] : B] \leq_s k^{s^2+s}$ for every $x \in A$. On the other hand, if $B \not\subset C$ then $A = (B \cdot C)[a]$ with some a ([7, Th. 1]). There exists then a finite subset F of C such that $\{a_1, \dots, a_s\} \subset B[F, a]$. If $x = \sum_{i=1}^s c'_i a_i$ is an element of A and $F' = \{c'_1, \dots, c'_s\}$, it is obvious that $B[x] \subset B[F, F', a]$. Hence, $[B[x] : B]_r \leq [B[F, F', a] : B]_r = [B[a] \cdot B[F, F'] : B] \leq [B[a] : B] \cdot k^{2F+s}$. One may remark here $[B : Z] \leq [A : C] < \infty$ ([10, Lemma] or Prop. 3).

(1) \Rightarrow (2). If $B \subset C$ then A/C is of bounded degree, and then $[A : C] < \infty$ by [4, Lemma 4]. Next, assume that $B \not\subset C$. By Prop. 1, there exists some $B' = \sum_{i=1}^n D' e'_{ij} \in \mathcal{R}_{i,j}^0$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic and of bounded degree. Since $[D' : V_{B'}(D')] \leq [B' : V_{B'}(B')] < \infty$ by [10, Lemma], $V_A(\{e'_{ij}\})$ is left algebraic and of bounded degree over the field $V_{B'}(B') \cdot C$. Hence, again by [4, Lemma 4], $[V_A(\{e'_{ij}\}) : C] < \infty$, namely, $[A : C] < \infty$.

3. Left and right dimensionalities

Proposition 4. *Let $[B : Z] < \infty$ (or $[A : C] < \infty$). If B is a regular subring then $[A : B]_L = [A : B]_R$, provided we do not distinguish between two infinite dimensions.*

Proof. Assume that $[A : B]_L < \infty$. Since $B \cdot V = B \otimes_Z V$ is a simple intermediate ring of A/C and $[A : C] < \infty$ by [10, Lemma], we obtain $B \cdot V = V_A^1(B \cdot V) = V_A(C_0)$ and $[B \cdot V : B] = [V : Z]$. Hence, $[A : B]_L = [A : B \cdot V]_L \cdot [B \cdot V : B] = [C_0 : C] \cdot [B \cdot V : B] = [A : B \cdot V]_R \cdot [B \cdot V : B] = [A : B]_R$, which proves our assertion.

As an application of Th. 2, we shall prove the following that contains [6, Th. 5.2].

Theorem 3. *Let $[B : Z] < \infty$, and let A be left algebraic and of bounded degree over B . If $Z \cdot C$ is a separable field extension of Z then $[A : B]_L = [A : B]_R < \infty$.*

Proof. By Th. 2, there holds $[A : B]_L \leq [A : C] \cdot [Z \cdot C : Z] < \infty$. Since the simple ring $V_A(Z) = V_A(Z \cdot C)$ is $B \otimes_Z V$ by Wedderburn' theorem, V is a simple ring. Now, our assertion is a consequence of Prop. 4.

Throughout the rest of this note, \mathfrak{A} and \mathfrak{G} will denote the absolute endomorphism ring $\text{Hom}(A, A)$ of A and the group of all B -ring automorphisms of A , respectively. We now consider some results which can be proved for Galois extensions A/B .

Proposition 5. *Let $\mathfrak{G}A_R$ and $\mathfrak{G}A_L$ be dense in $V(B_L)$ and $V_{\mathfrak{A}}(B_R)$, respectively. Let B' be an intermediate ring of A/B left (or right) finite over B such that A is B' - A -irreducible and A - B' -irreducible. In order that $[B' : B]_L = [B' : B]_R$, it is necessary and sufficient that $[V : V_A(B')]_R = [V : V_A(B')]_L$.*

Proof. If $\sigma|B' = \tau \cdot \tilde{v}|B'$ for some $v \in V$ ($\sigma, \tau \in \mathfrak{G}$), we write $\sigma|B' \sim \tau|B'$. Evidently, the relation \sim is an equivalence relation in $\mathfrak{G}|B'$, and the number of the equivalence classes w. r. t. \sim denoted as $(\mathfrak{G}|B' : \tilde{V})$ coincides with the number of $B'_R \cdot A_R$ -homogeneous components of $(\mathfrak{G}|B')V_L A_R$ as well as with that of $B'_L \cdot A_L$ -homogeneous components of $(\mathfrak{G}|B')V_R A_L$. Hence, by [8, Lemmas 1.3, 1.4, 1.5], we obtain $[B' : B]_L = (\mathfrak{G}|B' : \tilde{V}) \cdot [V : V_A(B')]_R$ and $[B' : B]_R = (\mathfrak{G}|B' : \tilde{V}) \cdot [V : V_A(B')]_L$. Comparing those above, we readily see our conclusion.

Combining Prop. 5 with [9, Cor. 2 (b)], we obtain the next :

Corollary 3. *Let $J(\mathfrak{G}, A) = B$, and let A be $B \cdot V$ - A -irreducible and*

A-B-V-irreducible. Let B' be an intermediate ring of A/B left (or right) finite over B such that A is B' - A -irreducible and A - B' -irreducible. In order that $[B' : B]_L = [B' : B]_R$, it is necessary and sufficient that $[V : V_A(B')]_R = [V : V_A(B')]_L$.

The next is a partial extension of [9, Th. 3 (b)].

Theorem 4. *Let a division ring A be Galois over B , and $[V : C_0] < \infty$. If B' is an intermediate ring of A/B left finite over B then $[B' : B]_L = [B' : B]_R$.*

Proof. Since $[V : V_A(B')]_R \leq [B' : B]_L < \infty$, we obtain $[V : V_A(B')]_R = [V : V_A(B')]_L$ by Prop. 4. Hence, by Cor. 3, $[B' : B]_L = [B' : B]_R$.

4. The notion of QG-1 extensions

Let \mathfrak{S} be a (multiplicative) sub-semigroup of $\mathfrak{A} = \text{Hom}(A, A)$.

Definition 1. \mathfrak{S} is said to be a left (resp. right) Galois semigroup of A/B if $V_{\mathfrak{A}}(\mathfrak{S}) \cap A_L = B_L$ and $V_{\mathfrak{A}}(\mathfrak{S}) \cap A_R \supset B_R$ (resp. $V_{\mathfrak{A}}(\mathfrak{S}) \cap A_L \supset B_L$ and $V_{\mathfrak{A}}(\mathfrak{S}) \cap A_R = B_R$). If \mathfrak{S} is a left and right Galois semigroup of A/B , we call \mathfrak{S} a Galois semigroup of A/B .

Definition 2. Let \mathfrak{S} be a left (resp. right) Galois semigroup of A/B . If A is $B \cdot V$ - A -irreducible and $\mathfrak{S}A_R$ is a subring of \mathfrak{A} (resp. A is $A \cdot B \cdot V$ -irreducible and $\mathfrak{S}A_L$ is a subring of \mathfrak{A}), then we call A/B a left (resp. right) QG-1 extension with respect to \mathfrak{S} , and \mathfrak{S} a left (resp. right) Galois semigroup belonging to the left (resp. right) QG-1 extension A/B . Moreover, if A/B is a left and right QG-1 extension with respect to a Galois semigroup \mathfrak{S} then we call A/B a QG-1 extension with respect to \mathfrak{S} .

Finally we shall present a proposition which is useful in our consideration.

Proposition 6. *Assume that A/B is a left QG-1 extension w. r. t. a left Galois semigroup \mathfrak{S} containing \bar{V} . Let M be a B - B -submodule of A possessing a linearly independent finite left B -basis.*

(a) $(\mathfrak{S} | M)V_R$ possesses a linearly independent V_R -basis that forms at the same time a linearly independent A_R -basis of $(\mathfrak{S} | M)A_R$, so that there holds $[M : B]_L = [(\mathfrak{S} | M)A_R : A_R]_R = [(\mathfrak{S} | M)V_R : V_R]_R$.

(b) Let B be a division ring, and a a non-zero element of M . The right V -module $a\mathfrak{S}V_R$ possesses a linearly independent right V -basis. In particular, if $[M : B]_L = [a\mathfrak{S}V_R : V]_R$ then $M = BaB$.

Proof. (a) is essentially [9, Lemma 6], and the argument used in

the proof of [6, Lemma 2.2] enables us to obtain the remaining.

5. Left QG-1 extensions of division rings

In the present section, we shall state without proof the extensions of [4, Th. 1] and [4, Th. 2] to left QG-1 extensions of division rings. In fact, by the validity of Prop. 6 and [6, Lemma 6.6], the respective proofs of them proceed in the same way as those of [4, Th. 1] (and [4, Cor. 2]) and [4, Th. 2] did.

Theorem 5. *Let a division ring A be a left QG-1 extension of B , and $[B:Z] = \infty$. If M is a B - B -submodule of A left finite over B then $M = BaB$ for some $a \in M$.*

Theorem 6. *Let a division ring A be a left QG-1 extension of B , and $[B:Z] = \infty$. If A/B is left algebraic then it is left locally finite.*

Remark. In [5, Th. 1], we have seen that if A/B is Galois and left algebraic and of bounded degree then $[A:B] < \infty$. However, the following example shows that [5, Th. 1] can not be extended to QG-1 extensions. Let $\phi = \text{GF}(p)$ (p a prime), and x, y, z, \dots an infinite number of indeterminates. If $B = \phi(x, y, z, \dots)$ and $A = \phi(x^{1/p}, y^{1/p}, z^{1/p}, \dots)$ then A/B is not finite dimensional but algebraic and of bounded degree.

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