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On the Differential Submodules of Modules

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ON THE DIFFERENTIAL SUBMODULES OF MODULES

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§1. Introduction.

In [17], A. Seidenberg proved that if R is a Noetherian Ritt algebra, then any differential ideal of R has a primary decomposition of a differential version. This is extended by W. C. Brown and W. E. Kuan [5], and S. Sato [12], under the assumption that the ring R is Noetherian. In [6], we extend this result for differential ideals of rings which may be non-Noetheian. Furthermore we showed some detailed results for differential ideals. In this paper, we extend some results of higher derivations of rings introduced in [6] to modules, using similar methods to those of [6].

In §2, we show some of the basic facts of a module over a commutative ring. In §3, we consider the problem of determining conditions under which the weak prime divisors of a differential submodule are also differential. In §4, we study the class of modules in which primary decomposition of a differential version holds. In paticular we show that if M is a strongly Laskerian module over a commutative ring, then any differential submodule of M has a primary decomposition of a differential version.

§2. Preliminaries.

In this section we collect some definitions and results for later use. All rings in this paper are assumed to be commutative with a unit element and all modules are assumed to be unitary. Furthermore we always denote a ring by R and an R-module by M.

Let t be an indeterminate over R and $R[[t]]$ the formal power series ring over R. Put $R_m = R[[t]]/(t^{m+1})$ and $M_m = M \otimes_R R_m$ $(m = 1, 2, ...)$. Then M_m is an R_m-module. Furthermore we put $M[[t]] = \varprojlim M_m$. Then $M[[t]]$ is an $R[[t]]$-module. Particularly put $R_\infty = R[[t]]$ and $M_\infty = M[[t]]$ (cf. [11], p.28).

A prime ideal P of R is called a *weak associated pime of M* if there exists $x \in M$ such that P is a minimal element of the set of prime ideals containing $\mathrm{ann}_R(x)$ (the annihilator of x); We denote by $\mathrm{Ass}_R^f(M)$ the set of weak associated primes of M (cf. [3, IV, §1, Exercise17]). For a submodule N of M, the weak associated primes of the R-module M/N are referred to as the *weak prime divisors of* N.

We say that $a \in R$ is a *zero-divisor* of M if there exists a non-zero $x \in M$ such that $ax = 0$. The set of zero-divisors of M is written $Z_R(M)$.

For a submodule Q of M, if $\mathrm{Ass}_R^f(M/Q)$ consists of one element, then we say that Q is *primary* in M. Furthermore if $\mathrm{Ass}_R^f(M/Q) = \{P\}$, then we say that Q is *P-primary* in M (cf. [3, IV, §2, Exercise 12]).

Let M be a finitely generated R-module. We say that a P-primary submodule Q of M is *strongly primary* in M if $\mathrm{ann}_R(M/Q)$ contains a power of P (cf. [3, IV, §2, Exercise 27]).

We say that M is a *(strongly) Laskerian* R-module if M is finitely generated as an R-module and every submodule of M can be written as an intersection of a finite number of (strongly) primary submodules. We say a ring is *(strongly) Laskerian* if it has the property as a module over itself . It is well known that if a module is Laskerian, or strongly Laskerian, then so is any factor module, and any quotient module with respect to a multiplicative subset in the ring. Particularly, a ring with a faithful module of one of these types is also a ring of that type (cf. [3, IV, §2, Exercise 23, 28], [8], [9]).

Let S be a multiplicative subset of R, that is, S is a subset of R which contains the product ab for all $a, b \in S$, and which contains 1 but not 0. Let $f : M \longrightarrow S^{-1}M$ be the natural mapping defined by $f(x) = x/1$ for $x \in M$. For a submodule N of M, the inverse image $f^{-1}(S^{-1}N)$ of $S^{-1}N$ under f is called the *saturation* of N in M with respect to S, and denoted by $\mathrm{sat}_S(N)$. For a prime ideal P of R, $\mathrm{sat}_P(N)$ denote the saturation of N in M with respect to $R - P$.

The following proposition is needed to prove Theorem (3.1).

Proposition (2.1). *Let R be a ring, t an indeterminate over R and M a strongly Laskerian R-module. If Q is a primary submodule of M with $Ass_R^f(M/Q) = \{P\}$, then $Q[[t]]$ is a primary submodule of $M[[t]]$ with $Ass_{R[[t]]}^f(M[[t]]/Q[[t]]) = \{P[[t]]\}$.*

Proof. Replacing $M[[t]]$ by $M[[t]]/Q[[t]](= M/Q[[t]])$, we may assume $Q[[t]] = (0)$. Thus $Q = (0)$ is a primary submodule of M with $\mathrm{Ass}_R^f(M) =$

$\{P\}$, whence $\mathrm{ann}_R(M)$ is a P-primary ideal of R. Furthermore we may suppose that $\mathrm{ann}_{R[[t]]}(M[[t]]) = (0)$. Then we have $\mathrm{ann}_R(M) = (0)$ and so (0) is a P-primary ideal of R. Suppose that $a \in R[[t]] - P[[t]]$. Then we shall show that a is not a zero-divisor of $M[[t]]$. Write $a = a_0 + a_1t + \cdots$, where $a_0, a_1, ..., a_{m-1} \in P$ and $a_m \notin P$. Since $P = \sqrt{(0)}$, we have that $a_0, a_1, ..., a_{m-1}$ are nilpotent and so $b := a_0 + a_1t + \cdots + a_{m-1}t^{m-1}$ is nilpotent. Therefore it is enough to show that $a - b$ is not a zero-divisor of $M[[t]]$. Since $a-b = a_mt^m + \cdots$ and $a_m \notin P = Z_R(M)$, we have that $a-b$ is not a zero-divisor of $M[[t]]$. Now suppose that $ax = 0$ and $x \neq 0$ ($a \in R[[t]]$ and $x \in M[[t]]$). Then we shall show that $a \in \sqrt{\mathrm{ann}_{R[[t]]}(M[[t]])}$. Since $\mathrm{ann}_{R[[t]]}(M[[t]])$ is $P[[t]]$-primary, $\sqrt{\mathrm{ann}_{R[[t]]}(M[[t]])} = P[[t]]$. If $a \notin P[[t]]$, then a is not a zero-divisor of $M[[t]]$, which is a contradiction. Thus (0) is a primary submodule of $M[[t]]$ with $\mathrm{Ass}_{R[[t]]}(M[[t]]) = \{P[[t]]\}$.

A *derivation* of R is an additive endomorphism $d : R \longrightarrow R$ such that $d(ab) = d(a)b + ad(b)$ for every $a, b \in R$. The set of all derivations of R is denoted by $Der(R)$.

For $m \leq \infty$ we define a *higher derivation* of length m of R to be a sequence $d = (d_0, d_1, ..., d_m)$ of additive endomorphisms $d_n : R \longrightarrow R$, satisfying the conditions $d_0 = 1$ (the identity mapping of R) and $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ for $1 \leq n \leq m$ and $a, b \in R$. The set of all higher derivations of length m of R is denoted by $HDer^m(R)$. Note that the set $HDer^m(R)$ has a group structure (cf. [10]).

A *derivation* of M is an ordered pair (d, D), satisfying the following two conditions:

(1) $d \in Der(R)$ and

(2) $D : M \longrightarrow M$ is an additive endomorphism such that $D(ax) = d(a)x + aD(x)$ for $a \in R$ and $x \in M$.

We denote the set of all derivations of an R-module M by $Der(R, M)$. It becomes an R-module in a natural way.

For $m < \infty$, a *higher derivation* of length m of M is an ordered pair (d, D), satisfying the following two conditions:

(1) $d = (d_0, d_1, ..., d_m) \in HDer^m(R)$

(2) $D = (D_0, D_1, ..., D_m)$ is a sequence of additive endomorphisms $D_n : M \longrightarrow M$ such that $D_0 = 1$ (the identity mapping of M) and

$D_n(ax) = \sum_{i+j=n} d_i(a)D_j(x)$ for $a \in R, x \in M$ and $1 \leq n \leq m$.

We denote the set of all higher derivations of length m of an R-module M by $HDer^m(R, M)$.

We say that an ordered pair (d, D) is a *higher derivation* of length ∞ of M if $d = (d_0, d_1, ...)$ and $D = (D_0, D_1, ...)$ are infinite sequences such that $((d_0, d_1, ..., d_m), (D_0, D_1, ..., D_m)) \in HDer^m(R, M)$ for every $0 \leq m < \infty$. The set of all higher derivations of length ∞ of an R-module M is denoted by $HDer^\infty(R, M)$.

For any $(d, D) \in HDer^m(R, M)(m \leq \infty)$, put

$$f_d(a) = \sum_{k=0}^{m}(\sum_{i+j=k} d_i(a_j))t^k \quad \text{for} \quad a = \sum_{i=0}^{m} a_i t^i \in R_m \ (a_i \in R).$$

Then f_d is an automorphism of the ring R_m such that $f_d(a) \equiv a \bmod t$ $(a \in R)$ and $f_d(t) = t$.

Furthermore put

$$g_D(x) = \sum_{k=0}^{m}(\sum_{i+j=k} D_i(x_j))t^k \quad \text{for} \quad x = \sum_{i=0}^{m} x_i t^i \in M_m \ (x_i \in M).$$

Then g_D satisfies the following four conditions.

(1) $g_D : M_m \longrightarrow M_m$ is a bijection.
(2) $g_D(x + y) = g_D(x) + g_D(y)$ $(x, y \in M_m)$.
(3) $g_D(x) \equiv x \bmod t$ $(x \in M)$.
(4) $g_D(ax) = f_d(a)g_D(x)$ $(a \in R_m, x \in M_m)$.

We note that for (d, D), (f_d, g_D) is uniquely determined. Conversely, let $g : M_m \longrightarrow M_m$ be a mapping satisfying the above conditions (1), (2), (3) and (4). Then, for an ordered pair (f_d, g), there exists a unique ordered pair $(d, D) \in HDer^m(R, M)$ such that $g = g_D$. Furthermore note that the sets $HDer^m(R, M)(m \leq \infty)$ have a group structure like that of $HDer^m(R)$ and a higher derivation of a module has a unique extension to the localizations (cf. [13],[14],[15]).

§3. Weak prime divisors of differential submodules.

Let R be a ring and M an R-module. Let $(d, D) \in Der(R, M)$. An ideal I of R is called *d-differential* if $d(I) \subset I$, and a submodule N of M is called (d,D)-*differential* if $D(N) \subset N$. Similarly, let $(d,D) \in HDer^m(R,M)$ $(m \leq \infty)$. An ideal I of R is called *d-differential* if $d_i(I) \subset I$ for all $i \geq 0$, and a submodule N of M is called (d,D)-*differential* if $D_i(N) \subset N$ for all $i \geq 0$.

In this section we consider the problem of detemining conditions under which the weak prime divisors of a differential submodules are also differential.

Theorem (3.1). *Let R be a ring and M a strongly Laskerian R-module. Suppose N is a submodule of M with $Ass_R^f(M/N) = \{P_1, ..., P_n\}$ and $(d,D) \in HDer^\infty(R,M)$. If N is (d,D)-differential, then $P_1, ..., P_n$ are d-differential.*

Proof. Let $N = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition. Put $\mathrm{Ass}_R^f(M/Q_i) = \{P_i\}$ $(i = 1, \ldots, n)$. Then $\mathrm{Ass}_R^f(M/N) = \{P_1, ..., P_n\}$. Let t be an indeterminate over R. Then we have that $N[[t]] = Q_1[[t]] \cap \cdots \cap Q_n[[t]]$, and each $Q_i[[t]]$ is a $P_i[[t]]$-primary submodule of $M[[t]]$ by Proposition (2.1). Therefore we have that

$$\mathrm{Ass}_{R[[t]]}^f(M[[t]]/N[[t]]) = \{P_1[[t]], ..., P_n[[t]]\}.$$

In the group $HDer^\infty(R,M)$, we have that $HDer^\infty(R,M) \ni (d,D)^{-1} = (d^{-1}, D^{-1})$, where $d^{-1} = (1, -d_1, -d_2 + d_1^2, \cdots)$ and $D^{-1} = (1, -D_1, -D_2 + D_1^2, \cdots)$. Since N is (d,D)-differential, N is $(d,D)^{-1}$-differential. Thus we have that $g_D(N[[t]]) \subset N[[t]]$ and $g_{D^{-1}}(N[[t]]) = g_D^{-1}(N[[t]]) \subset N[[t]]$, where $g_D : M[[t]] \to M[[t]]$ is the mapping corresponding to D. Hence we get $g_D(N[[t]]) = N[[t]]$. It is clear that $g_D(N[[t]]) = g_D(Q_1[[t]]) \cap \cdots \cap g_D(Q_n[[t]])$, and each $g_D(Q_i[[t]])$ is a $f_d(P_i[[t]])$-primary submodule of $M[[t]]$, where $f_d : R[[t]] \to R[[t]]$ is the mapping corresponding to d. It follows that

$$\mathrm{Ass}_{R[[t]]}^f(M[[t]]/N[[t]]) = \{f_d(P_1[[t]]), ..., f_d(P_n[[t]])\}.$$

Therefore, for any i, $f_d(P_i[[t]]) = P_j[[t]]$ for some j. Hence we can easily check that $i = j$, and so $f_d(P_i[[t]]) = P_i[[t]]$. Consequently P_i is d-differential.

Next we examine the problem on the Laskerian case. We show the following lemma by making use of the Krull intersection theorem for Laskerian modules (cf.[8, Corollary 3.2]).

Lemma (3.2). *Let R be a ring containing the rational numbers and M a Laskerian R-module. Suppose $(d, D) \in Der(R, M)$ and a is an element of the Jacobson radical of R. If $d(a)$ is a unit in R, then $a \notin Z_R(M)$.*

Proof. If $ax = 0$ $(x \in M)$, then $x \in \bigcap_{n=1}^{\infty} a^n M$. By Corollary 3.2 of [8], we have $\bigcap_{n=1}^{\infty} a^n M = (0)$, and thus $x = 0$.

Proposition (3.3). *Let R be a ring containing the rational numbers and M a Laskerian R-module. Suppose N is a submodule of M with $Ass_R^f(M/N) = \{P_1, ..., P_n\}$ and $(d, D) \in Der(R, M)$. If N is (d, D)-differential, then $P_1, ..., P_n$ are d-differential.*

Proof. If $P(\in \mathrm{Ass}_R^f(M/N))$ is not d-differential, then there exists $a \in P$ such that $d(a) \notin P$. Now we consider the R-module M/N. Let $\bar{D} : M/N \to M/N$ be the mapping defined by $\bar{D}(x + N) = D(x) + N$ $(x \in M)$. Then we have $(d, \bar{D}) \in Der(R, M/N)$. We further consider the $S^{-1}R$-module $S^{-1}(M/N)$, where $S = R - P$. Let $(d^*, \bar{D}^*) \in Der(S^{-1}R, S^{-1}(M/N))$ be a unique extension of $(d, \bar{D})$. Put $b = a/1$ $(\in S^{-1}P)$. Then $d^*(b)$ is a unit in $S^{-1}R$, since $d^*(b) \notin S^{-1}P$. Therefore $b \notin Z_{S^{-1}R}(S^{-1}(M/N))$ by Lemma (3.2). On the other hand, we may assume that $P_i \cap S = \phi$ $(i = 1, \ldots, t)$ and $P_i \cap S \neq \phi$ $(i = t+1, \ldots, n)$. Then, we have $\mathrm{Ass}_{S^{-1}R}^f(S^{-1}(M/N)) = \{S^{-1}P_1, ..., S^{-1}P_t\}$. It follows that $Z_{S^{-1}R}(S^{-1}M/S^{-1}N) = S^{-1}P_1 \cup \cdots \cup S^{-1}P_t$. Since $P = P_i$ for some i $(1 \leq i \leq t)$ and $a \in P$, we have $b \in S^{-1}P \subset Z_{S^{-1}R}(S^{-1}(M/N))$. Thus we get a contradiction.

Proposition (3.4). *Let R be a ring containing the rational numbers and M a Laskerian R-module. Suppose N is a submodule of M with $Ass_R^f(M/N) = \{P_1, ..., P_n\}$ and $(d, D) \in HDer^m(R, M)$ $(m \leq \infty)$. If N is (d, D)-differential, then $P_1, ..., P_n$ are d-differential.*

Proof. For $(d_0, d_1, \ldots)$ and $(D_0, D_1, \ldots)$, put $\delta_1 = d_1$, $\delta_2 = d_2 - \frac{1}{2}d_1^2, \ldots$ and $\Delta_1 = D_1$, $\Delta_2 = D_2 - \frac{1}{2}D_1^2, \ldots$. Then $(\delta_1, \Delta_1), (\delta_2, \Delta_2), \ldots$ are derivations of M (cf. [7], [1], [2], [13]). Since N is (d, D)-differential, N is (δ_r, Δ_r)-differential ($r = 1, \dots, m$). Therefore $P_1, ..., P_n$ are δ_r-differential for all r by Proposition (3.3), and thus each P_i is d-differential.

Proposition (3.5). *Let R be a ring of characteristic 0 and let M be a Laskerian R-module. Suppose that N is a submodule of M and $(d, D) \in HDer^m(R, M)$ ($m \leq \infty$). Put $\{P_1, ..., P_t\} = \{P \in \mathrm{Ass}^f_R(M/N) | P \cap \mathbf{Z} = (0)\}$, where $\mathbf{Z}$ ($\subset R$) is the rational integers. If N is (d, D)-differential, then $P_1, ..., P_t$ are d-differential.*

Proof. Put $S = \mathbf{Z} - \{0\}$. Then S is a multiplicative subset of R. Furthermore $S^{-1}R$ contains the rational numbers and $S^{-1}M$ is a Laskerian $S^{-1}R$-module. Let $(d^*, D^*) \in HDer^m(S^{-1}R, S^{-1}M)$ be a unique extension of (d, D). Since $S^{-1}N$ is (d^*, D^*)-differential and $\mathrm{Ass}^f_{S^{-1}R}(S^{-1}M/S^{-1}N) = \{S^{-1}P_1, ..., S^{-1}P_t\}$, it follows from Proposition (3.4) that $S^{-1}P_1, ..., S^{-1}P_t$ are d^*-differential. Therefore $P_1, ..., P_t$ are also d-differential. In fact, for any $S^{-1}P \in \mathrm{Ass}^f_{S^{-1}R}(S^{-1}M/S^{-1}N)$ and for any $a \in P$, we have $d^*_n(a/1) \in S^{-1}P$ for all n. Thus we have $d_n(a) \in P$, because $d^*_n(a/1) = d_n(a)/1$.

§4. Primary decomposition of differential submodules.

In this section we study the class of modules in which primary decomposition of a differential version holds.

Proposition (4.1). *Let R be a ring and M a strongly Laskerian R-module. Suppose N is a submodule of M and $(d, D) \in HDer^\infty(R, M)$. If N is (d, D)-differential, then N can be expressed as an irredundant intersection of a finite number of primary submodules of M which are (d, D)-differential.*

Proof. Let $N = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition, where Q_i ($i = 1, \cdots, n$) is strongly primary submodules of M. Put $\mathrm{Ass}^f_R(M/Q_i) = \{P_i\}$ ($i = 1, ..., n$). Then there exists an integer $k \geq 1$ such that $P_i^k M \subset Q_i$ for all i. Put $N_i^* = P_i^k M + N$. Then we have $N \subset N_i^* \subset Q_i$. By Theorem (3.1), each P_i is d-differential, and hence N_i^* is also (d, D)-differential. Furthermore we have that $P_i^k \subset \mathrm{ann}_R(M/N_i^*) \subset \mathrm{ann}_R(M/Q_i) \subset P_i$ for all i. It follows that P_i is minimal among the prime ideals containing $\mathrm{ann}_R(M/N_i^*)$. Put $Q_i^* = \mathrm{sat}_{P_i}(N_i^*)$. Then Q_i^* is a primary submodule of M with $\mathrm{Ass}^f_R(M/Q^*_i) = \{P_i\}$ and Q^*_i is a (d, D)-differential. Therefore we have that $N \subset Q^*_i \subset Q_i$. Thus we get $N = Q^*_1 \cap \cdots \cap Q^*_n$.

In case of characteristic $q \neq 0$, we have the following theorem.

Theorem (4.2). *Let R be a ring of characteristic $q \neq 0$ and M a finitely generated R-module. Suppose N is a decomposable submodule of M and $(d, D) \in HDer^m(R, M)$ $(m < \infty)$. If N is (d, D)-differential, then N can be expressed as an irredundant intersection of a finite number of primary submodules of M which are (d, D)-differential.*

Proof. Let $N = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of N. Put $\mathrm{Ass}^f_R(M/Q_i) = \{P_i\}$ and $I_i = \mathrm{ann}_R(M/Q_i)$ $(i = 1, ..., n)$. Then each I_i is a P_i-primary ideal of R. Let $I_i^{(t)}$ be the ideal of R generated by the set $\{a^t | a \in I_i\}$, where $t = (m!)q$. Since $d_n(a^t) = 0$ for all $a \in R$ and $1 \leq n \leq m$ by [6, Lemma 2], $I_i^{(t)}$ is d-differential. Hence the submodule $I_i^{(t)}M$ of M is (d, D)-differential. Furthermore we have $I_i^{(t)}M \subset I_iM \subset Q_i$. Put $N_i = I_i^{(t)}M + N$ $(i = 1, ..., n)$. Then for each i, N_i is (d, D)-differential, $N \subset N_i \subset Q_i$ and $I_i^{(t)} \subset \mathrm{ann}_R(M/N_i) \subset I_i$. Therefore P_i is minimal among the prime ideals containing $\mathrm{ann}_R(M/N_i)$. It follows that $\mathrm{sat}_{P_i}(N_i)$ is a P_i-primary submodule of M. Put $Q'_i = \mathrm{sat}_{P_i}(N_i)$. Then we have that Q'_i is (d, D)-differential and $N \subset Q'_i \subset Q_i$. Thus we get $N = Q'_1 \cap \cdots \cap Q'_n$.

Next we consider the case of characteristic zero.

Proposition (4.3). *Let R be a ring containing the rational numbers and M a strongly Laskerian R-module. Suppose N is a submodule of M and $(d, D) \in HDer^m(R, M)$ $(m < \infty)$. If N is (d, D)-differential, then N can be expressed as an irredundant intersection of a finite number of primary submodules of M which are (d, D)-differential.*

Proof. Any weak associated prime of N is (d, D)-differential by Proposition (3.4). Thus we can obtain the proof in almost the same way as Proposition (4.1). Therefore we omit the proof.

Proposition (4.4). *Let R be a ring of characteristic* 0 *and M a finitely generated R-module. Suppose N is a decomposable submodule of*

M and $(d, D) \in HDer^m(R, M)$ $(m < \infty)$. If N is (d, D)-differential and $\mathrm{ann}_R(M/N) \cap \mathbf{Z} \neq (0)$, where $\mathbf{Z}(\subset R)$ is the rational integers, then N can be expressed as an irredundant intersection of a finite number of primary submodules of M which are (d, D)-differential.

Proof. Put $I = \mathrm{ann}_R(M/N)$, $\bar{R} = R/I$ and $\bar{M} = M/N$. Suppose $I \cap \mathbf{Z} = (q)$ $(q \neq 0)$. Then the ring $\bar{R}$ is of characteristic q and $\bar{M}$ is an $\bar{R}$-module. Since the ideal I is d-differential and N is (d, D)-differential, (d, D) induces a higher derivation $(\bar{d}, \bar{D}) \in HDer^m(\bar{R}, \bar{M})$ in the natural way, that is, $\bar{d}_n(a + I) = d_n(a) + I$ $(a \in R)$ and $\bar{D}_n(x + N) = D_n(x) + N$ $(a \in N)$. Since (0) is a $(\bar{d}, \bar{D})$-differential decomposable submodule of $\bar{M}$, there are $(\bar{d}, \bar{D})$-differential primary submodules $Q'_1, ..., Q'_n$ of $\bar{M}$ such that $(0) = Q'_1 \cap \cdots \cap Q'_n$ by Theorem (4.2). Let $f : M \to \bar{M}$ be the natural mapping, defined by $f(x) = x + N$. Put $Q_i = f^{-1}(Q'_i)$. Then we have that each Q_i is a (d, D)-differential primary submodule of M and $N = Q_1 \cap \cdots \cap Q_n$.

The following theorem is a main result in the case of characteristic zero.

Theorem (4.5). *Let R be a ring of characteristic 0 and M a strongly Laskerian R-module. Suppose that N is a submodule of M and $(d, D) \in HDer^m(R, M)$ $(m < \infty)$. If N is (d, D)-differential, then N can be expressed as an irredundant intersection of a finite number of primary submodules of M which are (d, D)-differential.*

Proof. Put $I = \mathrm{ann}_R(M/N)$. We may assume that $I \cap \mathbf{Z} = (0)$ by Proposition (4.4), where $\mathbf{Z}(\subset R)$ is the rational integers. Let $N = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition such that $P_i \cap \mathbf{Z} = (0)$ $(i = 1, ..., t)$ and $P_i \cap \mathbf{Z} \neq (0)$ $(i = t + 1, ..., n)$, where $\{P_i\} = \mathrm{Ass}^f_R(M/Q_i)$ $(i = 1, ..., n)$. Put $N_1 = Q_1 \cap \cdots \cap Q_t$ and $N_2 = Q_{t+1} \cap \cdots \cap Q_n$. Then we have $\mathrm{ann}_R(M/N_2) \cap \mathbf{Z} \neq (0)$.

First we consider the $S^{-1}R$-module $S^{-1}M$, where $S = \mathbf{Z} - \{0\}$. The ring $S^{-1}R$ contains the rational numbers and $S^{-1}N$ is a strongly Laskerian $S^{-1}R$-module. Let $(d^*, D^*) \in HDer^m(S^{-1}R, S^{-1}M)$ be a unique extension of (d, D). Then $S^{-1}N(= S^{-1}N_1)$ is a (d^*, D^*)-differential submodule of $S^{-1}M$. Hence $S^{-1}N$ can be written as an intersection $Q'_1 \cap \cdots \cap Q'_r$ of primary submodules Q'_i of $S^{-1}M$ which are (d^*, D^*)-differential by Proposition (4.3). Put $Q^*_i = f^{-1}(Q'_i)$ $(i = 1, ..., r)$, where $f : M \to S^{-1}M$

is the natural mapping, defined by $f(x) = x/1$. Then we have that $N_1 = Q_1^* \cap \cdots \cap Q_r^*$ and $Q_1^*, \ldots, Q_r^*$ are (d, D)-differential primary submodules of M.

Next we consider the $\bar{R}$-module $\bar{M}$, where $\bar{R} = R/I$ and $\bar{M} = M/N$. Then $\bar{R}$ is a ring of characteristic 0 and $\bar{M}$ is a strongly Laskerian $\bar{R}$-module. Put $N_1/N = \bar{N}_1$ and $N_2/N = \bar{N}_2$. Since $N = N_1 \cap N_2$, we have $\bar{N}_1 \cap \bar{N}_2 = (0)$. Furthermore we have $\mathrm{ann}_{\bar{R}}(\bar{M}/\bar{N}_2) = \mathrm{ann}_{\bar{R}}(M/N_2) = \overline{\mathrm{Ann}_R(M/N_2)}$, and hence $\mathrm{ann}_{\bar{R}}(\bar{M}/\bar{N}_2) \cap \mathbf{Z} = (q)$ for some $q \neq 0$. Put $\bar{J} = q\bar{M}$. Then we have that $\bar{N}_2 \supset \bar{J}$ and $\mathrm{ann}_{\bar{R}}(\bar{M}/\bar{J}) \cap \mathbf{Z} \neq (q)$, and so $\bar{N}_1 \cap \bar{J} = (0)$. Since I is d-differential and N is (d, D)-differential, (d, D) induces the higher derivation $(\bar{d}, \bar{D}) \in HDer^m(\bar{R}, \bar{M})$ in the natural way. The submodule $\bar{J}$ is clearly $(\bar{d}, \bar{D})$-differential. It follows from Proposition (4.4) that $\bar{J}$ can be written as an intersection $Q_1'' \cap \cdots \cap Q_s''$ of primary submodules Q_i'' of $\bar{M}$ which are $(\bar{d}, \bar{D})$-differential. Put $Q_i^{**} = g^{-1}(Q_i'')$ $(i = 1, \ldots, s)$ and $J = g^{-1}(\bar{J})$, where $g : M \to \bar{M}$ is the natural mapping, defined by $g(x) = x + N$. Then $Q_1^{**}, \ldots, Q_s^{**}$ are primary and (d, D)-differential. Therefore we have $N = g^{-1}(0) = g^{-1}(\bar{N}_1 \cap \bar{J}) = g^{-1}(\bar{N}_1) \cap g^{-1}(\bar{J}) = N_1 \cap Q_1^{**} \cap \cdots \cap Q_s^{**} = Q_1^* \cap \cdots \cap Q_r^* \cap Q_1^{**} \cap \cdots \cap Q_s^{**}$. This completes the proof.

As the case of arbitrary characteristic, we have the following theorem, by (4.1), (4.2) and (4.5).

Theorem (4.6). *Let R be a ring and M a strongly Laskerian R-module. Suppose N is a submodule of M and $(d, D) \in HDer^m(R, M)$ $(m \leq \infty)$. If N is (d, D)-differential, then N can be expressed as an irredundant intersection of a finite number of primary submodules of M which are (d, D)-differential.*

The following result is a generalization of Theorem 6 of [12] to the strongly Laskerian case.

Corollary (4.7). *Let R be a ring and M a strongly Laskerian R-module. Suppose N is a submodule of M and $(d, D) \in Der(R, M)$. If N is (d, D)-differential, then N can be expressed as an irredundant intersection of a finite number of primary submodules of M which are (d, D)-differential.*

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