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(σ, τ)-DERIVATIONS IN PRIME RINGS*

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Throughout, R will represent a prime ring with center C , and I a non-zero right ideal of R . Let σ, τ be ring automorphisms of R . We set $C_{\sigma, \tau} = \{c \in R \mid c\sigma(x) = \tau(x)c \text{ for all } x \in R\}$, $[x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x$ and $(x, y)_{\sigma, \tau} = x\sigma(y) + \tau(y)x$; in particular, $C_\tau = C_{1, \tau}$, $[x, y]_\tau = [x, y]_{1, \tau}$ and $(x, y)_\tau = (x, y)_{1, \tau}$. Needless to say, $C_1 = C$, $[x, y]_1 = [x, y] = xy - yx$ and $(x, y)_1 = (x, y) = xy + yx$. Let $d: x \rightarrow x'$ be a (σ, τ) -derivation of R , namely an additive endomorphism of R such that $(xy)' = x'\sigma(y) + \tau(x)y'$ for all $x, y \in R$. Given a subset U of R , we set $[U]_{\sigma, \tau} = \{u \in U \mid [u', u]_{\sigma, \tau} \in C_{\sigma, \tau}\}$ and $(U)_{\sigma, \tau} = \{u \in U \mid (u', u)_{\sigma, \tau} \in C_{\sigma, \tau}\}$; in case d is a $(1, \tau)$ -derivation, we write $[U]_\tau = [U]_{1, \tau}$ and $(U)_\tau = (U)_{1, \tau}$.

We consider the following conditions:

- a) R is commutative and $\sigma = \tau$.
- a)* R is a commutative ring of characteristic 2 and $\sigma = \tau$.
- b) $[a', a]_{\sigma, \tau} = 0$ for all $a \in I$.
- c) $(a', a)_{\sigma, \tau} = 0$ for all $a \in I$.
- d) $I = [I]_{\sigma, \tau}$, that is, $[a', a]_{\sigma, \tau} \in C_{\sigma, \tau}$ for all $a \in I$.
- e) $I = (I)_{\sigma, \tau}$, that is, $(a', a)_{\sigma, \tau} \in C_{\sigma, \tau}$ for all $a \in I$.
- f) $I = [I]_{\sigma, \tau} \cup (I)_{\sigma, \tau}$.

If d is a (σ, σ) -derivation of R , then $\sigma^{-1}d$ is a usual derivation, and so the next is immediate by [3, Theorem 2]: *If d is a non-zero (σ, σ) -derivation, then a) and f) are equivalent.*

In the present paper, we shall generalize [2, Proposition 2 and Theorem 1] by proving the following theorems.

Theorem 1. *Let R be a prime ring. Let $d: x \rightarrow x'$ be a non-zero (σ, τ) -derivation of R , and I a non-zero right ideal of R . Then a) and b) are equivalent, and a)* and c) are equivalent.*

Theorem 2. *Let R be a prime ring of characteristic not 2. Let $d: x \rightarrow x'$ be a non-zero (σ, τ) -derivation of R , and I a non-zero right ideal of R . Then*

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a), b), d) and e) are equivalent.

Theorem 3. *Let R be a prime ring of characteristic not 2. Let $d: x \rightarrow x'$ be a non-zero (σ, τ) -derivation of R , and I a non-zero ideal of R . Then a), b), d), e) and f) are equivalent.*

Proof of Theorem 1. Obviously, $\sigma^{-1}d$ is a $(1, \sigma^{-1}\tau)$ -derivation of R . So, without loss of generality, we may assume that d is a $(1, \tau)$ -derivation. As is easily seen, $I' \neq 0$ and a) and a)* imply b) and c), respectively. Assume b) (resp. c)). Linearlizing the identity b) (resp. c)) on I , we obtain $[a', b]_{\tau} + [b', a]_{\tau} = 0$ (resp. $(a', b)_{\tau} + (b', a)_{\tau} = 0$) for all $a, b \in I$. Replacing b by ab in the above equation, we get $0 = [a', ab]_{\tau} + [(ab)', a]_{\tau} = \tau(a)[a', b]_{\tau} + [a', a]_{\tau}b + [a'b + \tau(a)b', a]_{\tau} = \tau(a)([a', b]_{\tau} + [b', a]_{\tau}) + a'ba - \tau(a)a'b = a'[b, a] + [a', a]_{\tau}b = a'[b, a]$, that is, $a'[b, a] = 0$ (resp. $a'(b, a) = 0$) for all $a, b \in I$. Replace b by bx ($x \in R$) in the above equation to obtain $0 = a'b[x, a]$, i.e., $a'I[x, a] = 0$. This proves that either $I'I = 0$ or $I \subseteq C$. Assume now that $I \not\subseteq C$ (and so $I'I = 0$). Since b) (resp. c)) implies that $\tau(a)a' = 0$ for all $a \in I$, we get $0 = \tau(ab)(ab)' = \tau(aba)b'$ for all $a, b \in I$. Hence $axbax\tau^{-1}(b') = 0$ for all $x \in R$. Thus, either $bI = 0$ or $b' = 0$ for every $b \in I$, by [4, Lemma 2]; therefore $I^2 = 0$. This contradiction shows that $I \subseteq C$ and I is an ideal of R . Now, for the case b), we can apply the argument employed in the last part of the proof of [2, Proposition 2] to see that a) holds. On the other hand, for the case c), we can easily see that $\tau(a) = -a$ for all $a \in I$, Then $-ab = \tau(ab) = \tau(a)\tau(b) = ab$ for all $a, b \in I$, and so $2I^2 = 0$. Hence R is of characteristic 2 and $\tau = 1$.

In preparation for proving Theorems 2 and 3, we need some lemmas.

Lemma 1. *Let d be a non-zero $(1, \tau)$ -derivation of R .*

(1) *Let $a, b \in [I]_{\tau}$ (resp. $(I)_{\tau}$). Then $a+b \in [I]_{\tau}$ (resp. $(I)_{\tau}$) if and only if $a-b \in [I]_{\tau}$ (resp. $(I)_{\tau}$).*

(2) *If $b \in (I)_{\tau}$ then $[b', b^2]_{\tau} = 0$.*

(3) *Let $a, b \in [I]_{\tau}$. If R is of characteristic not 3 and f) holds, then either $a+b \in [I]_{\tau}$ or $a, b, a+b, a-b \in (I)_{\tau}$; in particular, if $a \in I \setminus (I)_{\tau}$ and $b \in [I]_{\tau}$ then $a+b \in [I]_{\tau}$.*

(4) *If $I \neq [I]_{\tau}$, then there is no positive integer n such that $b^n = 0$ for all $b \in I \setminus [I]_{\tau}$.*

Proof. (1) follows from $[a'-b', a-b]_\tau = -[a'+b', a+b]_\tau + 2([a', a]_\tau + [b', b]_\tau)$ (resp. $(a'-b', a-b)_\tau = -(a'+b', a+b)_\tau + 2((a', a)_\tau + (b', b)_\tau)$), and (2) is obvious by $[x, y^2]_\tau = [(x, y)_\tau, y]_\tau$ for all $x, y \in R$. In order to see (3), it is enough to take (1) into account and follow the proof of [1, Lemma 5]. The proof of (4) is quite similar to that of [3, Lemma 2 (2)].

Lemma 2. *Let d be a non-zero $(1, \tau)$ -derivation of R . Suppose that R is of characteristic not 2 and f) holds.*

- (1) *If $b \in I \setminus [I]_\tau$, then $(b^2)' = \tau(b^2)b' = b'b^2 = 0$ and $b^2 \neq 0$.*
- (2) *If $C \cap I \neq 0$, then d) holds.*
- (3) *If $b \in I \setminus [I]_\tau$ and x is an element of R with $x' = 0$, then $\tau(b)b'xb^3 = 0$.*

Proof. (1) Since $(b^2)' = (b', b)_\tau \in C_\tau$ and $[b', b^2]_\tau = 0$ by Lemma 1 (2), we have $[(b^2+b)', b^2+b]_\tau = [(b^2-b)', b^2-b]_\tau = [b', b]_\tau \notin C_\tau$, namely $b^2+b \notin [I]_\tau$ and $b^2-b \notin [I]_\tau$. By Lemma 1 (1), these together with $(b^2+b) - (b^2-b) = 2b \in [I]_\tau$ show that $2b^2 = (b^2+b) + (b^2-b) \in (I)_\tau$, and so $b^2 \in (I)_\tau$. Hence $2(b^2)'b^2 = ((b^2)', b^2)_\tau \in C_\tau$ by $(b^2)' \in C_\tau$, i.e., $(b^2)'b^2 = \tau(b^2)(b^2)' \in C_\tau$. Furthermore, by Lemma 1 (2), $0 = (b^2)'[(b^2+b)', (b^2+b)^2]_\tau = 2(b^2)'[b', b^3]_\tau = 2(b^2)'\tau(b^2)[b', b]_\tau$. Now, by $(b^2)'b^2 \in C_\tau$ and $(b^2)' \in C_\tau$, $0 = [(b^2)'b^2, x]_\tau = \tau([b^2, x])(b^2)'$ for all $x \in R$. Hence, as is easily seen, either $b^2 \in C$ or $(b^2)' = 0$. If $b^2 \in C$ then, by $(b^2)'\tau(b^2)[b', b]_\tau = 0$ and $(b^2)' \in C_\tau$, $\tau(b^2)R(b^2)'R[b', b]_\tau = 0$, and so $(b^2)' = 0$, in any case. Since $b^2+b \notin [U]_\tau$, we have also $0 = ((b^2+b)^2)' = ((b^2+b)', (b^2+b))_\tau = (b', b^2)_\tau = 2b'b^2$. Hence $b'b^2 = 0$ and $\tau(b^2)b' = b'b^2 - [b', b^2]_\tau = 0$.

Finally, we shall prove that $b^2 \neq 0$. Suppose, to the contrary, that $b^2 = 0$. Then $\tau(b)b'b = \tau(b)(b^2)' = 0$. Let $x \in R$. Since $c = (b+bx)'(b+bx) \mp \tau(b+bx)(b+bx)' \in C_\tau$, we see that $0 = \tau(b^2)c = \tau(b)cb = \tau(b)(b'+b'x+\tau(b)x')bxb = \tau(b)b'xbxb$, and so $\tau(b)b' = 0$ by [4, Lemma 2]. But, this forces a contradiction $[b', b]_\tau = (b^2)' - 2\tau(b)b' = 0$.

(2) First, we claim that $(c^2)' \in C_\tau$ and $\tau(c^2) = c^2$ for any $c \in C \cap I$. Let $x \in R$. Then $[c', x]_\tau = [x', c]_\tau$ by $(cx)' = (xc)'$. If $[c', c]_\tau \in C_\tau$ (resp. $(c', c)_\tau \in C_\tau$) then $0 = [[c', c]_\tau, x]_\tau = (c - \tau(c))[c', x]_\tau = (c - \tau(c))[x', c]_\tau = (c - \tau(c))^2x'$ (resp. $0 = [(c', c)_\tau, x]_\tau = (c^2 - \tau(c^2))x'$). Now, $R' \neq 0$ shows that $\tau(c^2) = c^2$, in any case. Hence $0 = (c^2x)' - (xc^2)' = [(c^2)', x]_\tau$, and so $(c^2)' \in C_\tau$.

Now, suppose, to the contrary, that there exists $b \in I \setminus [I]_\tau$. Take any

non-zero $c_0 \in C \cap I$, and put $c = c_0^2 (\neq 0)$. Then $\tau(c) = c$ and $c' \in C_\tau$ by the above claim. Hence $[(b+c)', b+c]_\tau = [b', b]_\tau \in C_\tau$, namely $b+c \notin [I]_\tau$. Recalling that $((b+c)')' = 0 = (b^2)'$ by (1), we get $0 = [((b+c)')', b]_\tau = 2[b'c + \tau(b)c' + cc', b]_\tau = 2c[b', b]_\tau$. Since $2c$ is a non-zero central element, this forces a contradiction $[b', b]_\tau = 0$.

(3) Since $c = (b+bx^2)'(b+bx^2) \mp \tau(b+bx^2)(b+bx^2)' \in C_\tau$ and $(b^2)' = \tau(b^2)b' = 0$ by (1), we get $0 = \tau(b^2)cR = \tau(b^2)Rc$. Hence $c = 0$, because $b^2 \neq 0$ by (1). Now, $\tau(b)b'b = \tau(b)(b^2)' = 0$ and $0 = \tau(b)c = \tau(b)(b'+b'xb^2)(b+bx^2) = \tau(b)b'xb^3 + \tau(b)b'xb^3xb^2$. Similarly, $0 = -\tau(b)b'xb^3 + \tau(b)b'xb^3xb^2$, and therefore $2\tau(b)b'xb^3 = 0$. Hence $\tau(b)b'xb^3 = 0$.

Proof of Theorem 2. We may, and shall, assume that d is a $(1, \tau)$ -derivation. In view of Theorem 1, it suffices to show that each of d) and e) implies b).

d) \Leftrightarrow b). The proof of [2, Theorem 1] still works under the present hypothesis.

e) \Rightarrow b). Suppose, to the contrary, that b) does not hold. For any $a \in I$, we have $2(a', a)_\tau a^2 = ((a', a)_\tau, a^2)_\tau = ((a^2)', a^2)_\tau \in C_\tau$. So, for any $x \in R$, $0 = [(a', a)_\tau a^2, x]_\tau = (a', a)_\tau [a^2, x]$, and therefore $(a', a)_\tau R[a^2, x] = 0$. Hence either $(a', a)_\tau = 0$ or $a^2 \in C$. By Theorem 1, $(b^2)' = (b', b)_\tau \neq 0$ for some $b \in I$, and hence b^2 is a non-zero element of $C \cap I$, and so d) holds, by Lemma 2 (2); hence b) holds by the above. This is a contradiction.

Lemma 3. *Let d be a $(1, \tau)$ -derivation of R . If R is of characteristic 3 and I is a non-zero ideal, then a), b), d), e) and f) are equivalent.*

Proof. In view of Theorem 2, it suffices to show that f) implies d). First, we claim that $(a^3)' = 0$ for all $a \in I$. Actually, if $a \in [I]_\tau$ then $(a^3)' = a'a^2 + \tau(a)a'a + \tau(a)\tau(a)a' = a'a^2 + (a'a - [a', a]_\tau)a + \tau(a)(a'a - [a', a]_\tau) = 3a'a^2 - 3[a', a]_\tau a = 0$; if $a \notin [I]_\tau$ then $(a^3)' = (a^2)'a + \tau(a^2)a' = 0$ by Lemma 2 (1).

Now, suppose, to the contrary, that $I \neq [I]_\tau$. Then, by Lemma 1 (4), there exists $b \in I \setminus [I]_\tau$ with $b^3 \neq 0$. Recalling that $(a^3)' = 0$ for all $a \in I$, we see that $\tau(b)b'(ax)^3b^3 = 0$ for all $a \in I$ and $x \in R$ (Lemma 2 (3)). Since $b^3 \neq 0$, [5, Theorem] proves that $\tau(b)b'I = 0$. Thus $\tau(b)b' = 0$, which together with $(b^2)' = 0$ (Lemma 2 (1)) forces a contradiction $[b', b]_\tau = 0$.

We are now in a position to complete the proof of Theorem 3.

Proof of Theorem 3. We may, and shall, assume that d is a $(1, \tau)$ -derivation. In view of Theorem 2 and Lemma 3, it remains only to show that f) implies d) or e) under the hypothesis that R is of characteristic not 2 or 3.

Suppose, to the contrary, that neither d) nor e) holds. Then there exist $a \in [I]_{\tau} \setminus (I)_{\tau}$ and $b \in (I)_{\tau} \setminus [I]_{\tau}$. If $a+b \in (I)_{\tau}$ then $(a+b) - b = a \notin (I)_{\tau}$ shows that $a+2b = (a+b) + b \in (I)_{\tau}$ (Lemma 1 (1)), and so $2b = a + 2b - a \in [I]_{\tau}$ (Lemma 1 (3)), a contradiction. On the other hand, if $a+b \notin (I)_{\tau}$ then $b = (a+b) - a \in [I]_{\tau}$ (Lemma 1 (3)), again a contradiction.

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