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## On Commutativity of Rings with Generalized Derivations

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# On Commutativity of Rings with Generalized Derivations

Nadeem ur Rehman

## Abstract

The concept of derivations as well as of generalized inner derivations have been generalized as an additive function  $F : R \rightarrow R$  satisfying  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in R$ , where  $d$  is a derivation on  $R$ , such a function  $F$  is said to be a generalized derivation. In the present paper we have discussed the commutativity of prime rings admitting a generalized derivation  $F$  satisfying (i)  $[F(x), x] = 0$ , (ii)  $F([x, y]) = [x, y]$ , and (iii)  $F(x \circ y) = x \circ y$  for all  $x, y$  in some appropriate subset of  $R$ .

**KEYWORDS:** prime rings, generalized derivations, derivations, ideals, Lie ideals and commutativity.

## ON COMMUTATIVITY OF RINGS WITH GENERALIZED DERIVATIONS

NADEEM-UR-REHMAN

ABSTRACT. The concept of derivations as well as of generalized inner derivations have been generalized as an additive function  $F: R \rightarrow R$  satisfying  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in R$ , where  $d$  is a derivation on  $R$ , such a function  $F$  is said to be a generalized derivation. In the present paper we have discussed the commutativity of prime rings admitting a generalized derivation  $F$  satisfying (i)  $[F(x), x] = 0$ , (ii)  $F([x, y]) = [x, y]$ , and (iii)  $F(x \circ y) = x \circ y$  for all  $x, y$  in some appropriate subset of  $R$ .

### 1. INTRODUCTION

Let  $R$  denote an associative ring with center  $Z(R)$ . For any  $x, y \in R$ , the symbol  $[x, y]$  stands for the commutator  $xy - yx$  and the symbol  $x \circ y$  denotes for anti-commutator  $xy + yx$ . Recall that a ring  $R$  is called prime if for any  $a, b \in R$ ,  $aRb = (0)$  implies that either  $a = 0$  or  $b = 0$ . An additive mapping  $d: R \rightarrow R$  is called a derivation if  $d(xy) = d(x)y + xd(y)$  holds for all  $x, y \in R$ . For a fixed  $a \in R$ , the mapping  $I_a: R \rightarrow R$  given by  $I_a(x) = [a, x]$  is a derivation which is said to be inner derivation.

Many analysts have studied generalized derivation in the context of algebras on certain normed spaces (see [10] for reference). By a generalized derivation on an algebra  $A$  one usually means a map of the form  $x \mapsto ax + xb$ , where  $a$  and  $b$  are fixed elements in  $A$ . We prefer to call such maps generalized inner derivations for the reason they present a generalization of the concept of inner derivations (i.e. the maps of the form  $x \mapsto ax - xa$ ). Now in a ring  $R$ , let  $F$  be a generalized inner derivation of  $R$  given by  $F(x) = ax + xb$ . Notice that  $F(xy) = F(x)y + xI_b(y)$ , where  $I_b(y) = yb - by$  is an inner derivation.

Motivated by these observation Hvala [10] introduced the notions of generalized derivations in rings. An additive mapping  $F: R \rightarrow R$  is called a generalized derivation if there exists a derivation  $d: R \rightarrow R$  such that  $F(xy) = F(x)y + xd(y)$  holds for all  $x, y \in R$ . Obviously, every derivation generalized inner derivation and left multiplier (i.e. an additive mapping  $F: R \rightarrow R$  such that  $F(xy) = F(x)y$  for all  $x, y \in R$ ) are generalized derivations.

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*Key words and phrases*. prime rings, generalized derivations, derivations, ideals, Lie ideals and commutativity.

In the present paper we shall attempt to generalize some known results for derivations to generalized derivations.

## 2. PRELIMINARY RESULTS

Throughout the present paper, we shall make use of the following two basic identities without any specific mention:

$$\begin{aligned}x \circ (yz) &= (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z, \\(xy) \circ z &= x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].\end{aligned}$$

We begin with the following known results which will be used extensively to prove our theorems.

**Lemma 2.1** ([4, Lemma 4]). *If  $U \not\subseteq Z(R)$  is Lie ideal of a 2-torsion free prime ring  $R$  and  $a, b \in R$  such that  $aUb = 0$ , then  $a = 0$  or  $b = 0$ .*

**Lemma 2.2** ([4, Lemma 5]). *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$ . If  $d$  is a nonzero derivation of  $R$  such that  $d(U) = 0$ , then  $U \subseteq Z(R)$ .*

**Lemma 2.3** ([2, Theorem 7]). *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$ . If  $d$  is a nonzero derivation of  $R$  such that  $[u, d(u)] \in Z(R)$  for all  $u \in U$ , then  $U \subseteq Z(R)$ .*

**Lemma 2.4** ([3, Theorem 4]). *Let  $R$  be a prime ring and  $I$  a nonzero left ideal of  $R$ . If  $R$  admits a nonzero derivation  $d$  such that  $[x, d(x)]$  is central for all  $x \in I$ , then  $R$  is commutative.*

**Lemma 2.5** ([11, Lemma 3]). *If a prime ring  $R$  contains a nonzero commutative right ideal, then  $R$  is commutative.*

Now, we prove the following.

**Lemma 2.6.** *Let  $R$  be a 2-torsion free prime ring and  $U$  be a nonzero Lie ideal of  $R$ . If  $U$  is a commutative Lie ideal of  $R$ , i.e.  $[u, v] = 0$  for all  $u, v \in U$ , then  $U \subseteq Z(R)$ .*

*Proof.* Since  $U$  is a commutative Lie ideal of  $R$ , i.e.

$$(2.1) \quad [u, v] = 0, \quad \text{for all } u, v \in U.$$

Replacing  $v$  by  $[u, r]$  in (2.1), we get  $[u, [u, r]] = 0$  for all  $u \in U, r \in R$ . Again replace  $r$  by  $rs$ , to get  $[u, [u, rs]] = 0$  for all  $u \in U, r, s \in R$ , that is

$$[u, [u, r]]s + r[u, [u, s]] + 2[u, r][u, s] = 0, \quad \text{for all } u \in U, r, s \in R.$$

This implies that  $2[u, r][u, s] = 0$  for all  $u \in U, r, s \in R$ . Since  $\text{char}(R) \neq 2$ , we get  $[u, r][u, s] = 0$ . Replacing  $s$  by  $sr$ , we get  $[u, r]R[u, r] = (0)$  for all  $u \in U, r \in R$ . Thus primeness of  $R$  forces that  $[u, r] = 0$  for all  $u \in U, r \in R$ , and hence  $U \subseteq Z(R)$ .  $\square$

## 3. LIE IDEALS AND GENERALIZED DERIVATIONS OF PRIME RINGS

**Theorem 3.1.** *Let  $R$  be a 2-torsion free prime ring and  $U$  be a nonzero Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $R$  admits a nonzero generalized derivation  $F$  with  $d$  such that  $[F(u), u] = 0$  for all  $u \in U$ , and if  $d \neq 0$ , then  $U \subseteq Z(R)$ .*

*Proof.* We have

$$(3.1) \quad [F(u), u] = 0, \quad \text{for all } u \in U.$$

Linearizing (3.1) and using (3.1), we obtain

$$(3.2) \quad [F(u), v] + [F(v), u] = 0, \quad \text{for all } u, v \in U.$$

Notice that  $vw + wv = (v + w)^2 - v^2 - w^2$  for all  $v, w \in U$ . Since  $u^2 \in U$  for all  $u \in U$ ,  $vw + wv \in U$ . Also  $vw - wv \in U$  for all  $v, w \in U$ . Hence we find that  $2vw \in U$  for all  $v, w \in U$ . Replacing  $v$  by  $2vu$  in (3.2) and use (3.1) and (3.2), to get

$$(3.3) \quad v[d(u), u] + [v, u]d(u) = 0, \quad \text{for all } u, v \in U.$$

Again replacing  $v$  by  $2wv$  in (3.3) and using (3.3), we get  $[w, u]vd(u) = 0$  for all  $u, v, w \in U$ , and hence  $[w, u]Ud(u) = (0)$  for all  $u, w \in U$ . Thus for each  $u \in U$ , by Lemma 2.1 we find that either  $[w, u] = 0$  or  $d(u) = 0$ . Now, let  $A = \{u \in U \mid d(u) = 0\}$  and  $B = \{u \in U \mid [w, u] = 0 \text{ for all } w \in U\}$ . Then  $A$  and  $B$  are additive subgroups of  $U$  and  $U = A \cup B$ . But a group can not be a union of two its proper subgroups, and hence  $U = A$  or  $U = B$ . If  $U = A$ , then  $d(u) = 0$  for all  $u \in U$ . Thus by Lemma 2.2, we get the required result. On the other hand if  $[w, u] = 0$  for all  $w, u \in U$ , then by Lemma 2.6, we get  $U \subseteq Z(R)$ . This completes the proof of the theorem.  $\square$

Using the same techniques with necessary variations, we can prove the following corollary even without the characteristic assumption on the ring.

**Corollary 3.2.** *Let  $R$  be a prime ring. If  $R$  admits a nonzero generalized derivation  $F$  with  $d$  such that  $[F(x), x] = 0$  for all  $x \in R$ , and if  $d \neq 0$ , then  $R$  is commutative.*

In a recent paper, Daif and Bell [7] established that a semiprime ring  $R$  must be commutative if it admits a derivation  $d$  such that  $d([x, y]) = [x, y]$  for all  $x, y \in R$ . Further, Ashraf and Rehman [1] extended the mentioned result for Lie ideals of  $R$ . In the present section we generalize this result for generalized derivations and Lie ideals of  $R$ .

**Theorem 3.3.** *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F([u, v]) = [u, v]$  for all  $u, v \in U$ , and if  $F = 0$  or  $d \neq 0$ , then  $U \subseteq Z(R)$ .*

*Proof.* Given that  $F$  is a generalized derivation of  $R$  such that  $F([u, v]) = [u, v]$  for all  $u, v \in U$ . If  $F = 0$ , then  $[u, v] = 0$  for all  $u, v \in U$ . Thus by Lemma 2.6, we get the required result.

Now, onward we assume that  $F \neq 0$ . Suppose on contrary that  $U \not\subseteq Z(R)$ . For any  $u, v \in U$ , we have  $F([u, v]) = [u, v]$ . This can be rewritten as

$$(3.4) \quad F(u)v + ud(v) - F(v)u - vd(u) = [u, v], \quad \text{for all } u, v \in U.$$

Replacing  $v$  by  $2vu$  in (3.4) and using the fact that  $\text{char}(R) \neq 2$ , we find that

$$F(u)vu + ud(v)u + [u, v]d(u) - F(v)u^2 - vd(u)u = [u, v]u, \quad \text{for all } u, v \in U,$$

and hence application of (3.4) gives that  $[u, v]d(u) = 0$  for all  $u, v \in U$ . Again replace  $v$  by  $2wv$ , to get  $[u, w]vd(u) = 0$  for all  $u, v, w \in U$ , and hence  $[u, w]Ud(u) = (0)$  for all  $u, w \in U$ . Thus for each  $u \in U$ , by Lemma 2.1, either  $[u, w] = 0$  or  $d(u) = 0$ . Now, let  $U_1 = \{u \in U \mid [u, w] = 0 \text{ for all } w \in U\}$  and  $U_2 = \{u \in U \mid d(u) = 0\}$ . Then  $U_1$  and  $U_2$  both are additive subgroups of  $U$  and  $U_1 \cup U_2 = U$ . Thus either  $U_1 = U$  or  $U_2 = U$ . If  $U_1 = U$ , then  $[u, w] = 0$  for all  $u, w \in U$ . Hence by Lemma 2.6, we get  $U \subseteq Z(R)$ , contradiction. On the other hand if  $U_2 = U$ , then  $d(u) = 0$  for all  $u \in U$ . Thus by Lemma 2.2, we get  $U \subseteq Z(R)$ , again a contradiction. This completes the proof of the theorem.  $\square$

Using the same techniques with necessary variations we get the following.

**Theorem 3.4.** *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F([u, v]) + [u, v] = 0$  for all  $u, v \in U$ , and if  $F = 0$  or  $d \neq 0$ , then  $U \subseteq Z(R)$ .*

**Corollary 3.5.** *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F(uv) = uv$  for all  $u, v \in U$ , and if  $F = 0$  or  $d \neq 0$ , then  $U \subseteq Z(R)$ .*

*Proof.* For any  $u, v \in U$ ,  $F(uv - vu) = F(uv) - F(vu) = uv - vu$ , and hence by Theorem 3.3, we get the required result.  $\square$

Similarly, in view of the Theorem 3.4, we get the following.

**Corollary 3.6.** *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F(uv) = vu$  for all  $u, v \in U$ , and if  $F = 0$  or  $d \neq 0$ , then  $U \subseteq Z(R)$ .*

**Theorem 3.7.** *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F(u \circ v) = u \circ v$  for all  $u, v \in U$ , and if  $F = 0$  or  $d \neq 0$ , then  $U \subseteq Z(R)$ .*

*Proof.* If  $F = 0$ , then we have

$$(3.5) \quad u \circ v = 0, \quad \text{for all } u, v \in U.$$

Replacing  $v$  by  $2vw$  in (3.5) and using (3.5), we have  $2v[u, w] = 0$  for all  $u, v, w \in U$ . This implies that  $v[u, w] = 0$  for all  $u, v, w \in U$ . Again replace  $v$  by  $[u, r]$ , to get  $[u, r][u, w] = 0$  for all  $u, w \in U, r \in R$ . For any  $s \in R$ , replacing  $r$  by  $rs$ , we get  $[u, r]R[u, w] = (0)$  for all  $u, w \in U, r \in R$ . Thus, in particular we have  $[u, w]R[u, w] = (0)$  for all  $u, w \in U$ . Thus primeness of  $R$  yields that  $[u, w] = 0$  for all  $u, w \in U$ , and hence by Lemma 2.6, we get the required result.

Therefore now onward we shall assume that  $F \neq 0$ . Suppose on contrary that  $U \not\subseteq Z(R)$ . For any  $u, v \in U$ , we have  $F(u \circ v) = u \circ v$ . This can be rewritten as

$$(3.6) \quad F(u)v + ud(v) + F(v)u + vd(u) = u \circ v, \quad \text{for all } u, v \in U.$$

Replacing  $v$  by  $2vu$  in (3.6), we find that

$$(F(u)v + ud(v) + F(v)u + vd(u) - u \circ v)u + (u \circ v)d(u) = 0, \quad \text{for all } u, v \in U.$$

Thus an application of (3.6) gives that  $(u \circ v)d(u) = 0$  for all  $u, v \in U$ . Again replace  $v$  by  $2wv$ , to get  $[u, w]vd(u) = 0$  for all  $u, v, w \in U$ . Note that the arguments given in the proof of Theorem 3.3 are still valid in the present situation and hence repeating the same process we get the required result.  $\square$

Using similar arguments one can also prove the following.

**Theorem 3.8.** *Let  $R$  be a 2-torsion free prime ring and  $U$  a nonzero Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F(u \circ v) + u \circ v = 0$  for all  $u, v \in U$ , and if  $F = 0$  or  $d \neq 0$ , then  $U \subseteq Z(R)$ .*

#### 4. IDEALS AND GENERALIZED DERIVATIONS OF PRIME RINGS

In the hypothesis of Theorems 3.7 and 3.8, if we choose the underlying subset as an ideal instead of a Lie ideal, then we can prove the following result even without the characteristic assumption on the ring.

**Theorem 4.1.** *Let  $R$  be a prime ring and  $I$  a nonzero ideal of  $R$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F(x \circ y) = x \circ y$  holds for all  $x, y \in I$ , and if  $F = 0$  or  $d \neq 0$ , then  $R$  is commutative.*

*Proof.* For any  $x, y \in I$ , we have  $F(x \circ y) = x \circ y$ . If  $F = 0$ , then  $x \circ y = 0$  for all  $x, y \in I$ . Replacing  $y$  by  $yz$  and using the fact that  $xy = -yx$ , we find that  $y[x, z] = 0$  for all  $x, y, z \in I$ , and hence  $IR[x, z] = (0)$  for all  $x, z \in I$ . Since  $I \neq (0)$  and  $R$  is prime, we get  $[x, z] = 0$  for all  $x, z \in I$ , and hence by Lemma 2.5,  $R$  is commutative. Hence onward we assume that  $F \neq 0$ . For any  $x, y \in I$ , we have  $F(x \circ y) = x \circ y$ . This can be rewritten as

$$(4.1) \quad F(x)y + xd(y) + F(y)x + yd(x) = x \circ y, \quad \text{for all } x, y \in I.$$

Replacing  $y$  by  $yx$  in (4.1), we get

$$(F(x)y + xd(y) + F(y)x + yd(x) - (x \circ y))x + (x \circ y)d(x) = 0, \quad \text{for all } x, y \in I.$$

In view of (4.1) the above relation yields that  $(x \circ y)d(x) = 0$  for all  $x, y \in I$ . Again replace  $y$  by  $zy$ , to get  $z(x \circ y)d(x) + [x, z]yd(x) = 0$  for all  $x, y, z \in I$ , and hence  $[x, z]IRd(x) = (0)$  for all  $x, z \in I$ . Thus primeness of  $R$  forces that for each  $x \in I$  either  $d(x) = 0$  or  $[x, z]I = (0)$  for all  $z \in I$ . The set of  $x \in I$  for which these two properties hold are additive subgroups of  $I$  whose union is  $I$ , and therefore  $d(x) = 0$  for all  $x \in I$  or  $[x, z]I = (0)$  for all  $x, z \in I$ . If  $[x, z]I = (0)$  for all  $x, z \in I$ , then  $[x, z]RI = (0)$ . Since  $I \neq (0)$ , we find that  $[x, z] = 0$  for all  $x, z \in I$ , and hence again by Lemma 2.5,  $R$  is commutative. On the other hand if  $d(x) = 0$  for all  $x \in I$ , then implies that  $[d(x), x] = 0$  for all  $x \in I$ , and hence by Lemma 2.4,  $R$  is commutative.  $\square$

Using similar arguments as used in the above theorem, we can prove the following.

**Theorem 4.2.** *Let  $R$  be a prime ring and  $I$  a nonzero ideal of  $R$ . If  $R$  admits a generalized derivation  $F$  with  $d$  such that  $F(x \circ y) + x \circ y = 0$  holds for all  $x, y \in I$ , and if  $F = 0$  or  $d \neq 0$ , then  $R$  is commutative.*

**Remark.** In view of the above results, it is an obvious question is whether these results can be extended to left multiplier (i.e. a generalized derivation with  $d = 0$ ). Unfortunately, we are unable to extend these results to the case where  $F$  is a left multiplier. We leave as an open question whether or not these results can be extended in the setting of left multiplier.

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