Mathematical Journal of Okayama University

Volume 13, *Issue* 2 1967 *Article* 3

NOVEMBER 1968

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A NOTE ON GROUP RINGS OF p-GROUPS

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Following [1], a ring B (with 1) is called *left perfect* if B is semi-primary and the (Jacodson) radical $\mathfrak{R}(B)$ of B is left T-nilpotent. As was noted in [4], every left perfect primary ring is a complete matrix ring over a local ring, where we take the term "local ring" in the sense that the set of all non-units of the ring (with 1) forms an ideal. One may remark here that the characteristic of a left perfect local ring is either 0 or a power of a prime. The principal aim of the present note is to give a theorem concerning group rings of p-groups over a left perfect primary ring, which will play an important role in [4].

Let B be a left perfect local ring, and G a group different from $\{1\}$. Then, the mapping $\psi: \sum \sigma x_\sigma \longrightarrow \sum \bar{x}_\sigma$ defines a ring homomorphism of the group ring GB onto the division ring $\bar{B}=B/\mathfrak{R}(B)$, where $\bar{x}$ means the residue class of x module $\mathfrak{R}(B)$. The kernel Δ of ψ coincides with $\sum(1-\sigma)B+G\cdot\mathfrak{R}(B)$, and will be called the *fundamental ideal* of GB. Under these notations, we shall prove the following, whose second assertion contains [2; Th, 2.2] and [3; Lemma 3].

Lemma. (a) *Let G be a finite group. If α is a unit of GB whenever $\psi(\alpha)$ is non-zero, then the order of G and the characteristic of B are powers of a prime p.*

(b) *The following conditions are equivalent*: (1) *Δ is nilpotent, and* (2) *$\mathfrak{R}(B)$ is nilpotent and the order of G and the characteristic of B are powers of a prime p.*

Proof. (a) Let G be of order n. If B is of characteristic 0 then $\psi(\sum_{\sigma\in G}\sigma)=n\cdot 1\neq 0$, because $\mathfrak{R}(B)$ is a nil-ideal. But, for any $\tau\in G$ different from 1 we have $(\sum\sigma)(1-\tau)=0$. This contradiction shows that B is of characteristic p^c (p a prime). Now, suppose that $n=p^e n'$ with $(n', p)=1$ and $n'>1$. Then, for any prime divisor p' of n' we can find a p'-Sylow group G' of G. Since $\psi(\sum_{\sigma'\in G'}\alpha')$ is a power of p', it is a non-zero element of $\bar{B}$. While, $(\sum\sigma')(1-\tau')=0$ for any $\tau'\in G'$ different from 1, which is a contradiction.

(b) (1)$\Longrightarrow$(2): Suppose $\Delta^{m-1}\neq 0$ and $\Delta^m=0$. Let $\sum\sigma x_\sigma$ be a non-zero element of Δ^{m-1}. Then, for any $\tau\in G$ we have $(\sum\sigma x_\sigma)(1-\tau)=0$, namely, $\sum\sigma x_\sigma=\sum\sigma x_{\sigma\tau^{-1}}$. Hence, $x_\sigma=x_{\sigma\tau^{-1}}$ for every σ. Taking $\sigma=\tau$, if follows $x_\tau=x_1$. Consequently, G must be of finite order. Now, our implication is obvious by (a).

(2) ⇒ (1): Let G be of order p^e, B of characteristic p^c, and $\mathfrak{N}(B)^n = 0$. In case $e=1$, noting that $(1-\sigma)^{pc}=0$ for every $\sigma \in G$, it will be easy to see that $\Delta^{pc+n}=0$. We can proceed therefore with the induction with respect to e. Let $e>1$, and G' a subgroup of the center of G whose order is p. Then, $\lambda: \sum \sigma x_\sigma \longrightarrow \sum \bar{\sigma}\bar{x}_\sigma$ defines a ring homomorphism of GB onto $\bar{G}\bar{B}$, where $\bar{G}=G/G'$. Obviously. Ker λ is the ideal generated by the fundamental ideal Δ' of $G'B$. Accordingly, if $\Delta'^{m'}=0$ then (Ker $\lambda)^{m'}=0$. Now, noting that $\lambda(\Delta)$ is contained in the fundamental ideal $\bar{\Delta}$ of $\bar{G}\bar{B}$ and $\bar{\Delta}^{\bar{m}}=0$ for some $\bar{m}$, we readily obtain $\Delta^{\bar{m}m'}=0$.

The next will be easily seen (cf. [2; Th. 2.3]).

Corollary. *Let B be a local ring with the nilpotent radical, and G a group different from* $\{1\}$. *Then, Δ is locally nilpotent if and only if the characteristic of B is a power of a prime p and G is a locally finite p-Group.*

If B is a left perfect primary ring, then the center Z of B is a perfect local ring. Now, we shall prove the following:

Theorem. *Let B be a left perfect primary ring with the center Z, G a finite group, and $G' \neq \{1\}$ a normal subgroup of G. If $\bar{G}=G/G'$ then the following conditions are equivalent*: (1) $\sum \sigma x_\sigma$ *is a unit of* GB *whenever* $\sum \bar{\sigma} x_\sigma$ *is a unit of* $\bar{G}B$, (2) $G'Z$ *is a local ring, and* (3) *the order of G' and the characteristic of B are powers of a prime p.*

Proof. The mapping $\varphi: \sum \sigma x_\sigma \longrightarrow \sum \bar{\sigma} x_\sigma$ is a ring homomorphism of GB onto $\bar{G}B$ and Ker $\varphi = \sum_{\substack{\sigma \in G \\ \sigma' \in G'}} \sigma(1-\sigma')B$. Further, the mapping $\psi': \sum \sigma' z_{\sigma'} \longrightarrow \sum \bar{z}_{\sigma'}$ is a ring homomorphism of $G'Z$ onto the field $\bar{Z}= Z/\mathfrak{N}(Z)$ where $\bar{z}$ means the residue class of z modulo $\mathfrak{N}(Z)$, and Ker $\psi' = \sum (1-\sigma')Z + G' \cdot \mathfrak{N}(Z)$.

(1) ⇒ (2): If α is an arbitrary element of Ker φ then $1-\alpha$ is a unit as an inverse image of 1 relative to φ, and so Ker φ is contained in $\mathfrak{N}(GB)$. Since GB is left perfect by [4; Prop. 3.3 (b)], Ker φ is a nil-ideal, whence we see that $\sum (1-\sigma')Z$ is a nil-ideal of $G'Z$. On the other hand, it is known that $G' \cdot \mathfrak{N}(Z)$ is contained in $\mathfrak{N}(G'Z)$. Hence, Ker ψ' coincides with the radical of $G'Z$ and $G'Z$ is a local ring.

(2) ⟹ (3): Since $G'Z$ is a local ring, $\mathfrak{N}(G'Z)$ coincides with Ker ψ'. If follows therefore every inverse image relative to ψ' of a non-zero elemetn of Z is a unit of $G'Z$. Accordingly, it follows (3) by Lemma (a).

(3) ⟹ (1): Let P be the subring of B generated by 1. Then, P is a local subring of Z with the nilpotent radical and $\sum (1-\sigma')P$ is a nilpotent ideal of $G'P$ by Lemma (b). Now, one will readily see that Ker φ is nilpo-

tent, and then our implication is obvious.

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(Received April 1, 1968)