Mathematical Journal of Okayama University

Volume 39, *Issue* 1 1997 *Article* 6

JANUARY 1997

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Hiroaki Komatsu* Kazuyuki Tanabe†

*Okayama University

†Okayama University

Math. J. Okayama Univ. **39** (1997), 57–59

ON RINGS HAVING A FAITHFUL NOETHERIAN MODULE

HIROAKI KOMATSU and KAZUYUKI TANABE

It is well known that every commutative ring with an identity element and having a faithful Noetherian module is Noetherian (cf. [1, p.261]). In this paper, we consider a noncommutative ring with or without an identity element which has a faithful Noetherian or Artinian module, and we shall show that every nil ideal of bounded index of such a ring is nilpotent.

For a nil ideal I of a ring R, $n(I)$ is defined to be the supremum of the nilpotency indeces of the elements in I.

Theorem 1. *If a ring R has a faithful Noetherian left R-module M, then every nil ideal I of R with $n(I) < \infty$ is nilpotent.*

Proof. For a R-submodule N of M and an ideal A of R, $A * N$ is defined to be the set of all $m \in M$ satisfying $Am \subseteq N$. It is easy to see the following properties.

(a) $A * N$ is a R-submodule of M containing N.

(b) $(AB) * N = B * (A * N)$ for any ideals A, B.

We shall prove the theorem by induction on $n(I)$. If $n(I) = 1$ then $I = 0$. Let k be a positive integer and assume that every nil ideal I with $n(I) \leq k$ is nilpotent. Let I be a nil ideal of R with $n(I) = k+1$. Let J be the ideal of R generated by all x^k where $x \in I$. Since M is Noetherian, there exists a positive integer n such that $J^n * 0 = J^{n+1} * 0$.
Now suppose that $J^n * 0 \neq M$. Then the set

$$\{(J^n K) * 0 \mid K \text{ is an ideal of } R,\ (J^n K) * 0 \neq M\} \quad (\ni J)$$

has a maximal member $(J^n L) * 0$, where L is an ideal of R. Let x be an arbitrary element in I. For any $r \in R$, we see that $0 = (x^k r + x)^{k+1} = x^k r x^k + x^k r x^k y$ for some $y \in I$. Since y is nilpotent, we have $x^k r x^k = 0$. Therefore, the ideal (x^k) of R generated by x^k is nilpotent. It is obvious that $(x^k) * \big((J^n L) * 0\big) \supseteq (J^n L) * 0$. If $(x^k) * \big((J^n L) * 0\big) = (J^n L) * 0$, then we have $(x^k)^2 * \big((J^n L) * 0\big) = (x^k) * \Big((x^k) * \big((J^n L) * 0\big)\Big) = (x^k) * \big((J^n L) * 0\big) = (J^n L) * 0$. Continuing this method, we have $(x^k)^\nu * \big((J^n L) * 0\big) = (J^n L) * 0$ for any positive integer ν, which implies a contradiction $(J^n L) * 0 = M$, because (x^k) is nilpotent. Hence, we have $(x^k) * \big((J^n L) * 0\big) \supsetneq (J^n L) * 0$.

Since $(x^k)*((J^nL)*0) \neq (J^nL(x^k))*0$, the maximality of $(J^nL)*0$ implies that $(J^nL(x^k))*0 = M$, i.e., $J^nL(x^k) = 0$. Hence, we have $J^nLJ = 0$. Let A denote the annihilator of $M/((J^nL)*0)$. Then we have $J \subseteq A$, and so $n((I+A)/A) \leq k$. Hence, by the assumption of our induction, there exists a positive integer m such that $I^m \subseteq A$, i.e., $I^m * ((J^nL)*0) = M$. Therefore, we have $I*((J^nL)*0) \supsetneq (J^nL)*0$, and again by the maximality of $(J^nL)*0$, we have $(J^nLI)*0 = M$, i.e., $J^nLI = 0$. Then, for any $x \in I$ and $a \in J^nL$, we see that $0 = (x+a)^{k+1} = x^ka$. Hence, we have $J^{n+1}L = 0$. This implies that $(J^nL)*0 = L*(J^n*0) = L*(J^{n+1}*0) = (J^{n+1}L)*0 = M$, a contradiction. Thus we have shown that $J^n * 0 = M$, i.e., $J^n = 0$.
Let A_i denote the annihilator of $J^{i-1}M/J^iM$ $(i = 1, \ldots, n)$. Since $J \subseteq A_i$, by the assumption of our induction, there exist positive integers m_i such that $I^{m_i}J^{i-1}M \subseteq J^iM$ $(i = 1, \ldots, n)$. Putting $l = m_1 + \cdots + m_n$, we have $I^lM = 0$, that is, $I^l = 0$.

Theorem 2. *If a ring R has a faithful Artinian left R-module M, then every nil ideal I of R with $n(I) < \infty$ is nilpotent.*

Proof. We shall prove the theorem by induction on $n(I)$. If $n(I) = 1$ then $I = 0$. Let k be a positive integer and assume that every nil ideal I with $n(I) \leq k$ is nilpotent. Let I be a nil ideal of R with $n(I) = k+1$. Let J be the ideal of R generated by all x^k where $x \in I$. Since M is Artinian, there exists a positive integer n such that $J^nM = J^{n+1}M$.
Now suppose that $J^n \neq 0$. Then the set

$$\{KJ^nM \mid K \text{ is an ideal of } R,\ KJ^nM \neq 0\} \quad (\ni J)$$

has a minimal member LJ^nM, where L is an ideal of R. Let x be an arbitrary element in I. As was shown in the proof of Theorem 1, the ideal (x^k) of R generated by x^k is nilpotent. Since $LJ^nM \neq 0$, it follows that $(x^k)LJ^nM \subsetneq LJ^nM$, and so $(x^k)LJ^n = 0$ by the minimality of LJ^nM. Hence, we have $JLJ^n = 0$. Let A denote the annihilator of LJ^nM. Then we have $J \subseteq A$, and so $n((I+A)/A) \leq k$. Hence, by the assumption of our induction, there exists a positive integer m such that $I^m \subseteq A$, i.e., $I^mLJ^n = 0$. Therefore, we have $ILJ^nM \subsetneq LJ^nM$, and the minimality of LJ^nM implies that $ILJ^n = 0$. Then, for any $x \in I$ and $a \in LJ^n$, we see that $0 = (a+x)^{k+1} = ax^k$. Hence, we have $LJ^{n+1} = 0$. This implies that $LJ^nM = LJ^{n+1}M = 0$, a contradiction. Thus we have shown that $J^n = 0$. The rest of the proof is quite similar to the proof of Theorem 1.

In [2], the second author of this paper established amen rings. According to [2], a ring R is said to be *amen* if every Artinian left R-module is Noetherian and every Noetherian left R-module is Artinian. By making use of Theorems 1 and 2, we can give an easy proof to [2, Proposition 2].

Corollary 3 ([2, Proposition 2]). *Let R be a ring and I a nil ideal of R with $n(I) < \infty$. If R/I is amen then R is amen.*

Proof. Let M be an Artinian left R-module. By Theorem 2, we have $I^n M = 0$ for some positive integer n. Since R/I is amen, all $I^{i-1}M/I^iM$ are Noetherian ($i = 1, \ldots, n$). Hence, M is Noetherian. Similarly, every Noetherian left R-module is Artinian.

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HIROAKI KOMATSU
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
OKAYAMA UNIVERSITY
OKAYAMA, 700 JAPAN

KAZUYUKI TANABE
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
OKAYAMA UNIVERSITY
OKAYAMA, 700 JAPAN

(*Received February 2, 1998*)