Mathematical Journal of Okayama University

Volume 26, *Issue* 1 1984 *Article* 2

JANUARY 1984

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Math. J. Okayama Univ. 25 (1984), 11—12

ON A THEOREM OF Y. TSUSHIMA

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Let p be a fixed prime number, let G be a finite p-solvable group with a p-Sylow subgroup P of order p^a $(a \geq 1)$ and let $t(G)$ be the nilpotency index of the radical of a group algebra of G over a field of characteristic p. Recently, Y. Tsushima [3] has proved that if $t(G) = a(p-1)+1$ and P is regular then P is elementary abelian. Unfortunately his proof is correct only when p is not a Fermat prime. A cause of his mistake is in the part of an application of [1, Theorem A (ii)]. It should be noted that the first part of [1, Theorem A (ii)] used essentially in his paper easily follows from [1, Theorem B]. At this point of view we shall present the next proposition which shall give a refinement of his theorem and a generalization of [2, Corollary 13]. Moreover this proof shall give an improvement of his proof.

Proposition. *Assume that P is non-abelian and regular. If $t(G) = a(p-1)+1$ then p is a Fermat prime and a 2-Sylow subgroup of $G/O_{p'}(G)$ is non-abelian.*

Proof. We argue by induction on $|G|$. We may assume $O_{p'}(G) = 1$ by the inequality $t(G) \geq t(G/O_{p'}(G)) \geq a(p-1)+1$ (see [4]). We set $U = O_p(G) \neq 1$. By the inequality $t(G) \geq t(G/U)+t(U)-1 \geq a(p-1)+1$ (see [4]), U is elementary abelian and it may be assumed by induction that P/U is abelian. Since P is regular, it follows from this that $(xy)^p = x^p y^p$ for all x, $y \in P$ and so p is odd as P is non-abelian. For all $y \in U$ and $x \in P$, we have

$$y^{x^{p-1}+\cdots+x+1} = y^{x^{p-1}} \cdots y^x y = x^{-p}(xy)^p = 1$$

where $y^{x^s} = x^{-s} y x^s$ and $x^{p-1}+\cdots+x+1$ is the sum of endomorphisms x^{p-1}, $\cdots, x, 1$ of U. Since G/U is a subgroup of $GL(U)$ (see [1, Lemma 1.2.5]), Hall-Higman's theorem [1, Theorem B] together with the last equation yields that $(X-1)^{p-1}$ is the minimal polynomial on U of an element of order p in P/U and this implies the result as p is odd.

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(*Received October 24, 1983*)