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ON QUATERNION KÄHLERIAN BLASCHKE MANIFOLDS

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1. Introduction. A Blaschke manifold may be characterized as a compact riemannian manifold whose tangent cut loci are spheres of constant radius at all points. A compact rank one symmetric space with its standard riemaniann metric, which will be denoted by *CROSS* throughout this paper, is an example of the Blaschke manifold. The famous Blaschke conjecture may be stated as follows :

A Blaschke manifold (M, g) is isometric to a *CROSS*.

See [1] for a detailed investigation of this topic.

H. Sato has worked on the topological Blaschke problem and shown that any Blaschke manifold is homeomorphic to a *CROSS* (see [2]). On the other hand, L. W. Green, M. Berger and others have proved the Blaschke conjecture to be true in cases where the Blaschke manifold is homeomorphic to the sphere or the real projective space (see [1], appendix). In the other cases this conjecture is still open.

The quaternion Kählerian structure on a $4n$-dimensional manifold M $(n \geq 2)$ is defined by the pair (Q, g), where Q is a certain subbundle of the $(1, 1)$-tensor bundle over M and g is a certain riemannian metric on M (see [3] for the details). Recently, S. Ishihara, S. Salamon and others have worked on the geometry of quaternion Kählerian manifolds. Their investigations show that the geometry has some analogy with the geometry of Kählerian manifolds. The standard quaternion projective space $(\boldsymbol{HP}^n, g_0)$ has the so-called standard quaternion Kählerian structure. However, it is not yet known whether there exist quaternion Kählerian structures other than the standard one on the manifolds which are homeomorphic to $\boldsymbol{HP}^n$ $(n \geq 2)$.

In this paper, we will discuss the Blaschke manifold which is equipped with a quaternion Kählerian structure.

Our main results in this paper may be stated as follows :

Theorem. *Let (M, g) be a $4n$-dimensional $(n \geq 2)$ compact riemannian manifold which is normalized so that the diameter is equal to $\pi/2$. Let S denote the scalar curvature of (M, g). Assume the following:*

(1) *There exists a subbundle Q of the tensor bundle of type $(1, 1)$ over M such that (M, Q, g) is a quaternion Kählerian manifold.*

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(2) *(M, g) is a Blaschke manifold.*

Then, $S \leq 16n(n+2)$, and the equality holds if and only if (M, g) is isometric to the standard quaternion projective space $(\boldsymbol{HP}^n, g_0)$.

Remark 1. It is known that when M is homeomorphic to the quaternion projective space $\boldsymbol{HP}^n (n \geq 2)$ and n is odd in the theorem above, "if and only if" is true even if condition (2) is omitted. In fact, in this situation, $S > 0$ implies that (M, g) is isometric to $(\boldsymbol{HP}^n, g_0)$ if n is odd (see [4]). The above theorem is, without any omissions, particularly suited for even integer n.

Remark 2. This theorem is analogous to Berger's theorem which describes the Blaschke condition on the Kählerian manifold $(\boldsymbol{CP}^n, g)$ $(n \geq 2)$ (see [1], p. 150).

2. Proof of theorem. To prove the theorem we use the notations and some results of [3]. Note that a quaternion Kählerian manifold whose dimension ≥ 8 is an Einstein space (see [3]).

For a unit tangent vector u to M at x, we take a geodesic segment $\gamma = \gamma(s)$ $(0 \leq s \leq \pi/2)$ of (M, g) with an arc-length parameter s such that $\gamma(\pi/4) = x$ and $\dot{\gamma}(\pi/4) = u$, and put $m = \gamma(0)$ and $m' = \gamma(\pi/2)$. Note that m' is a cut point of m. Let (J_1, J_2, J_3) be a canonical basis of Q at x. Then each J_i can be extended to a paralle tensor field along γ, which we denote by the same symbol J_i. Clearly the extended tensor fields J_1, J_2, J_3 give a canonical basis of Q at each point $\gamma(s)$ $(0 \leq s \leq \pi/2)$.

Now, we define vector fields X_1, X_2 and X_3 along γ by

$$X_i(s) = \sin(2s) J_i(\dot{\gamma}(s)) \qquad 0 \leq s \leq \pi/2$$

for each i. Note that $X_i(0) = X_i(\pi/2) = 0$. Let I be an index form along γ. Then, for each i, we have

$$0 \leq I(X_i, X_i) = \int_0^{\pi/2} 4\cos^2(2s)\,ds - \int_0^{\pi/2} \sigma(\dot{\gamma}(s), J_i(\dot{\gamma}(s)))\sin^2(2s)\,ds$$

where $\sigma(\dot{\gamma}(s), J_i(\dot{\gamma}(s)))$ is the sectional curvature determined by two vectors $\dot{\gamma}(s)$ and $J_i(\dot{\gamma}(s))$. Then, $\sum_{i=1}^3 I(X_i, X_i) \geq 0$ gives

$$3\int_0^{\pi/2} 4\cos^2(2s)\,ds \geq \int_0^{\pi/2} \sum_{i=1}^3 \sigma(\dot{\gamma}(s), J_i(\dot{\gamma}(s)))\sin^2(2s)\,ds.$$

It follows from (5.2) of § 5 in [3] that

$$3\int_0^{\pi/2} 4\cos^2(2s)\,ds \geq \frac{3}{4n(n+2)}\int_0^{\pi/2} S\cdot\sin^2(2s)\,ds.$$

The scalar curvature S of (M, g) is a constant function on M because (M, g) is an Einstein space. This inequality therefore gives

$$S \leq 16n(n+2).$$

Now, assume $S = 16n(n+2)$. Then, we have

$$I(X_i, X_i) = 0$$

for each i. This implies that each X_i is a Jacobi field along γ, namely

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}X_i + R(X_i, \dot\gamma)\dot\gamma = 0$$

where R is the curvature tensor on (M, g). Since $\nabla_{\dot\gamma}\nabla_{\dot\gamma}X_i = -4X_i$, it can be easily seen that

$$R(X_i, \dot\gamma)\dot\gamma = 4X_i.$$

Substituting $s = \pi/4$, we obtain

$$\sigma(u, J_i u) = 4$$

for each i. This implies that (M, Q, g) is of constant Q-sectional curvature. From theorem 5.5 in [3] it is therefore easy to see that (M, g) is a locally symmetric space. Moreover, $I(X_i, X_i) = 0$ implies that m' is the first conjugate point of m along γ. Then, because of Blaschke condition we can find that the cut locus and the first conjugate locus of each point coincide and hence M is simply connected. Thus, we see that (M, g) is a globally symmetric space. Since a Blaschke manifold can not contain any totally geodesic flat torus of dimension greater than one, (M, g) is a *CROSS*. Consequently, from theorem 14.51 and table 14.52 in [5] we can find that (M, g) is isometric to $(\boldsymbol{HP}^n, g_0)$.

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