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On decompositions into simple rings. II

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ON DECOMPOSITIONS INTO SIMPLE RINGS. II

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This is a natural sequel to [4]. The notation and terminology employed there will be used here. A ring R is said to be *fully left* (resp. *right*) *idempotent* if every left (resp. right) ideal of R is idempotent.

Theorem. *The following conditions are equivalent:*

(1) $R = \bigoplus_{\lambda \in \Lambda} R_\lambda$, *where* R_λ *is the complete ring of linear transformations of finite rank of a vector space over a division ring.*

(2) R *is a left s-unital semi-prime ring whose left ideals are left annihilators.*

(3) R *is a regular ring whose left ideals are left annihilators.*

(4) R *is a left s-unital semi-prime ring whose maximal left ideals are left annihilators.*

(5) R *is a regular ring whose maximal left ideals are left annihilators.*

(6) R *is a right s-unital, left V-ring whose maximal left ideals are left annihilators.*

(7) R *is a fully left idempotent ring whose maximal left ideals are left annihilators.*

Proof. The equivalence of (1)—(3) has been shown in [4, Theorem 1]. However, we shall prove our theorem without making use of the above equivalence.

The implications (3)$\Longrightarrow$(2)$\Longrightarrow$(4), (3)$\Longrightarrow$(5)$\Longrightarrow$(4) and (7)$\Longrightarrow$(4) are obvious, and (1)$\Longrightarrow$(3) is a direct consequence of [2, Theorem IV. 16. 3].

(4)$\Longrightarrow$(1): We shall prove first that the left singular ideal Z of R is 0. To see this, we assume $Z \neq 0$ and take a left ideal T that is maximal with respect to $Z \cap T = 0$. Then T is contained in $l(Z) = r(Z)$. Since $Z \cap l(Z) = 0$, T must coincide with $l(Z)$. Moreover, we have $Z \oplus T = R$. In fact, if not, by [4, Lemma 1 (a)] and the hypothesis, $Z \oplus T$ is contained in $l(a)$ for some non-zero $a \in R$. This implies a contradiction $a \in r(Z \oplus T) = l(Z) \cap l(T) = T \cap l(T) = 0$. Again by [4, Lemma 1 (a)] and the hypothesis, $T\ (\neq R)$ is contained in a maximal left ideal $l(u)$ where $u \in r(T) = Z$. Since Ru is isomorphic to $R/l(u)$, Ru is a minimal left ideal, which implies a contradiction $Ru = Re$ for some idempotent $e \in Z$. Hence, we have $Z = 0$,

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and then every maximal left ideal of R is a direct summand of ${}_RR$. By [4, Lemma 1 (b)], ${}_RR$ is then completely reducible, and R is a left V-ring. Accordingly, every left ideal of R is a left annihilator. Now, let R_λ be an arbitrary homogeneous component of R. Then, as is well-known, R_λ is a (non-trivial) simple ring and every left ideal of R_λ is a left annihilator in R_λ. Hence, again by [2, Theorem IV. 16. 3], R_λ is the complete ring of linear transformations of finite rank of a vector space over a division ring.

(4) $\Longrightarrow$ (6) : This is almost evident by the proof of (4) $\Longrightarrow$ (1).

(6) $\Longrightarrow$ (7) : Let I be an arbitrary left ideal of R. If $I^2 \neq I$, there exists a maximal left ideal M such that $M \supseteq I^2$ and $M \not\supseteq I$. Now, let a be an arbitrary element of I not contained in M. Then, $a \in aR = a(M+I) \subseteq M + I^2 = M$, a contradiction. Hence, every left ideal of R is idempotent.

Combining our theorem with [4, Corollary 1], we readily obtain

Corollary 1. *The following conditions are equivalent*:

(1) *R is a direct sum of artinian simple rings.*

(2) *R is a left (or right) s-unital semi-prime ring whose maximal left ideals are left annihilators and maximal right ideals are right annihilators.*

(3) *R is a regular ring whose maximal left ideals are left annihilators and maximal right ideals are right annihilators.*

(4) *R is a right s-unital, left V-ring whose maximal left ideals are left annihilators and maximal right ideals are right annihilators.*

(5) *R is a fully left (or right) idempotent ring whose maximal left ideals are left annihilators and maximal right ideals are right annihilators.*

The next is also a combination of our theorem and [4, Corollary 2].

Corollary 2. *Let R be a ring with* 1. *Then the following conditions are equivalent*:

(1) *R is artinian semi-simple.*

(2) *Every maximal left ideal of R is a direct summand of ${}_RR$.*

(3) *R is a left non-singular ring whose essential left ideals are left annihilators.*

(4) *R is a semi-prime ring whose essential left ideals are left annihilators.*

(5) *R is a regular ring whose essential left ideals are left annihilators.*

(6) *R is a semi-prime ring whose maximal left ideals are left annihilators.*

(7) *R has no proper essential left ideals.*

(8) *R is a left V-ring whose maximal left ideals are left annihilators.*

(9) *R is a fully left idempotent ring whose maximal left ideals are left annihilators.*

(2′) — (9′) *The left-right analogues of* (2) — (9).

Remark. A left R-module ${}_RN$ is said to be *s-unital* if for any $u \in N$ there holds $u \in Ru$. Obviously, every irreducible left R-module is s-unital. If R is a left s-unital ring, slightly modifying the proofs of [1, Theorem 1.6] and [3, Theorem 2.1], we can see that the following conditions are equivalent:

(1) Every irreducible left R-module N has the property that for each pair of left s-unital modules ${}_RA$, ${}_RB$ with ${}_RA \subseteq {}_RB$ each $f \in \mathrm{Hom}({}_RA, {}_RN)$ can be extended to an element of $\mathrm{Hom}({}_RB, {}_RN)$.

(2) Every left s-unital module ${}_RN$ has zero radical, i.e., the intersection of all maximal submodules of ${}_RN$ is 0.

(3) R is a left V-ring.

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