Mathematical Journal of Okayama University

Volume 46, *Issue* 1 2004 *Article* 28

JANUARY 2004

Hasse Principle" for Finite p-Groups with Cyclic Subgroups of Index p2

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Math. J. Okayama Univ. **46** (2004), 31–38

"HASSE PRINCIPLE" FOR FINITE p-GROUPS WITH CYCLIC SUBGROUPS OF INDEX p^2

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1. Introduction

Let G be a group. A map $f\colon G \longrightarrow G$ satisfying $f(xy) = f(x)f(y)^x$ for every $x, y \in G$, where $f(y)^x = xf(y)x^{-1}$, is called a cocycle of G. Let f be a cocycle of G. If, for every $x \in G$, there exists $a \in G$ such that $f(x) = a^{-1}a^x$ then f is called a local coboundary, and if there exists $a \in G$ such that $f(x) = a^{-1}a^x$ for every $x \in G$ then f is called a (global) coboundary. G is said to enjoy "Hasse principle" if every local coboundary of G is a coboundary. Abelian groups trivially enjoy "Hasse principle". It is known that a finite group G enjoys "Hasse principle" if and only if every conjugacy preserving automorphism of G is an inner automorphism ([6], Theorem 3.1).

Some types of groups enjoying "Hasse principle" are known ([1], [2], [3], [5], [6], [7], [8], [9]). For finite p-groups, it is known that the following groups enjoy "Hasse principle".

(1) finite p-groups with cyclic subgroups of index p ([1]);
(2) extraspecial p-groups ([1]);
(3) finite p-groups of order p^4 ([2]).

Among the known results, the following are useful for our study:

Theorem 1 ([2])**.** *Metacyclic groups enjoy "Hasse principle".*

Theorem 2 ([3])**.** *Let H be a central subgroup of G. If G/H is generated by xH and yH $(x, y \in G)$ and every element of G/H can be written as $x^r y^s H$, then G enjoys "Hasse principle".*

Recently, M. Kumar and L. R. Vermani [3] proved that for an odd prime p, every non-abelian finite p-group of order p^m having a normal cyclic subgroup of order p^{m-2} but having no element of order p^{m-1} enjoys "Hasse principle". Further they have described that there are fourteen 2-groups (up to isomorphism) of order 2^m of the above type and they showed that twelve of them enjoy "Hasse principle" but remaining two do not enjoy "Hasse principle". In [4], for any prime p, all finite non-abelian p-groups of order p^m having cyclic subgroups of order p^{m-2} but having no element of order p^{m-1} are classified. From the result we see that there is a missing group in a description in [3], which is given by

$$\langle a, b \mid a^{2^{m-2}} = 1,\ b^4 = a^{2^{m-3}},\ b^{-1}ab = a^{-1}\rangle$$

(see [4], Remark 3 (1)). This group is metacyclic, and so enjoys "Hasse principle". Further, two groups given in [3], Theorem 3.4 are isomorphic (see [4], Remark 3 (2)).

In this note we report that every non-abelian p-group of order p^m having a cyclic subgroup of order p^{m-2} but having no normal cyclic subgroup of order p^{m-2} and no element of order p^{m-1} enjoys "Hasse principle". From now on suppose that G is a non-abelian p-group of this type.

(I) For an odd prime p, there are seven possibilities about G. Using notation given in [4], we here list these groups:

$$G_1 = \langle x, y, z \mid x^{p^{m-2}} = 1,\ y^p = z^p = 1,\ xy = yx,\ z^{-1}xz = xy,\ yz = zy\rangle \quad (m \geq 3);$$

$$G_5 = \langle x, y, z \mid x^{p^{m-2}} = 1,\ y^p = z^p = 1,\ xy = yx,\ z^{-1}xz = xy,\ z^{-1}yz = x^{p^{m-3}}y\rangle \quad (m \geq 4);$$

$$G_6 = \langle x, y, z \mid x^{p^{m-2}} = 1,\ y^p = z^p = 1,\ xy = yx,\ z^{-1}xz = xy,\ z^{-1}yz = x^{rp^{m-3}}y\rangle \quad (m \geq 4),$$

where r is a quadratic nonresidue mod p.

$$G_7 = \langle x, y, z \mid x^{p^{m-2}} = 1,\ y^p = z^p = 1,\ y^{-1}xy = x^{1+p^{m-3}},\ z^{-1}xz = xy,\ yz = zy\rangle \quad (m \geq 4);$$

$$G_9 = \langle x, y \mid x^{p^{m-2}} = 1,\ y^{p^2} = 1,\ y^{-1}xy = x^{1+p}\rangle \quad (m \geq 5);$$

$$G_{10} = \langle x, y \mid x^{p^2} = 1,\ x^{p^{p-3}} = y^{p^2},\ y^{-1}xy = x^{1-p}\rangle \quad (m \geq 6);$$

$$G_{11} = \langle x, y, z \mid x^9 = 1,\ y^3 = 1,\ z^3 = x^3,\ xy = yx,\ z^{-1}xz = xy,\ z^{-1}yz = x^6y\rangle$$

By Theorem 2, G_1 enjoys "Hasse principle", and because G_9 and G_{10} are metacyclic by Theorem 1, they also enjoy "Hasse principle".

(II) For $p = 2$, there are twelve possibilities about G. Again, using notation in [4], we list these groups:

$$G_5 = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = z^2 = 1,\ xy = yx,\ z^{-1}xz = xy,\ yz = zy\rangle \quad (m \geq 4);$$

$$G_9 = \langle x, y \mid x^{2^{m-2}} = 1,\ y^4 = 1,\ x^{-1}yx = y^{-1}\rangle \quad (m \geq 5);$$

$$G_{13} = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = z^2 = 1,\ xy = yx,\ z^{-1}xz = x^{-1}y,\ yz = zy\rangle \quad (m \geq 5);$$

$$G_{14} = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = 1,\ z^2 = x^{2^{m-3}},\ xy = yx,\ z^{-1}xz = x^{-1}y,\ yz = zy\rangle \quad (m \geq 5);$$

$$G_{17} = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = z^2 = 1,\ y^{-1}xy = x^{1+2^{m-3}},\ z^{-1}xz = xy,\ yz = zy\rangle \quad (m \geq 5);$$

$$G_{18} = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = 1,\ z^2 = y,\ y^{-1}xy = x^{1+2^{m-3}},\ z^{-1}xz = x^{-1}y\rangle \quad (m \geq 5);$$

$$G_{21} = \langle x, y \mid x^{2^{m-2}} = 1,\ x^{2^{m-3}} = y^4,\ x^{-1}yx = y^{-1}\rangle \quad (m \geq 6);$$

$$G_{22} = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = z^2 = 1,\ xy = yx,\ z^{-1}xz = x^{1+2^{m-4}}y,\ z^{-1}yz = x^{2^{m-3}}y\rangle \quad (m \geq 6);$$

$$G_{23} = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = z^2 = 1,\ xy = yx,\ z^{-1}xz = x^{-1+2^{m-4}}y,\ z^{-1}yz = x^{2^{m-3}}y\rangle \quad (m \geq 6);$$

$$G_{24} = \langle x, y, z \mid x^{2^{m-2}} = 1,\ y^2 = z^2 = 1,\ y^{-1}xy = x^{1+2^{m-3}},\ z^{-1}xz = x^{-1+2^{m-4}}y,\ yz = zy\rangle \quad (m \geq 6);$$

$$G_{25} = \langle x, y, z \mid x^{2^{m-2}} = 1, y^2 = 1,\ z^2 = x^{2^{m-3}},\ y^{-1}xy = x^{1+2^{m-3}},\ z^{-1}xz = x^{-1+2^{m-4}}y,\ yz = zy\rangle \quad (m \geq 6);$$

$$G_{26} = \langle x, y, z \mid x^8 = 1,\ y^2 = 1,\ z^2 = x^4,\ y^{-1}xy = x^5,\ z^{-1}xz = xy,\ yz = zy\rangle$$

By Theorem 2, G_5, G_{13} and G_{14} enjoy "Hasse principle". Because G_9 and G_{21} are metacyclic, they also enjoy "Hasse principle".

In what follows, we denote by $\mathrm{Aut}_c\, G$ and $\mathrm{Inn}\, G$ the set of automorphisms which preserves each conjugacy class of G and the inner automorphism group of G, respectively.

2. The case p odd

In [2], it has been shown that if every $f \in \mathrm{Aut}_c\, G$ that fixes one of the generating elements of G is in $\mathrm{Inn}\, G$, then G enjoys "Hasse principle". Let p be an odd prime. We here show that the groups G_5, G_6, G_7 and G_{11} given in (I) enjoy "Hasse principle".

G_5 and G_6 enjoy "Hasse principle".

Proof. Let $f \in \mathrm{Aut}_c\, G_5$ such that $f(z) = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_5$ with $0 \leq i, r < p^{m-2}$, $0 \leq j, k, s, t < p$ such that $f(x) = a^{-1}xa, f(y) = b^{-1}yb$, and so

$$f(x) = z^{-k}y^{-j}x^{-i} \cdot x \cdot x^i y^j z^k = z^{-k}xz^k,$$
$$f(y) = z^{-t}y^{-s}x^{-r} \cdot y \cdot x^r y^s z^t = z^{-t}yz^t.$$

As $z^{-1}xz = xy$ and $z^{-1}yz = x^{p^{m-3}}y$ we have

$$z^{-k}xz^k = x^{1+(1+2+\cdots+(k-1))p^{m-3}}y^k = x^{1+\frac{k(k-1)}{2}p^{m-3}}y^k.$$

We also have $z^{-t}yz^t = x^{tp^{m-3}}y$. Therefore $f(x) = x^{1+\frac{k(k-1)}{2}p^{m-3}}y^k$, $f(y) = x^{tp^{m-3}}y$. Since f is an automorphism,

$$f(z)^{-1}f(x)f(z) = f(z^{-1}xz) = f(xy) = f(x)f(y).$$

We have

$$\begin{aligned}
f(z)^{-1}f(x)f(z) &= z^{-1}(x^{1+\frac{k(k-1)}{2}p^{m-3}}y^k)z \\
&= (z^{-1}xz)^{1+\frac{k(k-1)}{2}p^{m-3}}(z^{-1}yz)^k \\
&= x^{1+(k+\frac{k(k-1)}{2})p^{m-3}}y^{1+k}, \\
f(x)f(y) &= x^{1+\frac{k(k-1)}{2}p^{m-3}}y^k x^{tp^{m-3}}y \\
&= x^{1+(t+\frac{k(k-1)}{2})p^{m-3}}y^{1+k}.
\end{aligned}$$

Therefore the following congruence holds:

$$1+\Big(k+\frac{k(k-1)}{2}\Big)p^{m-3} \equiv 1+\Big(t+\frac{k(k-1)}{2}\Big)p^{m-3} \pmod{p^{m-2}}.$$

From this it follows that $k \equiv t \pmod{p}$. Then because $0 \le k, t < p$, we have $k = t$. Thus we have $f(x) = z^{-k}xz^k$, $f(y) = z^{-k}xz^k$, $f(z) = z^{-k}zz^k$. This shows that $f \in \operatorname{Inn} G_5$, and so G_5 enjoys "Hasse principle". By an analogous argument we can show that G_6 enjoys "Hasse principle". □

In the rest of the paper, we proceed with a similar argument as above. Given $f \in \operatorname{Aut}_c G$, the image $f(g)$ of $g \in G$ will be denoted by $\overline{g}$.

G_7 enjoys "Hasse principle".

Proof. Let $f \in \operatorname{Aut}_c G_7$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_7$ with $0 \le i, r < p^{m-2}$, $0 \le j, k, s, t < p$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have $\overline{x} = x^{1+jp^{m-3}}y^k$, $\overline{y} = x^{-rp^{m-3}}y$, $\overline{z} = z$. Since f is an automorphism, $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y}$. Because $\overline{x}\,\overline{y} = x^{1+(j-r)p^{m-3}}y^{k+1}$, $\overline{z}^{-1}\overline{x}\,\overline{z} = x^{1+jp^{m-3}}y^{k+1}$, we have

$$\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y} \iff rp^{m-3} \equiv 0 \pmod{p^{m-2}}.$$

Thus we have $\overline{x} = x^{1+jp^{m-3}}y^k, \overline{y} = y, \overline{z} = z$. Therefore setting $u = y^j z^k$, we have

$$f(x) = u^{-1}xu, \quad f(y) = u^{-1}yu, \quad f(z) = u^{-1}zu,$$

and so $f \in \operatorname{Inn} G_7$. □

G_{11} enjoys "Hasse principle".

Proof. Let $f \in \mathrm{Aut}_c G_{11}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{11}$ with $0 \le i, r < 9$, $0 \le j, k, s, t < 3$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have $\overline{x} = x^{1+3k(k-1)}y^k$, $\overline{y} = x^{6t}y$, $\overline{z} = z$. Since f is an automorphism, $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y}$. Because $\overline{x}\,\overline{y} = x^{1+6t+3k(k-1)}y^{k+1}$, $\overline{z}^{-1}\overline{x}\,\overline{z} = x^{1+6k+3k(k-1)}y^{k+1}$, we have $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y} \Longleftrightarrow k = t$. Thus we have $\overline{x} = x^{1+3k(k-1)}y^k$, $\overline{y} = x^{6k}y$, $\overline{z} = z$. Therefore setting $u = z^k$, we have

$$f(x) = u^{-1}xu, \quad f(y) = u^{-1}yu, \quad f(z) = u^{-1}zu,$$

and so $f \in \mathrm{Inn}\, G_{11}$. □

3. The case $p = 2$

We here show that the groups G_{17}, G_{18}, G_{22}, G_{24}, G_{25} and G_{26} given in (II) enjoy "Hasse principle".

G_{17} *enjoys "Hasse principle".*

Proof. Let $f \in \mathrm{Aut}_c G_{17}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{17}$ with $0 \le i, r < 2^{m-2}$, $0 \le j, k, s, t < 2$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have $\overline{x} = x^{1+j2^{m-3}}y^k$, $\overline{y} = x^{r2^{m-3}}y$, $\overline{z} = z$. Since f is an automorphism, we have $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y}$. Because $\overline{x}\,\overline{y} = x^{1+(j+r)2^{m-3}}y^{k+1}$, $\overline{z}^{-1}\overline{x}\,\overline{z} = x^{1+j2^{m-3}}y^{k+1}$, $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y} \Longleftrightarrow r \equiv 0 \pmod 2$. Thus we have $\overline{x} = x^{1+j2^{m-3}}y^k$, $\overline{y} = y$, $\overline{z} = z$. Therefore setting $u = y^j z^k$, we have $\overline{x} = u^{-1}xu$, $\overline{y} = u^{-1}yu$, $\overline{z} = u^{-1}zu$, and so $f \in \mathrm{Inn}\, G_{17}$. □

G_{18} *enjoys "Hasse principle".*

Proof. Let $f \in \mathrm{Aut}_c G_{18}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{18}$ with $0 \le i, r < 2^{m-2}$, $0 \le j, k, s, t < 2$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have $\overline{x} = x^{j2^{m-3}+(-1)^k(1+k2^{m-4})+k^2 2^{m-4}}y^k$, $\overline{y} = x^{r2^{m-3}}y$, $\overline{z} = z$. Since f is an automorphism, we have $\overline{z}^2 = \overline{y}$. Because $y = \overline{z}^2 = x^{r2^{m-3}}y$, $\overline{z}^2 = \overline{y} \Longleftrightarrow r \equiv 0 \pmod 2$. Thus we have $\overline{x} = x^{j2^{m-3}+(-1)^k(1+k2^{m-4})+k^2 2^{m-4}}y^k$, $\overline{y} = y$, $\overline{z} = z$. Therefore setting $u = y^j z^k$, we have $\overline{x} = u^{-1}xu$, $\overline{y} = u^{-1}yu$, $\overline{z} = u^{-1}zu$, and so $f \in \mathrm{Inn}\, G_{18}$. □

G_{22} *enjoys "Hasse principle".*

Proof. Let $f \in \mathrm{Aut}_c G_{22}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{22}$ with $0 \le i, r < 2^{m-2}$, $0 \le j, k, s, t < 2$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have $\overline{x} = x^{1+k2^{m-4}}y^k$, $\overline{y} = x^{t2^{m-3}}y$, $\overline{z} = z$. Since f is an automorphism, we have $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{1+2^{m-4}}\,\overline{y}$. Because

$$\overline{x}^{1+2^{m-4}}\,\overline{y} = x^{1+t2^{m-3}+(1+k)2^{m-4}}y^{1+k}, \quad \overline{z}^{-1}\overline{x}\,\overline{z} = x^{1+k2^{m-3}+(1+k)2^{m-4}}y^{1+k},$$

$$\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{1+2^{m-4}}\,\overline{y} \Longleftrightarrow k = t.$$

Thus we have $\overline{x} = x^{1+k2^{m-4}}y^k$, $\overline{y} = x^{k2^{m-3}}y$, $\overline{z} = z$. Therefore setting $u = z^k$, we have $\overline{x} = u^{-1}xu$, $\overline{y} = u^{-1}yu$, $\overline{z} = u^{-1}zu$, and so $f \in \operatorname{Inn} G_{22}$. □

G_{23} enjoys "Hasse principle".

Proof. Let $f \in \operatorname{Aut}_c G_{23}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{23}$ with $0 \le i, r < 2^{m-2}$, $0 \le j, k, s, t < 2$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have

$$\overline{x} = \begin{cases} x & (k=0) \\ x^{-1+2^{m-4}}y & (k=1) \end{cases}, \quad \overline{y} = x^{t2^{m-3}}y, \quad \overline{z} = z.$$

Since f is an automorphism, we have $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\,\overline{y}$. If $k = 0$,

$$\overline{x}^{-1+2^{m-4}}\,\overline{y} = x^{-1+2^{m-4}+t2^{m-3}}y, \quad \overline{z}^{-1}\overline{x}\,\overline{z} = x^{-1+2^{m-4}}y.$$

Therefore $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\,\overline{y} \iff t = 0$. If $k = 1$,

$$\overline{x}^{-1+2^{m-4}}\,\overline{y} = x^{1+(t-1)2^{m-3}}, \quad \overline{z}^{-1}\overline{x}\,\overline{z} = x.$$

Therefore $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\,\overline{y} \iff t = 1$. Thus we have

$$\overline{x} = \begin{cases} x & (k=0) \\ x^{-1+2^{m-4}}y & (k=1) \end{cases}, \quad \overline{y} = x^{k2^{m-3}}y, \quad \overline{z} = z.$$

Therefore setting $u = z^k$, we have $\overline{x} = u^{-1}xu$, $\overline{y} = u^{-1}yu$, $\overline{z} = u^{-1}zu$, and so $f \in \operatorname{Inn} G_{23}$. □

G_{24} enjoys "Hasse principle".

Proof. Let $f \in \operatorname{Aut}_c G_{24}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{24}$ with $0 \le i, r < 2^{m-2}$, $0 \le j, k, s, t < 2$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have

$$\overline{x} = \begin{cases} x^{1+j2^{m-3}} & (k=0) \\ x^{-1+2^{m-4}+j2^{m-3}}y & (k=1) \end{cases}, \quad \overline{y} = x^{r2^{m-3}}y, \quad \overline{z} = z.$$

Since f is an automorphism, we have $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\,\overline{y}$. If $k = 0$,

$$\overline{x}^{-1+2^{m-4}}\,\overline{y} = x^{-1+2^{m-4}+(r-j)2^{m-3}}y, \quad \overline{z}^{-1}\overline{x}\,\overline{z} = x^{-1+2^{m-4}+j2^{m-3}}y.$$

Therefore $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\,\overline{y} \iff r \equiv 0 \pmod 2$. If $k = 1$,

$$\overline{x}^{-1+2^{m-4}}\,\overline{y} = x^{1+(r-j)2^{m-3}} = x^{1+(r+j)2^{m-3}}, \quad \overline{z}^{-1}\overline{x}\,\overline{z} = x^{j2^{m-3}}x = x^{1+j2^{m-3}}.$$

Therefore $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\,\overline{y} \iff r \equiv 0 \pmod 2$. Thus we have

$$\overline{x} = \begin{cases} x^{1+j2^{m-3}} & (k=0) \\ x^{-1+2^{m-4}+j2^{m-3}}y & (k=1) \end{cases}, \quad \overline{y} = y, \quad \overline{z} = z.$$

Therefore setting $u = y^j$, we have $\overline{x} = u^{-1}xu$, $\overline{y} = u^{-1}yu$, $\overline{z} = u^{-1}zu$, and so $f \in \operatorname{Inn} G_{24}$. □

G_{25} *enjoys "Hasse principle".*

Proof. Let $f \in \operatorname{Aut}_c G_{25}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{25}$ with $0 \le i, r < 2^{m-2}$, $0 \le j, k, s, t < 2$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have

$$\overline{x} = \begin{cases} x^{1+j2^{m-3}} & (k=0) \\ x^{-1+2^{m-4}+j2^{m-3}}y & (k=1) \end{cases}, \quad \overline{y} = x^{r2^{m-3}}y, \quad \overline{z} = z.$$

Since f is an automorphism, we have $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\overline{y}$. If $k = 0$,

$$\overline{x}^{-1+2^{m-4}}\overline{y} = x^{-1+2^{m-4}+(j+r)2^{m-3}}y, \quad \overline{z}^{-1}\overline{x}\,\overline{z} = x^{-1+2^{m-4}+j2^{m-3}}y.$$

Therefore $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\overline{y} \iff r \equiv 0 \pmod 2$. If $k = 1$,

$$\overline{x}^{-1+2^{m-4}}\overline{y} = x^{1+(r-j)2^{m-3}} = x^{1+(r+j)2^{m-3}}, \quad \overline{z}^{-1}\overline{x}\,\overline{z} = x^{1+j2^{m-3}}.$$

Therefore $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}^{-1+2^{m-4}}\overline{y} \iff r \equiv 0 \pmod 2$. Thus we have

$$\overline{x} = \begin{cases} x^{1+j2^{m-3}} & (k=0) \\ x^{-1+2^{m-4}+j2^{m-3}}y & (k=1) \end{cases}, \quad \overline{y} = y, \quad \overline{z} = z.$$

Therefore setting $u = y^j$, we have $\overline{x} = u^{-1}xu$, $\overline{y} = u^{-1}yu$, $\overline{z} = u^{-1}zu$, and so $f \in \operatorname{Inn} G_{25}$. □

G_{26} *enjoys "Hasse principle".*

Proof. Let $f \in \operatorname{Aut}_c G_{26}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{26}$ with $0 \le i, r < 8$, $0 \le j, k, s, t < 2$ such that $\overline{x} = a^{-1}xa$, $\overline{y} = b^{-1}yb$. We then have $\overline{x} = x^{1+4j}y^k$, $\overline{y} = x^{4r}y$, $\overline{z} = z$. Since f is an automorphism, we have $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y}$. Because $\overline{x}\,\overline{y} = x^{1+4j+4r}y^{1+k}$, $\overline{z}^{-1}\overline{x}\,\overline{z} = x^{1+4j}y^{1+k}$, $\overline{z}^{-1}\overline{x}\,\overline{z} = \overline{x}\,\overline{y} \iff r \equiv 0 \pmod 2$. Thus we have $\overline{x} = x^{1+4j}y^k$, $\overline{y} = y$, $\overline{z} = z$. Therefore setting $u = y^j z^k$, we have $\overline{x} = u^{-1}xu$, $\overline{y} = u^{-1}yu$, $\overline{z} = u^{-1}zu$, and so $f \in \operatorname{Inn} G_{26}$. □

Acknowledgement. The authors thank the referee for his or her comments.

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(*Received December 2, 2003*)