

Mathematical Journal of Okayama University

Volume 24, Issue 1

1982

Article 6

JUNE 1982

On the iterated Samelson product

Hideyuki Kachi*

*Shinshu University

Copyright ©1982 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

ON THE ITERATED SAMELSON PRODUCT

HIDEYUKI KACHI

1. Introduction. Let G be a topological group. The Samelson product $\langle \ , \ \rangle$ of G is a pairing

$$\pi_p(G) \times \pi_q(G) \longrightarrow \pi_{p+q}(G) \quad (p, q \geq 1)$$

defined as follows. Let $\alpha \in \pi_p(G)$, $\beta \in \pi_q(G)$ be represented by maps

$$f : (I^p, \dot{I}^p) \longrightarrow (G, e), \quad g : (I^q, \dot{I}^q) \longrightarrow (G, e).$$

Then $\langle \alpha, \beta \rangle \in \pi_{p+q}(G)$ is defined to be the element represented by the map

$$k : (I^p \times I^q, I^p \times \dot{I}^q \cup \dot{I}^p \times I^q) \longrightarrow (G, e),$$

where $k(x, y) = f(x)g(y)f(x)^{-1}g(y)^{-1}$ for $x \in I^p$ and $y \in I^q$.

With each element $\lambda \in \pi_1(G)$ we associate the operator

$$D_\lambda : \pi_r(G) \longrightarrow \pi_{r+1}(G)$$

defined by taking the Samelson product with λ . From the Jacobi identity, each of these operators constitutes a derivation, with respect to the Samelson product in $\pi_*(G)$. Suppose that $\pi_2(G) = 0$, as in the case when G is a Lie group. Then $2D_\lambda^2 = 0$; moreover, there is some evidence to support the following

Conjecture (I. M. James [6]). For some value of s , depending on λ but not on r , the operator

$$D_\lambda^s : \pi_r(G) \longrightarrow \pi_{r+s}(G),$$

defined by iteration of D_λ , is trivial.

Note that $D_\lambda = 0$ if λ can be represented by a loop within the center of G or G is commutative. In [5], James proved that the conjecture is true for the rotation group R_t and the generator of $\pi_1(R_t) \cong \mathbf{Z}_2$.

In this paper we show the corresponding result for the unitary group and the generator of its fundamental group by making use of the relation given by Bott [2] and Husseini [3].

Our result can be stated as follows;

Theorem 3.1. *Let $U(t)$ be the unitary group and μ the generator*

of $\pi_1(U(t)) \cong \mathbf{Z}$. Then the iterated operator

$$D_\mu^s : \pi_r(U(t)) \longrightarrow \pi_{r+s}(U(t))$$

is trivial for any r , where (i) $s = 2$ for t odd, (ii) $s = 5$ for $t \equiv 0 \pmod{8}$ and (iii) $s = 4$ otherwise.

Of course D_μ is trivial when $t = 1$. We can show that $D_\mu^3 : \pi_{2t}(U(t)) \longrightarrow \pi_{2t+3}(U(t))$ is non-trivial for $t \equiv 0 \pmod{8}$. But the author does not know of any example that D_μ^4 is non-trivial.

We also give an analogous result for the symplectic group $Sp(t)$ and the generator of $\pi_3(Sp(t))$ in §3. In §4 and §5, we study the relative Samelson product.

2. Preliminary. Let F denote the field of complex numbers C or the field of quaternions H . By F^n we denote an n -dimensional right vector space over F with a fixed basis and the usual inner product. The group of automorphisms of F^n which preserve this inner product is denoted by $G_n = G(F^n)$. As usual, $G(C^n) = U(n)$, $G(H^n) = Sp(n)$. Let $G_{n,k}$ be one of the complex Stiefel manifold $W_{n,k} = U(n)/U(n-k)$ or the symplectic Stiefel manifold $X_{n,k} = Sp(n)/Sp(n-k)$, depending on the field F . We denote the natural projection by $p : G_n \longrightarrow G_{n,k}$. The inclusion $i : G_n \longrightarrow G_{n+m}$ is the mapping which leaves the last m basic vectors fixed.

Let $k \geq 1$ and $m, n \geq k$. The intrinsic join operation, as defined in [4], is a pairing of $\pi_i(G_{n,k})$ with $\pi_j(G_{m,k})$ to $\pi_{i+j+1}(G_{n+m,k})$, where $i, j \geq 0$. The intrinsic join of $\alpha \in \pi_i(G_{n,k})$ with $\beta \in \pi_j(G_{m,k})$ is denoted by $\alpha * \beta$. This pairing is both bilinear and associative. Consider the case $k = 1$. The $G_{n,1}$ is homeomorphic to S^{dn-1} , where d is the dimension of F over the real field.

Lemma 2.1. *The homomorphism from $\pi_i(G_{m,1})$ to $\pi_{i+dn}(G_{n+m,1})$ defined by taking the intrinsic join with the generator of $\pi_{dn-1}(G_{n,1})$ is essentially the same as the dn -fold iterated suspension homomorphism.*

Bott [2] and Husseini [3] obtained a useful relation between the intrinsic join and the Samelson product:

Theorem 2.2. *For any $\alpha \in \pi_i(G_n)$ and $\beta \in \pi_j(G_m)$,*

$$\langle i_*\alpha, i_*\beta \rangle = \pm \partial((p_*\alpha) * (p_*\beta)),$$

as shown in the following diagram;

$$\begin{array}{ccc}
 & \pi_i(G_n) \times \pi_j(G_m) & \\
 p_* \times p_* \swarrow & & \searrow i_* \times i_* \\
 \pi_i(G_{n,k}) \times \pi_j(G_{m,k}) & & \pi_i(G_{n+m-k}) \times \pi_j(G_{n+m-k}) \\
 \downarrow * & & \downarrow \langle \cdot, \cdot \rangle \\
 \pi_{i-j+1}(G_{n-m,k}) & \xrightarrow{\partial} & \pi_{i+j}(G_{n+m-k})
 \end{array}$$

where ∂ denotes the boundary homomorphism in the homotopy exact sequence of the fiber space $p: G_{n+m} \longrightarrow G_{n+m,k}$ with fiber G_{n+m-k} .

3. The proof of the main Theorem.

Theorem 3.1. *Let $U(m)$ be the unitary group and μ be the generator of $\pi_1(U(m)) \cong \mathbf{Z}$. Then the iterated operator*

$$D_\mu^s: \pi_r(U(m)) \longrightarrow \pi_{r+s}(U(m))$$

is trivial for any r , where (i) $s = 2$ for m odd, (ii) $s = 5$ for $m \equiv 0 \pmod{8}$ and (iii) $s = 4$ otherwise.

Proof. Let $\mu' \in \pi_1(U(1))$ be a generator which satisfies $\mu = i_*\mu'$ for the inclusion $i: U(1) \longrightarrow U(m)$. Apply Theorem 2.2 for $\mu' \in \pi_1(U(1))$ and any $\beta \in \pi_r(U(m))$. Since $U(1)$ is identified with $W_{1,1}$, we have from Lemma 2.1

$$\begin{aligned}
 D_\mu(\beta) &= \langle i_*\mu', \beta \rangle \\
 &= \pm \partial E^2(p_*\beta) \in \pi_{r+1}(U(m)),
 \end{aligned}$$

where E is the suspension homomorphism. Hence

$$\begin{aligned}
 D_\mu^2(\beta) &= \langle \mu, \langle \mu, \beta \rangle \rangle = \pm D_\mu(\partial E^2(p_*\beta)) \\
 &= \pm \partial E^2(p_*\partial E^2(p_*\beta)).
 \end{aligned}$$

Now the composition

$$p_*\partial: \pi_{r+2}(S^{2m+1}) \longrightarrow \pi_{r+1}(U(m)) \longrightarrow \pi_{r-1}(S^{2m-1})$$

is the boundary homomorphism \mathcal{A} in the exact sequence of the fiber space $W_{m+1,2}$ over $W_{m+1,1} = S^{2m+1}$ with fiber $W_{m,1} = S^{2m-1}$. Then, as well known results,

$$\mathcal{A}(\iota_{2m+1}) = \begin{cases} 0 & \text{for } m \text{ odd} \\ \eta_{2m-1} & \text{for } m \text{ even} \end{cases}$$

where $\pi_{2m}(S^{2m-1}) = \{\eta_{2m-1}\} \cong \mathbf{Z}_2$ and $\pi_{2m+1}(S^{2m+1}) = \{\iota_{2m+1}\} \cong \mathbf{Z}$. Therefore we have

$$\begin{aligned} D_{\mu}^2(\beta) &= \pm \partial E^2(\mathcal{A}(E^2(p_*\beta))) \\ &= \pm \partial E^2(\mathcal{A}(\iota_{2m+1}) \circ E(p_*\beta)) \\ &= \begin{cases} 0 & \text{for } m \text{ odd} \\ \partial(\eta_{2m-1} \circ E^3(p_*\beta)) & \text{for } m \text{ even} \end{cases} \end{aligned}$$

Then, after two more steps, we obtain

$$\begin{aligned} D_{\mu}^4(\beta) &= \partial(\eta_{2m+1}^3 \circ E^5(p_*\beta)) \\ &= (\partial\eta_{2m+1}^3) \circ (E^4(p_*\beta)) \end{aligned}$$

for m even.

By Matsunaga [8], $\pi_{2m-3}(U(m))$ is generated by $\partial\nu_{2m+1}$ and its 2-primary components

$${}^2\pi_{2m-3}(U(m)) \cong \begin{cases} \mathbf{Z}_2 & \text{if } m \equiv 2 \pmod{4} \\ \mathbf{Z}_4 & \text{if } m \equiv 4 \pmod{8} \\ \mathbf{Z}_8 & \text{if } m \equiv 0 \pmod{8}, \end{cases}$$

where $\pi_{2m+4}(S^{2m+1}) = \{\nu_{2m+1}\} \cong \mathbf{Z}_{2^4}$ ($m \geq 2$). Since $\eta_{2m+1}^3 = 12\nu_{2m+1}$ it follows that

$$D_{\mu}^4(\beta) = 0$$

for $m \equiv 2, 4, 6 \pmod{8}$. After one more step, we have

$$D_{\mu}^5(\beta) = (\partial\eta_{2m+1}^4) \circ E^5(p_*\beta).$$

Thus we have

$$D_{\mu}^5(\beta) = 0$$

for $m \equiv 0 \pmod{8}$, since $\eta_{2m+1}^4 = 0$. Q.E.D.

Example. Let α_m be a generator of $\pi_{2m}(U(m)) \cong \mathbf{Z}_{m!}$. Specifically, we shall take α_m to be $\alpha_m = \partial\iota_{2m+1}$. Then we have

$$\begin{aligned} D_{\mu}(\alpha_{2m}) &= \alpha_{2m} \circ \eta_{4m} \neq 0 \\ D_{\mu}^2(\alpha_{2m}) &= \alpha_{2m} \circ \eta_{4m}^2 \neq 0 \\ D_{\mu}^3(\alpha_{2m}) &= \alpha_{2m} \circ \eta_{4m}^3 \quad (\text{If } m \equiv 0 \pmod{4} \text{ then } \alpha_{2m} \circ \eta_{4m}^3 \neq 0) \end{aligned}$$

and

$$D_{\mu}^4(\alpha_{2m}) = D_{\mu}(\alpha_{2m+1}) = 0.$$

Let

$$D_{\tau} : \pi_r(Sp(m)) \longrightarrow \pi_{r+3}(Sp(m))$$

be the operator defined by taking the Samelson product with the generator $\tau \in \pi_3(Sp(m)) \cong \mathbf{Z}$. Then we have

Theorem 3.2. *The iterated operator*

$$D_r^s : \pi_r(Sp(m)) \longrightarrow \pi_{r+3s}(Sp(m))$$

of D_r is trivial for any r , where (i) $s = 2$ if $m \equiv -1 \pmod{24}$, (ii) $s = 3$ if m is other odd and (iii) $s = 5$ if m is even.

Proof. $Sp(1)$ and $X_{1,1}$ are identified with S^3 . For the generator τ' of $\pi_3(Sp(1))$, the generator τ of $\pi_3(Sp(m))$ satisfies $\tau = i_*\tau'$ for the inclusion $i : Sp(1) \longrightarrow Sp(m)$. From Theorem 2.2 and Lemma 2.1, it follows that

$$(3.3) \quad D_r(\beta) = \langle i_*\tau', \beta \rangle = \pm \partial E^4(p_*\beta)$$

for any $\beta \in \pi_r(Sp(m))$.

Now the composition

$$p_*\partial : \pi_{r+4}(S^{4m-3}) \longrightarrow \pi_{r+3}(Sp(m)) \longrightarrow \pi_{r+3}(S^{4m-1})$$

is the boundary homomorphism Δ in the exact sequence of the fiber space $X_{m+1,2}$ over $X_{m+1,1} = S^{4m-3}$ with fiber $X_{m,1} = S^{4m-1}$.

Then we have

$$\Delta(\iota_{4m+3}) = \begin{cases} \omega & \text{for } m = 1 \\ ((m+1)\nu_{4m-1}) & \text{for } m \geq 2 \end{cases}$$

where $\pi_6(S^3) = \{\omega\} \cong \mathbf{Z}_{12}$. Thus we obtain

$$\begin{aligned} D_r^2(\beta) &= \pm \partial E^4(p_*\partial E^4(p_*\beta)) \\ &= \begin{cases} \pm \partial((E^4\omega) \circ (E^7 p_*\beta)) & \text{for } m = 1 \\ \pm \partial((m+1)\nu_{4m+3} \circ (E^7 p_*\beta)) & \text{for } m \geq 2. \end{cases} \end{aligned}$$

Thus $D_r^2(\beta) = 0$ for $m+1 \equiv 0 \pmod{24}$.

On iterating (3.3), we get

$$(3.4) \quad D_r^3(\beta) = \pm \partial((m+1)^2 \nu_{4m-3}^2 \circ E^{10} p_*\beta).$$

Since $2\nu_{4m+3}^2 = 0$, it follows from (3.4) that $D_r^3(\beta) = 0$ for m odd. After two more steps, we have $D_r^5(\beta) = 0$, since $\nu_{4m+3}^4 = 0$. Q.E.D.

Corollary 3.5 (Arkowitz-Curjel [1]). *If $1 \in \pi_3(S^3)$ is the homotopy class of the identity map, then $\langle \langle \langle 1, 1 \rangle, 1 \rangle, 1 \rangle = 0 \in \pi_{12}(S^3)$.*

4. The relative Samelson product. The definition and the material in this section are due to James [6]. Let H be a subgroup of the topological group G . The relative Samelson product $\langle \ , \ \rangle$ is a pairing

$$\pi_p(H) \times \pi_q(G, H) \longrightarrow \pi_{p+q}(G, H) \quad (p \geq 1, q \geq 2)$$

defined as follows. Let $\alpha \in \pi_p(H)$, $\beta \in \pi_q(G, H)$ be represented by maps

$$f : (I^p, \dot{I}^p) \longrightarrow (H, e), \quad g : (I^q, \dot{I}^q) \longrightarrow (G, H).$$

Then $\langle \alpha, \beta \rangle \in \pi_{p+q}(G, H)$ is represented by the map

$$k : (I^p \times I^q, I^p \times \dot{I}^q \cup \dot{I}^p \times I^q) \longrightarrow (G, H),$$

where $k(x, y) = f(x)g(y)f(x)^{-1}g(y)^{-1}$ for $x \in I^p$, $y \in I^q$.

The main relations between the ordinary and relative Samelson product are indicated in the following diagram :

$$(4.1) \quad \begin{array}{ccc} \pi_p(H) \times \pi_q(G, H) & \xrightarrow{\langle \cdot, \cdot \rangle} & \pi_{p+q}(G, H) \\ \downarrow 1 \times \partial & & \downarrow \partial \\ \pi_p(H) \times \pi_{q-1}(H) & \xrightarrow{\langle \cdot, \cdot \rangle} & \pi_{p+q-1}(H) \end{array}$$

$$(4.2) \quad \begin{array}{ccc} & \pi_p(H) \times \pi_q(G) & \\ i_* \times 1 \swarrow & & \searrow 1 \times j_* \\ \pi_p(G) \times \pi_q(G) & & \pi_p(H) \times \pi_q(G, H) \\ \downarrow \langle \cdot, \cdot \rangle & j_* \longrightarrow & \downarrow \langle \cdot, \cdot \rangle \\ \pi_{p+q}(G) & & \pi_{p+q}(G, H) \end{array}$$

The homomorphism i_* , j_* , ∂ , of course, are from the homotopy exact sequence of the pair (G, H) and the diagrams are commutative up to sign. We see from this that an element $\gamma \in \pi_r(H)$ determines a homomorphism of the homotopy exact sequence into itself raising dimension by r . On $\pi_*(H)$ we take the ordinary Samelson product with γ itself, on $\pi_*(G)$ the ordinary Samelson product with $i_*(\gamma)$, and on $\pi_*(G, H)$ the relative Samelson product with γ itself. And we denote by $D_{H,r}$, $D_{G,r}$ and $D_{G|H,r}$ each Samelson product respectively.

Lemma 4.3. *If $D_{H,r}^{\xi} = 0$ and $D_{G,r}^{\xi} = 0$, then $D_{G|H,r}^{\xi} = 0$.*

Proof. If $\alpha \in \pi_p(G, H)$, then $\partial D_{G|H,r}^{\xi}(\alpha) = \pm D_{H,r}^{\xi}(\partial\alpha) = 0$ by (4.1). Hence $D_{G|H,r}^{\xi}(\alpha) = j_*(\epsilon)$, by exactness, for some $\epsilon \in \pi_{p+r}(G)$. Thus, from (4.2), $D_{G|H,r}^{\xi}(\alpha) = D_{G|H,r}^{\xi}(j_*\epsilon) = \pm j_* D_{G,r}^{\xi}(\epsilon) = 0$. Q.E.D.

Proposition 4.4. *For the pair (G, H) and $\gamma \in \pi_r(H)$ in the following table, there exists an integer s for which the s -fold iterated operator $D_{G|H,r}^{\xi}$ of $D_{G|H,r} : \pi_p(G, H) \longrightarrow \pi_{p+r}(G, H)$ is trivial for any $p \geq 2$:*

(G, H)	$\gamma \in \pi_r(H)$	s
(i) $(U(n+k), U(n))$	γ is a generator of $\pi_1(U(n))$	10
(ii) $(R_{2n}, U(n))$	γ is a generator of $\pi_1(U(n))$	11
(iii) $(Sp(n+k), Sp(n))$	γ is a generator of $\pi_3(Sp(n))$	10.

Proof. Let γ be a generator of $\pi_r(H)$. Then the image of γ in $\pi_r(G)$ is a generator of $\pi_r(G)$. Thus from Lemma 4.3 and Theorems 3.1, 3.2, we obtain the results. Q.E.D.

5. The relative Samelson product on $(Sp(n), U(n))$. Consider the pair $(Sp(n), U(n))$. By the Bott periodicity, $\pi_{2m-1}(U(n+m))$ is infinite cyclic group for $n \geq 0$ and $\pi_{2n}(Sp(n+m), U(n+m))$ is infinite cyclic group for $m \geq 0$ if $n \equiv 1$ or $3 \pmod{4}$. The boundary homomorphism $\partial: \pi_{2n}(Sp(n+m), U(n+m)) \rightarrow \pi_{2n-1}(U(n+m))$ is an isomorphism for $n \equiv 1 \pmod{4}$ and maps a generator onto twice a generator for $n \equiv 3 \pmod{4}$. For the first non-stable range, we have

Lemma 5.1 (See [7]). *The following sequence*

$$0 \longrightarrow \pi_{2t+1}(Sp(t)) \xrightarrow{j_*} \pi_{2t+1}(Sp(t), U(t)) \xrightarrow{\partial} \pi_{2t}(U(t)) \longrightarrow 0$$

is exact and

$$\pi_{2t+1}(Sp(t), U(t)) = \begin{cases} \mathbf{Z}_{t!} & \text{if } t \equiv 0 \pmod{4} \\ \mathbf{Z} + \mathbf{Z}_2 & \text{if } t \equiv 1 \pmod{4} \\ \mathbf{Z}_{2 \times t!} & \text{if } t \equiv 2 \pmod{4} \\ \mathbf{Z} & \text{if } t \equiv 3 \pmod{4}. \end{cases}$$

We apply the diagram (4.1) to the pair $(Sp(n+m), U(n+m))$. Then, from Theorem 1 of Bott [2], we obtain

Proposition 5.2. *Let $m, n \geq 1$ with $n \equiv 1$ or $3 \pmod{4}$. Consider the relative Samelson product*

$$\langle \phi_m, \zeta_n \rangle \in \pi_{2m+2n+1}(Sp(n+m), U(n+m)),$$

where $\phi_m \in \pi_{2m+1}(U(n+m))$ and $\zeta_n \in \pi_{2n}(Sp(n+m), U(n+m))$ are generators. Let $\xi \in \pi_{2m+2n+1}(Sp(n+m), U(n+m))$ be a generator such that $\partial \xi$ is a generator of $\pi_{2n+2m}(U(n+m))$, then

$$\langle \phi_m, \zeta_n \rangle = \begin{cases} m!(n-1)! \xi & \text{mod image } j_* \text{ if } n \equiv 1 \pmod{4} \\ 2(m!(n-1)!) \xi & \text{mod image } j_* \text{ if } n \equiv 3 \pmod{4}. \end{cases}$$

where $j_*: \pi_{2n+2m+1}(Sp(n+m)) \rightarrow \pi_{2n+2m+1}(Sp(n+m), U(n+m))$.

By taking various n and m we obtain examples of non-trivial relative Samelson products in the case of $(Sp(t), U(t))$. Hence we deduce

Corollary 5.3. *If $t \geq 2$, then $U(t)$ is not homotopy normal in $Sp(t)$ in the sense of McCarty [9].*

REFERENCES

- [1] M. ARKOWITZ and C. R. CURJEL: Some properties of the exotic multiplications on the three sphere, *Quart. J. of Math.* **20** (1969), 171—176.
- [2] R. BOTT: A note on the Samelson product in the classical groups, *Comment. Math. Helv.* **34** (1960), 249—256.
- [3] S. Y. HUSSEINI: A note on the intrinsic join of Stietel manifolds, *Comment. Math. Helv.* **38** (1963), 26—30.
- [4] I. M. JAMES: The intrinsic join, *Proc. London Math. Soc.* **8** (1958), 507—535.
- [5] ———: Products between homotopy groups, *Composito Math.* **23** (1971), 329—345.
- [6] ———: *The Topology of Stiefel Manifolds*, London Math. Soc. Lecture Notes **24**, Cambridge Univ. Press, Cambridge, 1976.
- [7] H. KACHI: Homotopy groups of homogeneous space $Sp(n)/U(n)$, *J. Fac. Sci. Shinshu Univ.* **13** (1978), 35—41.
- [8] H. MATSUNAGA: On the groups $\pi_{2n+i}(U(n))$ for $i = 3, 4, 5$, *Mem. Fac. Sci. Kyushu Univ.* **15** (1961), 72—81.
- [9] G. S. McCARTY: Product between homotopy groups and the J -morphism, *Quart. J. of Math.* **15** (1964), 362—370.

DEPARTMENT OF MATHEMATICS
SHINSHU UNIVERSITY

(Received January 12, 1982)