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LINEARLY COMPACT DUAL-BIMODULES

Dedicated to Professor Kentaro Murata on his 70th birthday

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Let R and S be rings with identity and ${}_RQ_S$ an (R, S)-bimodule. In the previous paper [3], it is shown that if Q_S is quasi-injective and the canonical ring homomorphism $\lambda:\ R \to \mathrm{End}(Q_S)$ is surjective, then the pair of functors

$$H' = \mathrm{Hom}_R(-, Q) :\ {}_R\underline{M} \to \underline{N}_S \text{ and } H'' = \mathrm{Hom}_S(-, Q) :\ \underline{N}_S \to {}_R\underline{M}$$

defines a duality between ${}_R\underline{M}$ and $\underline{N}_S$, where ${}_R\underline{M}$ is the full subcategory of R-mod of finitely generated Q-torsionless R-modules and $\underline{N}_S$ is the full subcategory of mod-S whose objects are all the S-modules N such that there exists an exact sequence of the form $0 \to N \to Q^n \to Q^I$ for some $n > 0$ and some set I.

In this note, we shall give, in the first section, some characterizations of self-cogenerators and then point out that the linearly compactness of a ring is very closed to the existence of some kind of left dual-bimodules. Characterizing these left dual-bimodules, in the second section, we shall show that, for a left dual-bimodule ${}_RQ_S$ with ${}_RQ$ finitely generated, Q_S quasi-injective and λ surjective, Q_S is linearly compact if and only if the duality mentioned above can be extended to a duality between ${}_R\overline{FG}$ and $\overline{N}_S$(see bellow for the definition).

1. An S-module Q_S will be called a *self-cogenerator* provided that every right S-module isomorphic to a submodule of a factor module of Q^n, $n = 1, 2, \cdots$, is Q-torsionless [7, Definition 3.1]. Trivially each cogenerator in mod-S is a self-cogenerator. First, we shall give some characterizations of self-cogenerators. As is easily seen, we have

Lemma 1. *Let Q_S be an S-module. Suppose that*

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence of right S-modules such that N' and N'' are Q-torsionless and Q is N-injective. Then N is Q-torsionless.

Lemma 2. *Let Q_S be a quasi-injectie S-module. Suppose that every* *factor module of Q is Q-torsionless. Then every factor module of Q^n is also Q-torsionless for $n = 1, 2, \cdots$.*

Proof. We may show the case where $n = 2$. Let Q' be any submodule of Q^2 and let $p: Q^2 \to Q$ be the canonical projection. Then the induced homomorphism $\bar{p}: Q^2/Q' \to Q/p(Q')$ is an epimorphism with Ker $\bar{p}$ Q-torsionless. By assumption $Q/p(Q')$ is Q-torsionless and hence Q^2/Q' is also Q-torsionless by Lemma 1.

An (R, S)-bimodule Q will be called a *left dual-bimodule* provided that $\ell_R r_Q(A) = A$ for every left ideal A of R and $r_Q\ell_R(Q') = Q'$ for every S-submodule Q' of Q (see [3]). A ring that has the double annihilator property ([1, Exercise 24.11]) will be called a *dual ring*. Hence a dual ring R is a left dual-bimodule regarded as an (R, R)-bimodule. It is also a right dual-bimodule by defining symmetrically. In [3, Lemma 1.3], it is shown that if ${}_RQ_S$ is a left dual-bimodule, then every factor module of Q_S is Q-torsionless. Hence we have

Corollary 3. *Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective. Then Q_S is a self-cogenerator.*

Proposition 4. *Let ${}_RQ_S$ be an (R, S)-bimodule with Q_S quasi-injective and λ surjective. Then the following conditions are equivalent:*

(1) *Q_S is a self-cogenerator.*

(2) *Every factor module of Q^n is Q-torsionless for $n = 1, 2, \cdots$.*

(3) *Every factor module of Q_S is Q-torsionless.*

(4) *$r_Q\ell_R(Q') = Q'$ for every submodule Q' of Q_S.*

(5) *$\underline{N}_S = \{N_S | 0 \to N \to Q^n$ is exact for some $n > 0\}$.*

(6) *Every submodule of Q^n is Q-reflexive for $n = 1, 2, \cdots$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are evident. The equivalence of (3) and (4) follows from [3, Lemma 1.3] and (3) $\Rightarrow$ (5) follows from Lemma 2.

(5) $\Rightarrow$ (6). Let N_S be a submodule of Q^n. Then $N \in \underline{N}_S$ and $0 \to N \to Q^m \to Q^I$ is exact for some $m > 0$ and I. Since Q_S is Q-injective and Q-reflexive, Q is Q^m-injective and Q^m is Q-reflexive. Hence by [3, Lemma 3.1], N must be Q-reflexive.

(6) $\Rightarrow$ (1) also follows from [3, Lemma 3.1].

Remarks. (1) The equivalence of (1) and (3) of Proposition 4 has already shown in [5, Lemma 1.1].

(2) In case Q_S is a finitely cogenerated cogenerator, then a right S-module N is finitely cogenerated if and only if there is an $n>0$ such that $0 \to N \to Q^n$ is exact by [1, Exercise 10.3]. For example, each dual-bimodule ${}_RQ_S$ with Q_S injective and λ surjective is a finitely cogenerated cogenerator as an S-module by [3, Proposition 1.8 and Lemma 3.5].

Corollary 5. *For a ring R with R_R injective, the following conditions are equivalent:*

(1) *R_R is a self-cogenerator.*

(2) *Every finitely generated right R-module is torsionless.*

(3) *Every cyclic right R-module is torsionless.*

(3′) *Every simple right R-module is torsionless.*

(3″) *R_R is a cogenerator.*

(4) *$r_R\ell_R(A) = A$ for every right ideal A of R.*

(5) *$\underline{N}_R = \{N_R \mid 0 \to N \to R^n$ is exact for some $n > 0\}$.*

(5′) *$\underline{N}_R = \{N_R \mid N_R$ is finitely cogenerated$\}$.*

(6) *Every submodule of R_R^n is reflexive for $n = 1, 2, \cdots$.*

(6′) *Every finitely cogenerated right R-module is reflexive.*

Proof. (3) ⇒ (3′) and (3″) ⇒ (3) are evident and (3′) ⇒ (3″) follows from [1, Proposition 18.15].

Assume (5). Then since (3) and (5) are equivalent, R_R is an injective cogenerator and hence is a finitely cogenerated cogenerator by [6, Satz 3]. Assume (5′). Then since R is in $\underline{N}_R$, R_R is finitely cogenerated and injective. Hence it is a finitely cogenerated cogenerator again by [6, Satz 3]. Therefore, the equivalence (5) and (5′) follows from Remarks (2).

(6) ⇒ (6′) ⇒ (3′) are evident. Hence (6) and (6′) are equivalent.

A ring R is a *cogenerator ring* in case both ${}_RR$ and R_R are cogenerators [1, Exercise 24.10].

Corollary 6. *For a ring R with R_R injective, the following conditions are equivalent:*

(1) *R is a dual ring.*

(2) *${}_RR$ and R_R are self-cogenerators.*

(3) *R is a cogenerator ring.*

Proof. (1) ⇒ (3). As we shall show in Corollary 11, if R is a dual ring. then R_R injective is equivalent to ${}_RR$ being injective. Hence, from Corollaries 3 and 5 (1) ⇒ (3) follows.

(3) ⇨ (2) is evident.

(2) ⇨ (1). To prove (1) ⇨ (4) of Proposition 4, it is sufficient to assume that λ is surjective. Hence, in Corollary 5 (1) ⇨ (4) is always valid. Thus (2) implies (1).

Let N_S be an S-module, $(x_i)_I$ an indexed set of elements of N and $(N_i)_I$ an indexed set of submodules of N. Then the set of congruences $\{x \equiv x_i \pmod{N_i}\}$ is said to be *solvable* (*finitely solvable*), if there is a y in N (a y_F in N for each finite subset F of I) such that $y - x_i$ in N_i for each i in I ($y_F - x_i$ in N_i for each i in F).

If every finitely solvable set of congruences in N is solvable, then N will be called *linearly compact* ([7, Definition 2. 1]). Using this notion we can characterize left dual-bimodules.

Proposition 7. *Let ${}_RQ_S$ be an (R, S)-bimodule with Q_S linearly compact quasi-injective and λ an isomorphism. Then the following conditions are equivalent:*

(1) *Q is a left dual-bimodule.*

(2) *Q_S is a self-cogenerator and has essential socle.*

Moreover, if this is the case, ${}_RQ$ is injective and ${}_RR$ is linearly compact.

Proof. (1) ⇨ (2) follows from Corollary 3 and [3, Proposition 1. 8].

(2) ⇨ (1). By [7, Lemma 3. 7] every cyclic left R-module is Q-torsionless. Hence by Proposition 4 and [3, Lemma 1. 2] Q is a left dual-bimodule.

The last part follows from (2) by [7, Lemmas 3. 5 and 3. 7].

Corollary 8. *For a ring R with R_R linearly compact and injective, the following conditions are equivalent:*

(1) *R is a dual ring.*

(2) *R_R is a self-cogenerator and has essential socle.*

(3) *${}_RR$ is a self-cogenerator and has essential socle.*

Proof. As is remarked in the proof of Proposition 7, (2) implies that ${}_RR$ is linearly compact and injective. Hence, again by Proposition 7, (3) is equivalent to R being a dual ring.

As a consequence of Proposition 7 and [7, Theorem 3. 10], we have

Theorem 9. *A ring R is left linearly compact if and only if there exists a left dual-bimodule ${}_RQ_S$ such that Q_S is linearly compact quasi-injective and λ*

is surjective.

2. A subcategory of the module category will be called *finitely closed* if it is closed under submodules, factor modules and finite direct sums [4, p. 465]. Let ${}_RQ_S$ be an (R, S)-bimodule. Following [7], consider the full subcategory of R-mod consisting of all modules isomorphic to factor modules of submodules of R^n for $n = 1, 2, \cdots$. This is the full subcategory consisting of all modules isomorphic to submodules of factor modules of R^n for $n = 1, 2, \cdots$ and hence is equal to

$$\{ {}_RM \mid 0 \to M \to M' \text{ is exact for some } M' \in {}_RFG \}.$$

As is easily seen, this is the smallest one of the finitely closed subcategory containing either R or ${}_RFG$. We shall denote this by ${}_R\overline{FG}$, where ${}_RFG$ means the full subcategory of finitely generated left R-modules.

Similarly, the full subcategory of mod-S consisting of all modules isomorphic to factor modules of submodules of Q^n for $n = 1, 2, \cdots$ coincides with one consisting of all modules isomorphic to submodules of factor modules of Q^n for $n = 1, 2, \cdots$. This is the smallest one of the finitely closed subcategory containing either Q or the class of S-modules

$$\{ N_S \mid 0 \to N \to Q^n \text{ is exact for some } n > 0 \}.$$

By Proposition 4 this is the smallest one of the finitely closed subcategory containing $\underline{N}_S$ in case Q_S is a quasi-injective self-cogenerator and λ is surjective. Hence we shall denote this by $\overline{N}_S$. Furthermore, $\overline{N}_S$ also coincides with

$$\{ N_S \mid N' \to N \to 0 \text{ is exact for some } N' \in \underline{N}_S \}.$$

We are now ready to characterize those left dual-bimodules mentioned in Theorem 9 by means of a duality.

Theorem 10. *Let ${}_RQ_S$ be a left dual-bimodule with ${}_RQ$ finitely generated and λ surjective. Then the following conditions are equivalent:*

(1) *Q_S is a linearly compact quasi-injective module.*

(2) *The pair (H', H'') defines a duality between ${}_R\overline{FG}$ and $\overline{N}_S$.*

(3) *${}_RQ$ is an injective cogenerator.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 7 and [7, Theorem 3.8].

(2) $\Rightarrow$ (3). Assume (2). Then since ${}_RQ \in {}_R\overline{FG}$, we can apply [1, Exercise 20.5] to show that, for each $M \in {}_R\overline{FG}$, σ_M is an epimorphism by a

similar way as in [1, Theorem 23. 5]. In particular, every cyclic R-module is Q-reflexive by [3, Lemma 1. 2] and thus Q_S is quasi-injective by [3, Theorem 3. 2]. Furthermore, if $N \in \bar{N}_S$, then $H''(N)$ is in ${}_R\overline{FG}$ and hence $\sigma_{H''(N)}$ is an epimorphism, which shows that $N(\cong H'H''(N))$ is Q-reflexive [1, Proposition 20. 14]. In particular, every factor module of Q_S is Q-reflexive. Hence, by [2, Theorem 10], ${}_RQ$ is an injective cogenerator.

(3) $\Rightarrow$ (1) follows from [2, Theorem 10] and [7, Theorem 3. 6].

The following corollary follows from Theorem 10 and [3, Lemma 3. 5].

Corollary 11. *For a dual ring R, the following conditions are equivalent:*

(1) *R_R is linearly compact and is injective.*

(2) *${}_RR$ is lenearly compact and is injective.*

(3) *${}_RR_R$ defines a duality between ${}_R\overline{FG}$ and $\overline{FG}_R$.*

(4) *${}_RR$ is injective.*

(5) *R_R is injective.*

Finally, we shall remark that a cogenerator ring is also a dual ring satisfying the equivalent condition of Corollary 11 [1, Exercise 24. 12].

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