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ON SOME PROPERTIES OF GAUSSIAN COVARIANCE OPERATORS IN BANACH SPACES

Dedicated to Professor Hisao Tominaga on his 60th birthday

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1. **Introduction and preliminaries.** The aim of this paper is to give the general form of Gaussian covariance operators in certain Banach spaces in terms of p-summing operators. Let E be a Banach space with the dual E', and $R : E' \to E$ a linear operator. The operator R is called symmetric if $\langle Rx', y'\rangle = \langle Ry', x'\rangle$ for all $x', y' \in E'$. Every symmetric operator is continuous. The operator R is called positive if $\langle Rx', x'\rangle \geqq 0$ for all $x' \in E'$. The operator R is called a Gaussian covariance operator (Gaussian covariance) if it is a covariance operator of some Gaussian Radon probability measure on E. Every Gaussian covariance operator is symmetric and positive. For a symmetric positive operator $R : E' \to E$, there exists a continuous linear operator T from E' into an everywhere dense subspace of some Hilbert space H such that $R = T'T$, where T' denotes the conjugate of T (see Vakhania [16]). This expression is unique up to a unitary equivalence. The operator T is called the square root of R and denoted by $R^{1/2}$. Then a problem of our interest is how to find necessary and sufficient conditions on $R^{1/2}$ for which $R : E' \to E$ is a Gaussian covariance. As is well known, a necessary condition is given by the following: If $R : E' \to E$ is a Gaussian covariance, then $R^{1/2} : E' \to H$ is p-summing for every $p > 0$ in the sense of Pietsch [11]. Now we shall give a slight generalization of this result by introducing p-summing operators in locally convex spaces.

Let X be a locally convex space and A a subset of X. Denote by A^0 the polar of A, i.e., $A^0 = \{x' \in X' ; |\langle x, x'\rangle| \leqq 1$ for all $x \in A\}$. Then, by the Hahn-Banach theorem, the bipolar A^{00} of A is the closed convex balanced hull of A. A linear operator S from X into a normed space Y is called p-summing, $0 < p < \infty$, if there exists a neighborhood U of zero in X such that

$$\sum_{i=1}^n \|Sx_i\|^p \leqq \sup\{\sum_{i=1}^n |\langle x_i, x'\rangle|^p ; x' \in U^0\}$$

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for all $x_1, \ldots, x_n \in X$. The set of all such S will be denoted by $\pi_p(X, Y)$. Then, we have the inclusion $\pi_p(X, Y) \subset \pi_q(X, Y)$ for $0 < p \leqq q$. Let us denote by E'_c the dual of a Banach space E equipped with the compact convergence topology $c(E', E)$. In this case, $c(E', E)$ coincides with the topology of uniform convergence on compact convex balanced subsets of E, and so we have $(E'_c)' = E$. We show that if $R : E' \to E$ is a Gaussian covariance, then $R^{1/2} : E'_c \to H$ is p-summing for every $p > 0$.

Here we want to characterize Banach spaces E having the following property: A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is p-summing $(0 < p < \infty)$. It is known that E has this property for $p = 2$ if and only if it is of type 2 (see Chobanjan and Tarieladze [1] and Linde, Tarieladze and Chobanjan [4]). Under the assumption that E has an unconditional basis, it is also known that E has this property for $p = 1$ if and only if c_0 is not finitely representable in E (see Chobanjan and Tarieladze [1]). Remark that the second result is false if E does not have an unconditional basis. We shall extend their results to the general case $0 < p < \infty$.

In Section 3, we characterize the Banach space E having the above mentioned property. The main results of this section are stated as follows:

(1) E has this property for $p \in [2, \infty)$ if and only if it is of type 2.

(2) E has this property for $p \in (0, 1)$ if and only if it is of cotype $(2, p)$ in the sense of Mathé [7].

(3) Let $1 \leqq r < 2$. Then E has this property for some $p \in (r, 2)$ if and only if it is of stable type r and of cotype $(2, r)$.

In Section 4, we characterize Banach spaces E for which c_0 is not finitely representable in E. It is shown that if E has the above mentioned property for some $p \in (0, \infty)$, then c_0 is not finitely representable in E. But in general, the converse is not true. More precisely, there are Banach spaces E of stable type p and of cotype 2 which are not of cotype $(2, p)$, where $0 < p < 2$. Now we introduce the class GL. We say that a Banach space G is in the class GL if every 1-summing operator from G into any Banach space F factors through some L_1. It is known that if E has an unconditional basis, or more generally, if E has local unconditional structure (l.u.st.) in the sense of Gordon and Lewis [2], then both E and E' are in the class GL (GL-spaces). Suppose that E' is a GL-space. Then it is shown that c_0 is not finitely representable in E if and only if it has the above mentioned property for each (some) $p \in (0, 1)$.

Throughout the paper, we assume that all linear spaces are with real coefficients.

2. Type and cotype of Banach spaces. Let $0 < p \leqq 2$ and denote by $\{\varepsilon_n\}$ an i.i.d. sequence of Rademacher random variables (r.v.'s), and by $\{\theta_n^{(p)}\}$ an i.i.d. sequence of p-stable r.v.'s with the characteristic function (ch. f.) $\exp(-|t|^p)$.

A Banach space E is called of Rademacher type p (R-type p) if for each sequence $\{x_n\}$ in E, $\sum_n \|x_n\|^p < \infty$ implies the series $\sum_n x_n\varepsilon_n$ converges almost surely (a.s.), and, it is called of stable type p (s-type p) if $\sum_n \|x_n\|^p < \infty$ implies the series $\sum_n x_n\theta_n^{(p)}$ converges a.s. It is well known that s-type p implies R-type p (see Schwartz [14]), but in general, the converse is not true except for $p = 2$. For the case $p = 2$, s-type and R-type coincide, and so we call it type 2.

A Banach space E is called of Rademacher cotype q (R-cotype q), $2 \leqq q < \infty$, if for each sequence $\{x_n\}$ in E, the a.s. convergence of the series $\sum_n x_n\varepsilon_n$ implies $\sum_n \|x_n\|^q < \infty$, and, it is called of stable cotype q (s-cotype q), $0 < q \leqq 2$, if the a.s. convergence of the series $\sum_n x_n\theta_n^{(q)}$ implies $\sum_n \|x_n\|^q < \infty$. It is well known that every Banach space is of s-cotype q with $q < 2$ (see Maurey [8]). For the case $q = 2$, s-cotype and R-cotype coincide, and so we call it cotype 2.

Let E, F be Banach spaces. We say that E is finitely representable in F if for each $\lambda > 1$ and each finite dimensional subspace E_1 of E, there exists a finite dimensional subspace F_1 of F such that $d(E_1, F_1) \leqq \lambda$. Here

$$d(E_1, F_1) = \inf\{\|T\| \cdot \|T^{-1}\| ; T: E_1 \to F_1 \text{ is isomorphism}\}$$

denotes the Banach-Mazur distance.

It is well known that E is of R-cotype q for some $q \in [2, \infty)$ if and only if c_0 is not finitely represntable in E. For the details of type and cotype of Banach spaces, we refer to Maurey and Pisier [10] and Schwartz [14].

Now we introduce another notion of cotype which is very useful in our ensuing discussions. Let $0 < p \leqq 2$ and $0 < q \leqq 2$. Following Linde [5], we say that a linear operator $T: E' \to L_p$ is a Λ_p-operator if $\exp(-\|Tx'\|^p)$, $x' \in E'$, is the ch.f. of some Radon measure μ on E. Of course, the measure μ is symmetric and p-stable. Let τ_p denote the vector topology on E' defined by the family of seminorms (quasi-seminorms for $p < 1$)

$$x' \to \|Tx'\|, \ x' \in E',$$

where T varies over all Λ_p-operators from E' into any space L_p. We say

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that E is of cotype (q, p) if each τ_p-continuous linear operator $T: E' \to L_q$ is a Λ_q-operator. In other words, E is of cotype (q, p) if each τ_p-continuous q-stable symmetric cylindrical measure on E extends to a Radon measure. The notion of cotype (q, p) has been introduced by Mathé [7], and some interesting results are known for the case $0 < q \leqq p$ (see [6], [7]). But almost nothing is known about spaces of cotype (q, p) with $p < q \leqq 2$ and $q \geqq 1$. As remarked by Linde [6], the most interesting case seems to be $q = 2$ (and $p < 2$). We shall investigate spaces of cotype $(2, p)$ in Sections 3 and 4. Note that every Banach space is of cotype $(2, 2)$ (see [6] or [7]).

3. Gaussian covariance operators in Banach spaces. Let E be a Banach space, and μ a Gaussian Radon probability measure on E. Then there exists a symmetric positive operator $R: E' \to E$ such that

$$\langle Rx', y'\rangle = \int_E \langle x, x'\rangle\langle x, y'\rangle d\mu(x) - \int_E \langle x, x'\rangle d\mu(x) \int_E \langle x, y'\rangle d\mu(x)$$

for all $x', y' \in E'$. The operator $R: E' \to E$ is called the covariance operator of μ (see Vakhania [16]). As mentioned in Section 1, a symmetric positive operator $R: E' \to E$ is called a Gaussian covariance operator (Gaussian covariance) if it is a covariance operator of some Gaussian Radon probability measure on E. It is clear that by shifting the measure μ by an arbitrary element $x \in E$, the covariance operator remains unchanged. Thus, the operator $R: E' \to E$ is a Gaussian covariance if and only if there exists a symmetric Gaussian Radon probability measure μ on E such that

$$\langle Rx', y'\rangle = \int_E \langle x, x'\rangle\langle x, y'\rangle d\mu(x) \text{ for all } x', y' \in E'.$$

Let $R: E' \to E$ be a symmetric positive operator. Then, $R^{1/2}$ denotes the square root of R, that is, $R^{1/2}$ is a continuous linear operator from E' into an everywhere dense linear subspace of some Hilbert space H such that $R = (R^{1/2})' R^{1/2}$ (see Section 1). It is easy to see that if R is a Gaussian covariance, then there exist a separable Hilbert space H and a Λ_2-operator $T: E' \to H$ such that $R = T'T$. Note that the operators $R^{1/2}$ and T are unitary equivalent (see Vakhania [16, pp. 101]). In this case, T' is a continuous linear operator from H into E such that for the standard Gaussian cylindrical measure γ_H on H, $T'\gamma_H$ extends to a Radon measure on E, that is, $T': H \to E$ is γ_2-Radonifying.

First we give a necessary condition for a symmetric positive operator

$R: E' \to E$ to be a Gaussian covariance.

Theorem 1. *Let $R: E' \to E$ be a symmetric positive operator. If R is a Gaussian covariance, then $R^{1/2}: E'_c \to H$ is p-summing for every $p > 0$, that is, it is completely summing.*

Proof. Suppose that R is a Gaussian covariance. If we put $T = R^{1/2}$, then $T: E' \to H$ is a Λ_2-operator. Let μ be a Gaussian Radon probability measure on E with the ch.f. $\exp(-\|Tx'\|^2)$, $x' \in E'$. Take a compact convex balanced set K of E such that $\mu(K) > 0$. Then we have $\mu(\cup nK) = 1$ by the 0-1 law. For $0 < p < 1$, define the quasi-seminorm on E' by

$$\|x'\|_p = \left(\int_K |\langle x, x'\rangle|^p \, d\mu(x)\right)^{1/p}, \quad x' \in E'.$$

Let $\{x'_n\}$ be a sequence in E' such that $\|x'_n\|_p \to 0$. Then there is a subsequence $\{x'_{n_j}\}$ such that $x'_{n_j} \to 0$ μ-a.s. Evidently, $\|Tx'_{n_j}\| \to 0$. But this means that $T: E' \to H$ is continuous with respect to the quasi-seminorm $\|\cdot\|_p$, and so there is a constant $C > 0$ such that

$$\|Tx'\| \leqq C\left(\int_K |\langle x, x'\rangle|^p \, d\mu(x)\right)^{1/p}, \quad x' \in E'.$$

By Vladimirskii [17, Theorem 3], it follows that $T: E'_c \to H$ is p-summing, and the proof is completed.

Remark. Theorem 1 gives a slight generalization of the well known fact that every Λ_2-operator from E' into H is p-summing in the sense of Pietsch [11]. Here, E' is a Banach space equipped with the strong dual topology. Note that if $T: E'_c \to H$ is p-summing, then $T: E' \to H$ is p-summing, but in general, the converse is not true except for the case that E is reflexive.

Now we want to characterize Banach spaces E for which a symmetric positive operator $R: E' \to E$ is a Gaussian covariance if and only if $R^{1/2}: E'_c \to H$ is p-summing.

Lemma 1 (Linde [5]). *Let $0 < p \leqq 2$. Then the following assertions are equivalent.*

(1) *E is of stable type p.*

(2) *For each Radon probability measure μ on E of strong p-th order $\left(i.e. \int \|x\|^p \, d\mu(x) < \infty\right)$, there exists a Λ_p-operator $T: E' \to L_p$ such that*

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$$\| Tx' \| = \left(\int_E | \langle x, x' \rangle |^p d\mu(x) \right)^{1/p}, \quad x' \in E'.$$

Lemma 2. *Let $0 < p \leqq 2$ and suppose that E is of stable type p. Then a linear operator $T : E'_c \to H$ is p-summing if and only if it is τ_p-continuous.*

Proof. Let $T : E'_c \to H$ be a linear operator. If T is p-summing, then by Vladimirskii [17, Theorem 3], there exist a compact convex balanced set K of E and a Radon probability measure μ on K such that

$$\| Tx' \| \leqq \left(\int_K | \langle x, x' \rangle |^p d\mu(x) \right)^{1/p}, \quad x' \in E'.$$

Since E is of stable type p, by Lemma 1, it follows that $T : E' \to H$ is τ_p-continuous. Conversely, if T is τ_p-continuous, then there is a Λ_p-operator $S : E' \to L_p$ such that $\| Tx' \| \leqq \| Sx' \|$ for all $x' \in E'$. By the same way as in the proof of Theorem 1, $S : E'_c \to L_p$ is p-summing, and so is $T : E'_c \to H$. This completes the proof.

Remark. Without any additional assumption on E, if $T : E'_c \to H$ is τ_p-continuous, then it is completely summing.

Corollary 1. *For $0 < p < 1$, a linear operator $T : E'_c \to H$ is p-summing if and only if it is τ_p-continuouous.*

Proof. Since every Banach space is of stable type p with $p < 1$ (see [12]), the assertion follows from Lemma 2.

Proposition 1. *Let $0 < p \leqq 2$ and suppose that E is of stable type p. Then the following assertions are equivalent.*

(1) *E is of cotype $(2, p)$.*

(2) *A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is p-summing.*

Proof. (1) ⇒ (2) follows from Theorem 1 and Lemma 2. On the other hand, suppose that (2) holds. To prove (1), take a linear operator $T : E' \to L_2$ which is τ_p-continuous. Then, by Lemma 2, $T : E'_c \to L_2$ is p-summing. Let us put $R = T'T$. It is clear that $R : E' \to E$ is a symmetric positive operator, and $R^{1/2} : E'_c \to H$ is p-summing. By the assumption (2), it follows that R is a Gaussian covariance, that is, $R^{1/2}$ is a Λ_2-operator, and so is T.

But this means that E is of cotype $(2, p)$, and the proof is completed.

Since every Banach space is of stable type p with $p < 1$ (see [12]), by Proposition 1, we have

Theorem 2. *For $0 < p < 1$, the following assertions are equivalent.*

(1) *E is of cotype $(2, p)$.*

(2) *A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is p-summing.*

Lemma 3. *Let $0 < p \leqq 2$ and suppose that E has the following property: A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is p-summing. Then E is of Rademacher type p and of cotype $(2, r)$ for every $r > 0$.*

Proof. First we show that E is of Rademacher type p. Of course, we may assume $p > 1$, since every Banach space is of R-type p with $p \leqq 1$ (see [12]). Let $\{x_n\}$ be a sequence in E with $\sum_n \|x_n\|^p < \infty$. Then we define a continuous linear operator $T: l_{p'} \to E$ by $Te_n = x_n$ for all n, where e_n denotes the n-th unit vector of $l_{p'}$ $(1/p+1/p' = 1)$. Evidently, $T' : E'_c \to l_p$ is p-summing, and so is $(TJ)' : E'_c \to l_2$. Here, J denotes the natural injection from l_2 into $l_{p'}$. Let us put $R = TJ(TJ)'$. It is clear that $R : E' \to E$ is a symmetric positive operator, and $R^{1/2} : E'_c \to H$ is p-summing. By the assumption, it follows that R is a Gaussian covariance, that is, $R^{1/2}$ is a Λ_2-operator, and so is $(TJ)'$. Since $TJ: l_2 \to E$ is γ_2-Radonifying, the series $\sum_n x_n\varepsilon_n = \sum_n TJe_n\varepsilon_n$ converges a.s. in E (see [6]), where $\{\varepsilon_n\}$ is an i.i.d. sequence of Rademacher r.v.'s. But this means that E is of R-type p. On the other hand, by the same way as in the proof of Proposition 1, it follows that E is of cotype $(2, r)$ for every $r > 0$. This completes the proof.

Since every Banach space is of cotype $(2, 2)$ (see [7]), by Proposition 1 and Lemma 3, we have the following result due to Chobanjan and Tarieladze [1] (see also [4]).

Corollary 2. *The following assertions are equivalent.*

(1) *E is of type 2.*

(2) *A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is 2-summing.*

For $2 < p < \infty$, it is easy to see that a linear operator $T: E'_c \to H$ is

p-summing if and only if it is 2-summing (see Maurey [9, Proposition 74]). By Corollary 2, we have

Theorem 3. *For $2 \leqq p < \infty$, the following assertions are equivalent.*

(1) *E is of type 2.*

(2) *A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is p-summing.*

Finally, we shall consider the case $1 \leqq p < 2$.

Theorem 4. *For $1 \leqq p < 2$, the following assertions are equivalent.*

(1) *E is of stable type p and of cotype $(2, p)$.*

(2) *There is an $r \in (p, 2)$ such that a symmetric positive operator R : $E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is r-summing.*

Proof. Suppose that (1) holds. As is well known, stable type p implies stable type r for some $r \in (p, 2)$ (see e.g., [14]). Of course, cotype $(2, p)$ always implies cotype $(2, r)$ for every $r \in (p, 2)$. Thus, (2) follows from Proposition 1. On the other hand, suppose that (2) holds. Then, by Lemma 3, it follows that E is of Rademacher type r and of cotype $(2, p)$. But R-type r implies s-type p for every $p \in (0, r)$, proving (1). This completes the proof.

4. Gaussian covariance operators in GL-spaces. In this section, we introduce the class GL. After Gordon and Lewis [2], we say that a Banach space E is in the class GL (GL-space) if every 1-summing operator from E into any Banach space F factors through some space L_1. As was shown by Gordon and Lewis [2], if E has local unconditional structure (l.u.st.), then it is a GL-space. Let us remark that E has l.u.st. if and only if E' has it (see Pisier [13]). It is well known that if E has an unconditional basis, or more generally, if E has sufficiently many Boolean algebras of projections, then it has l.u.st., and in particular, both E and E' are GL-spaces. For the details of Banach spaces with l.u.st., we refer to Gordon and Lewis [2].

In [1], Chobanjan and Tarieladze has shown the following:

Theorem 5. *Suppose that E has an unconditional basis. Then the following assertions are equivalent.*

(1) *c_0 is not finitely representable in E.*

(2) *A symmetric positve operator $R : E' \to E$ is a Gaussian covariance*

if and only if $R^{1/2}: E' \to H$ *is* 1-*summing.*

In the following, we shall extend this result.

Theorem 6. *Let* $0 < p < 1$ *and suppose that* E *is of cotype* $(2, p)$. *Then* c_0 *is not finitely representable in* E.

In order to prove this theorem, we need the following lemmas.

Lemma 4 (Maurey and Pisier [10]). *Let* $\{\varepsilon_n\}$ (*resp.* $\{\gamma_n\}$) *be an* i.i.d. *sequence of Rademacher r.v.'s (resp. standard Gaussian r.v.'s). Then the following assertions are equivalent.*

(1) c_0 *is not finitely representable in* E.

(2) *For each sequence* $\{x_n\}$ *in* E, *the a.s. convergence of* $\sum_n x_n \varepsilon_n$ *implies the a.s. convergence of* $\sum_n x_n \gamma_n$.

Lemma 5 (Schwartz [14]). *Let* $E \subset L_0$ *be a linear subspace on which the* L_p *and* L_q *topologies are equivalent with* $p < q$. *Then* L_q *topology is equivalent to the* L_r *topology for all* $r < q$, *including* $r = 0$.

Proof of Theorem 6. Let $\{x_n\}$ be a sequence in E such that the series $\sum_n x_n \varepsilon_n$ converges a.s. Then we define a Radon probability measure on E by $\mu = \text{dist}(\sum_n x_n \varepsilon_n)$. Evidently, the measure μ is of strong r-th order for every $r > 0$, and in particular, $E' \subset L_r(E, \mu)$. By the Kahane inequality, we know that the topologies L_2 and L_r on E' are equivalent for every $r > 0$ (see Schwartz [14, Theorem 11.1]). Hence, by Lemma 5, it follows that the topologies L_2 and L_0 on E' are equivalent. Take a compact convex balanced set K of E with $\mu(K) > 0$. Then $\mu(\cup nK) = 1$ by the 0-1 law. By the same way as in the proof of Theorem 1, there is a constant $C > 0$ such that

$$(*) \quad \left(\int_E |\langle x, x'\rangle|^2 \, d\mu(x)\right)^{1/2} \leqq C\left(\int_K |\langle x, x'\rangle|^p \, d\mu(x)\right)^{1/p}$$

for all $x' \in E'$. Now we define a linear operator $T: l_2 \to E$ by $Te_n = x_n$ for all n, where e_n is the n-th unit vector of l_2. Then by the inequality (*), we have

$$\begin{aligned} \| T'x' \| &= (\textstyle\sum_n |\langle x_n, x'\rangle|^2)^{1/2} \\ &= \left(\int_E |\langle x, x'\rangle|^2 \, d\mu(x)\right)^{1/2} \leqq C\left(\int_K |\langle x, x'\rangle|^p \, d\mu(x)\right)^{1/p} \end{aligned}$$

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for all $x' \in E'$. But this means that $T' : E'_c \to l_2$ is p-summing (see [17, Theorem 3]), and so it is τ_p-continuous (see Corollary 1). Since E is of cotype $(2, p)$, it follows that $T' : E' \to l_2$ is a Λ_2-operator, that is, the series $\sum_n x_n\gamma_n = \sum_n Te_n\gamma_n$ converges a.s. (see [6]). Thus, the assertion follows from Lemma 4. This completes the proof.

Remark. Theorem 6 is a generalization of the well known fact that if E is of type 2, then c_0 is not finitely representable in E. Note that type 2 always implies cotype $(2, p)$ for every $p > 0$ (see [6]), but in general, the converse is not true.

Corollary 3. *Let $0 < p < \infty$ and suppose that E has the following property: A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is p-summing. Then c_0 is not finitely representable in E.*

Proof. The assertion follows from Lemma 3 and Theorem 6.

Theorem 7. *Let $0 < p \leqq 1$ and suppose that E' is a GL-space. Then the following assertions are equivalent.*

(1) *c_0 is not finitely representable in E.*

(2) *A symmetric positive operator $R : E' \to E$ is a Gaussian covariance if and only if $R^{1/2} : E'_c \to H$ is p-summing.*

Proof. Suppose that (1) holds. To prove (2), it suffices to show that if $T : E'_c \to H$ is 1-summing, then it is a Λ_2-operator. Suppose that $T : E'_c \to H$ is 1-summing. Evidently, $T : E' \to H$ is 1-summing in the sense of Pietsch [11], and $T'(H) \subset E$. Since E' is a GL-space, the operator $T : E' \to H$ is factorized by the bounded linear operators $V : E' \to L_1$ and $W : L_1 \to H$. By the assumption (1), it follows that E'' (bidual of E) is of Rademacher cotype q for some $q \in (2, \infty)$, and so $V' : L_\infty \to E''$ is r-summing for every $r \in (q, \infty)$ (see Maurey and Pisier [10]). But this implies that $T' : H \to E$ is r-summing. As is well known, every r-summing operator is r-Radonifying (see e.g., [14]), and in particular, $T' : H \to E$ is γ_2-Radonifying, that is, $T : E' \to H$ is a Λ_2-operator. Thus, (2) holds. On the other hand, (2) $\Rightarrow$ (1) follows from Corollary 3. This completes the proof.

Theorem 8. *Let $1 < p < 2$ and suppose that E' is a GL-space. Then the following assertions are equivalent.*

(1) *E is of stable type p.*

(2) *There is an $r \in (p, 2)$ such that a symmetric positive operator $R: E' \to E$ is a Gaussian covariance if and only if $R^{1/2}: E'_C \to H$ is r-summing.*

Proof. Let us remark that if E is of stable type p, then c_0 is not finitely representable in E (see e.g., [14]). Thus, the assertion follows from Theorems 2, 4 and 7.

Finally, we remark that Theorems 7 and 8 are false in the case where E' is not a GL-space. Such a counterexample is given by the following: Let H be an infinite dimensional separable Hilbert space, and denote by $c_p(H)$ a Banach space of all compact operators on H for which the c_p-norm $\| T \|_p = (\mathrm{trace}(T^* T)^{p/2})^{1/p}$ is finite $(1 \leqq p < \infty)$. It is well known that for $1 \leqq p < 2$, $c_p(H)$ is of Rademacher type p and of cotype 2 (see Tomczak-Jaegermann [15]). But in this case, we know that $c_p(H)$ is not of cotype $(2, r)$ for every $r \in (0, p)$ (see Proposition 1 and Kühn [3, Corollary 17]). Of course, if $2 \leqq p < \infty$, then $c_p(H)$ is of type 2, and so it is of cotype $(2, r)$ for every $r \in (0, 2)$. Let us mention that $c_p(H)$, $p \neq 2$, does not have l.u.st., as was shown by Gordon and Lewis [2]. It is well known that l_p is linearly isometric to a subspace of $c_p(H)$, and so $c_p(H)$ contains an infinite dimensional Banach subspace with l.u.st. However, we can prove that for each $p \in (1, 2)$, there exists a compact subset K of $c_p(H)$ such that every Banach subspace G of $c_p(H)$ with $K \subset G$, does not have l.u.st. In fact, G' is not a GL-space.

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