Mathematical Journal of Okayama University

Volume 46, *Issue* 1 2004 *Article* 11

JANUARY 2004

Principal Ideals in Ore Extensions

Wagner Cortes* Miguel Ferrero†

*Universidade Estadual
†Universidade Federal do Rio Grande do Sul

Principal Ideals in Ore Extensions

Wagner Cortes and Miguel Ferrero

Abstract

In this paper we prove that if R is a prime ring and I is an R-disjoint ideal of an Ore extension R[χ, δ,d], then I is closed and principal generated by a normal polynomial of minimal degree if and only if I contains a Sharma polynomial of minimal degree.

KEYWORDS: principal ideals, Sharma polynomials, Ore extensions.

Math. J. Okayama Univ. **46** (2004), 77–84

PRINCIPAL IDEALS IN ORE EXTENSIONS

Wagner CORTES and Miguel FERRERO

Abstract. In this paper we prove that if R is a prime ring and I is an R-disjoint ideal of an Ore extension $R[x;\sigma,d]$, then I is closed and principal generated by a normal polynomial of minimal degree if and only if I contains a Sharma polynomial of minimal degree.

Introduction

Let R be a commutative integral domain and $R[x]$ the polynomial ring over R in one indeterminate x. In [8], P. K. Sharma proved that if P is a nonzero R-disjoint prime ideal of $R[x]$ and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is a polynomial of minimal degree in P, then P is a principal ideal generated by f if and only if for any $b \in R$ such that $ba_i \in a_n R$ we necessarily have $b \in a_n R$. A polynomial satisfying this condition is called a Sharma polynomial in [4]. In this second paper the above result was extended to any R-disjoint closed ideal of $R[x]$, and some equivalent conditions were obtained for a polynomial to be a Sharma polynomial.

The result was also extended to some Ore extensions over a commutative domain in [9] and [2], where the authors proved similar results for an ideal to be a principal left ideal.

In this paper we consider Ore extensions $R[x;\sigma,d]$ over non necessarily commutative rings, where σ an automorphism and d a σ-derivation of R. An ideal I of $R[x;\sigma,d]$ is said to be principal if there exists a normal polynomial of minimal degree $f \in I$ such that $I = R[x;\sigma,d]f = fR[x;\sigma,d]$.

In the first section of the paper we prove the main results. A Sharma polynomial is defined as a normal polynomial which satisfies the condition given in the first paragraph of this introduction. Assume that R is a prime ring and I is an R-disjoint ideal of $R[x;\sigma,d]$. We show that I is closed and principal generated by a polynomial of minimal degree of I if and only if I contains a Sharma polynomial of minimal degree. In Section 2 a particular case is considered, namely, rings R satisfying unique factorization properties.

We point out that the results of the paper are as general as possible concerning ideals generated by normal polynomials of minimal degree in

---

Mathematics Subject Classification. Primary 16S36; Secondary 16P60.

Key words and phrases. principal ideals, Sharma polynomials, Ore extensions.

Both authors were partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil). Some results of this paper are contained in the Ph.D. thesis written by the first named author and presented to Universidade Federal do Rio Grande do Sul.

Ore extensions over prime rings, since we are not assuming any additional condition on R, σ and d.

1. Principal Ideals in Ore Extensions

Let R be a prime ring, σ an automorphism of R and d a σ-derivation of R, i.e., d is an additive mapping and $d(ab) = d(a)b + \sigma(a)d(b)$, for every $a, b \in R$. Denote by $R[x; \sigma, d]$ the Ore extension whose elements are the polynomials $\sum_{i=0}^{n} a_i x^i$, $a_i \in R$, with the usual addition and multiplication defined by $xa = \sigma(a)x + d(a)$, for all $a \in R$.

We denote by Q the symmetric quotient ring of R, unless otherwise stated. Recall that Q is the subring of the left Martindale quotient ring $Q(R)$ of R consisting of all the elements $q \in Q$ such that $qI \subseteq R$, for some non-zero ideal I of R. It is well-known that both σ and d have unique extensions to Q. Therefore, we can consider the over ring $Q[x; \sigma, d]$ of $R[x; \sigma, d]$.

As usual, a polynomial $f \in R[x; \sigma, d]$ is said to be normal if $R[x; \sigma, d]f = fR[x; \sigma, d]$. An ideal I of $R[x; \sigma, d]$ is said to be a principal ideal if there exists a normal polynomial of minimal degree $f \in I$ such that $I = fR[x; \sigma, d]$.

For a polynomial f, $\delta(f)$ stands the degree of f. If I is a non-zero R-disjoint (Q-disjoint) ideal of $R[x; \sigma, d]$ (resp. $Q[x; \sigma, d]$), the minimality of I, $Min(I)$, is defined as the smallest $n \geq 1$ such that I contains a polynomial of degree n.

It is well-known that for a non-zero R-disjoint ideal I of $R[x; \sigma, d]$ there exists a unique monic invariant polynomial $f_I \in Q[x; \sigma, d]$ with $\delta(f_I) = Min(I) = n$ and every $g(x) \in I$ of degree n is of the form $g(x) = af_I$, for some $a \in R$. Also, $I \subseteq Q[x; \sigma, d]f_I(x) \cap R[x; \sigma, d]$ ([7], Proposition 2.1 and Corollary 2.2). Recall that an invariant polynomial of $Q[x; \sigma, d]$ is just a normal polynomial. Hence, $Q[x; \sigma, d]f_I$ is a two-sided ideal of minimality n.

Given I as above, the ideal $[I] = Q[x; \sigma, d]f_I(x) \cap R[x; \sigma, d]$ is called the closure of I and I is said to be closed if $[I] = I$. In particular, if P is an R-disjoint prime ideal of $R[x; \sigma, d]$, under some additional assumption P is closed (see [6]).

Assume that $f \in R[x; \sigma, d]$ is a normal polynomial of minimal degree in $I = R[x; \sigma, d]f$. Since $R[x; \sigma, d]$ is a prime ring ([5], Theorem 4.4) we have that $rf = 0$ ($fr = 0$), $r \in R$, implies $r = 0$. Using this fact it easily follows that the leading coefficient of f is a normal element of R which is not a zero divisor. The polynomials of this type play a central role in this paper.

Following [4], a normal polynomial $f = \sum_{i=0}^{n} a_i x^i \in R[x; \sigma, d]$, $a_n \neq 0$, is said to be a *Sharma polynomial* if $ba_i \in a_n R$, for $0 \leq i \leq n$ and $b \in R$, implies $b \in a_n R$.

Sharma polynomials appear naturally in principal ideals which are closed. In fact, we have the following (cf. [8], Theorem 1; [4], Theorem 5).

Theorem 1.1. *Let R be a prime ring and $I = R[x;\sigma,d]f = fR[x;\sigma,d]$ a principal ideal which is closed, where $\delta(f) = Min(I) \geq 1$. Then f is a Sharma polynomial.*

Proof. Put $f = \sum_{i=0}^{n} a_i x^i \in R[x;\sigma,d]$, $a_n \neq 0$. Assume that there exists $b \in R$ with $ba_i \in a_nR$, for $0 \leq i \leq n$, and $b \notin a_nR$. Since a_n is a normal element of R it has an inverse $a_n^{-1} \in Q$. Also $q = a_n^{-1}b \notin R$ and $qa_i = a_n^{-1}ba_i \in R$, for $0 \leq i \leq n$. Thus $g = qf \in Q[x;\sigma,d]f_I(x) \cap R[x;\sigma,d] = [I]$. Hence $g \in I = R[x;\sigma,d]f$, since I is closed, and so $g = rf$, for some $r \in R$. Consequently $(q-r)f = 0$ and it follows that $q = r \in R$, a contradiction. Therefore f is a Sharma polynomial. □

Now we characterize Sharma polynomials. Assume that $f = \sum_{i=0}^{n} a_i x^i$ is a normal polynomial of minimal degree $n \geq 1$ in the ideal $I = R[x;\sigma,d]f$. Denote by $f_I(x) = \sum_{i=0}^{n} c_i x^i$ the monic invariant polynomial of $Q[x;\sigma,d]$ associated to the ideal I, where $c_n = 1$, and by $C(f)$ the set $\{a_0,...,a_n\}$. Put $C(f)^{-1} = \{q \in Q : qC(f) \subseteq R\}$ and $U = \{b \in R : bf_I \in R[x;\sigma,d]\}$. It is easy to see that U is a two-sided ideal of R since $rf_I = f_I\sigma^{-n}(r)$, for any $r \in R$.

The following result is an extension of ([4], Proposition 4).

Proposition 1.2. *Assume that $f = \sum_{i=0}^{n} a_i x^i \in R[x;\sigma,d]$ is a normal polynomial of minimal degree n in $I = R[x;\sigma,d]f$. Then the following conditions are equivalent:*

i) f is a Sharma polynomial;
ii) $C(f)^{-1} = R$;
iii) $U = a_nR = Ra_n$.

Proof. $(i) \Rightarrow (ii)$ Since a_n is a normal element of R, $a_n^{-1} \in Q$. Denote by $\varphi : R \to R$ the automorphism defined by $\varphi(r) = r'$, for any $r \in R$, where $r' \in R$ is such that $ra_n = a_nr'$. Thus φ can be extended to an automorphism of Q defined by $\varphi(q) = a_n^{-1}qa_n$, for all $q \in Q$, and $\varphi(a_nR) = a_nR$.

Suppose that $q \in C(f)^{-1}$ and let $p = \varphi(q)$. Thus $qa_i \in R$, for $0 \leq i \leq n$, and so $a_np\varphi(a_i) = qa_ia_n \in a_nR$, where $a_np \in R$. Hence, $\varphi^{-1}(a_np)a_i \in a_nR$, for all i, and by assumption we have $\varphi^{-1}(a_np) \in a_nR$. Consequently $qa_n = a_np \in a_nR = Ra_n$ and it follows that $q \in R$. Therefore $C(f)^{-1} = R$.

$(ii) \Rightarrow (iii)$ It is clear that $a_n \in U$. On the other hand, take $b \in U$ and consider $bf_I = \sum_{i=0}^{n} bc_i x^i \in R[x;\sigma,d]$. Note that since $f = a_nf_I$ we have that $a_i = a_nc_i$, for every i. So $ba_n^{-1}a_i = ba_n^{-1}a_nc_i = bc_i$, i.e., $ba_n^{-1} \in C(f)^{-1} = R$. Consequently, $b \in a_nR$.

$(iii) \Rightarrow (i)$ Assume that $b \in R$ and $ba_i \in a_nR$, for $0 \leq i \leq n$. Take $b' \in R$ with $ba_n = a_nb'$. We have $ba_i = ba_nc_i = a_nb'c_i$. Consequently, $b'c_i \in R$, for $0 \leq i \leq n$, and so $b'f_I \in R[x;\sigma,d]$. Hence, by assumption, $b' \in a_nR$ and it follows that $b \in a_nR$. Thus f is a Sharma polynomial. □

Mathematical Journal of Okayama University, Vol. 46 [2004], Iss. 1, Art. 11

Note that from Theorem 1.1 it follows that under the assumptions of Proposition 1.2, if we assume, in addition, that I is closed, then the conditions $(i), (ii), (iii)$ are satisfied.

To prove the converse of Theorem 1.1 we need some previous results. A σ-ideal H of R is an ideal with $\sigma(H) = H$. Given a σ-ideal H of R we denote by $H[x]$ the set of all the polynomials of $R[x; \sigma, d]$ whose coefficients are in H. First we prove the following which is well-known in polynomial rings when $H = 0$.

Lemma 1.3. *Assume that R is any (not necessarily prime) ring, $f \in R[x; \sigma, d]$ is a normal polynomial and H is a σ-ideal of R. Then there exists $h \in R[x; \sigma, d] \setminus H[x]$ with $hf \in H[x]$ if and only if there exists $b \in R \setminus H$ with $bf \in H[x]$.*

Proof. Put $f(x) = \sum_{i=0}^{n} a_i x^i$, $a_n \neq 0$, and take $h = \sum_{i=0}^{m} b_i x^i$ of minimal degree m such that $hf \in H[x]$ and $h \notin H[x]$. Assume, by contradiction, that $m \geq 1$.

It is clear that we may suppose that $b = b_m \notin H$, by the minimality of h. Since $ha_n f = hfg \in H[x]$, for some $g \in R[x; \sigma, d]$, and $b\sigma^m(a_n) \in H$, we have that $ha_n \in H[x]$. Hence $h(f - a_n x^n) \in H[x]$. As above it follows that $ha_{n-1} \in H[x]$ and an induction argument gives $ha_i \in H[x]$, for $0 \leq i \leq n$. Therefore $b\sigma^m(a_i) \in H$, for all i, and so $\sigma^{-m}(b)f \in H[x]$. □

Lemma 1.4. *Let I be an R-disjoint ideal of $R[x; \sigma, d]$ and $f \in R[x; \sigma, d]$ a normal polynomial of minimal degree in I. Then the leading coefficient of f generates a σ-ideal.*

Proof. We denote by σ and d again the natural extensions of σ and d to additive mappings of $R[x; \sigma, d]$. Put $f(x) = \sum_{i=0}^{n} a_i x^i$, $a_n \neq 0$. Since $xf = \sigma(f)x + d(f)$ we have that $\sigma(f)x = fh - d(f)$, for some $h \in R[x; \sigma, d]$. Also $\delta(h) = 1$ because a_n is not a zero divisor. Thus $\sigma(a_n) \in a_n R$ follows.

On the other hand, $x\sigma^{-1}(f) = fx + d(\sigma^{-1}(f)) = gf + d(\sigma^{-1}(f))$, for some $g \in R[x; \sigma, d]$ with $\delta(g) = 1$. It is well-known that polynomials in $R[x; \sigma, d]$ can be written with coefficients at right as well. Doing this and comparing the coefficients of x^{n+1} in the last relation we obtain $\sigma^{-n-1}(a_n) = b\sigma^{-n}(a_n)$, for some $b \in R$. Consequently $\sigma^{-1}(a_n) \in Ra_n$. □

Note that the ideal $a_n R$ can be a σ-ideal for some a_n which is not σ-invariant. Examples of this type can be easily given, even with a_n a central element.

Now we are in position to prove the converse of Theorem 1.1.

Theorem 1.5. *Let R be a prime ring and I an R-disjoint ideal of $R[x; \sigma, d]$. Assume that there exists a Sharma polynomial $f \in I$ with $\delta(f) = Min(I)$. Then I is closed and principal generated by f.*

Proof. Put $f = \sum_{i=0}^{n} a_i x^i$, where $a = a_n \neq 0$, and take $g \in [I]$. If $\delta(g) = n$, then $ag - f\sigma^{-n}(b) \in [I]$, where b is the leading coefficient of g. So $ag = rf$, for some $r \in R$. It is not difficult to show, by induction on $\delta(g)$, that there exists $k \geq 0$ such that $a^k g = hf$, for some $h \in R[x;\sigma,d]$.

It is clear that $R[x;\sigma,d]f \subseteq I \subseteq [I]$. For $g \in [I]$ take $l \geq 0$ with $a^l g = hf$, for some $h \in R[x;\sigma,d]$, and suppose that l is minimal with this property. If $l = 0$, then $g \in R[x;\sigma,d]f$. Assume that $l \geq 1$. Then $hf \in aR[x]$, where aR is a σ-ideal of R by the former lemma. Thus Lemma 1.3 implies that there exists $r \in R$ such that $rf \in aR[x]$ and $r \notin aR$. This contradicts the fact that f is a Sharma polynomial. □

Putting together Theorems 1.1 and 1.5 we have the following

Corollary 1.6. *Assume that R is a prime ring and let I be an R-disjoint ideal of $R[x;\sigma,d]$. Then I is closed and principal generated by a normal polynomial of minimal degree if and only if I contains a Sharma polynomial of minimal degree. Moreover, in this case any normal generator of minimal degree in I is a Sharma polynomial and conversely.*

Remark 1.7. Note that the ideal I to be closed is a necessary condition. In fact, suppose that R is a commutative domain and $0 \neq a \in R$ is not invertible. Then the ideal $I = a(x-1)R[x]$ is a principal ideal which is not closed and the generator $f = ax + a$ is not a Sharma polynomial.

When we consider skew polynomial rings of automorphism type (derivation type) we can obtain similar results in a slightly more general context. In fact, when the ring R is σ-prime (resp. d-prime), σ an automorphism (resp. d a derivation) of R, then we can take as Q the symmetric ring of σ-quotients (resp. d-quotients) of R. The results obtained in this section can be proved again in these cases with similar arguments. Thus we have the following

Theorem 1.8. *Assume that R is a σ-prime (d-prime) ring and let I be an R-disjoint ideal of $R[x;\sigma]$ (resp. $R[x;d]$). Then I is closed and principal generated by a normal polynomial of minimal degree if and only if I contains a Sharma polynomial of minimal degree.*

Note that Theorem 1.8 applies for any R-disjoint prime of $R[x;\sigma]$ (resp. $R[x;d]$), since R-disjoint prime ideals are closed in these cases.

Remark 1.9. We point out that in both cases a result corresponding to Proposition 1.2 can also be obtained with the same proof.

2. Rings with unique factorization properties

Throughout this section we consider rings which satisfy unique factorization properties in the sense of [1].

Let R be a ring with an automorphism σ. An element $p \in R$ is said to be σ-prime if it is normal and the ideal pR is a σ-prime ideal of R.

Following [1], we say that R is a *σ-unique factorization ring* (σ-ufr, for short) if R is σ-prime and any non-zero σ-prime ideal of R contains a σ-prime element.

Lemma 2.1. *Assume that R is a σ-ufr. Then any non-zero σ-ideal of R contains a product of σ-prime elements.*

Proof. Suppose there exists a non-zero σ-ideal which does not contain a product of σ-prime elements. By Zorn's Lemma there exists a maximal σ-ideal J with this property. It easily follows that J is σ-prime, a contradiction. □

To prove the next theorem we need the following

Lemma 2.2. *Assume that R is a σ-ufr and suppose p, q are σ-prime elements of R. Then the element $p' \in R$ with $p'p = pq$ is also σ-prime.*

Proof. Since R is σ-prime and pR, qR are σ-prime ideals it easily follows that p' is normal and $p'R$ is a σ-ideal.

Suppose that A, B are σ-ideals of R and $AB \subseteq p'R$. Then $ABp \subseteq Rp'p = pRq$. Denote by A' and B' the ideals of R with $pA' = Ap$ and $pB' = Bp$, respectively. It follows that A' and B' are also σ-ideals and $pA'B' \subseteq pRq$. Thus $A'B' \subseteq Rq$ and so either $A' \subseteq Rq$ or $B' \subseteq Rq$. Assume $A' \subseteq Rq$ and take $a \in A$. So $ap = pa'$, for some $a' \in A'$, and we have $ap = pa' \in pRq \subseteq p'Rp$. Hence $a \in p'R$, i. e., $A \subseteq p'R$. This completes the proof. □

Now we are in position to prove the main result of this section. In the proof Q denotes the symmetric ring of σ-quotients of R.

Theorem 2.3. *Assume that R is a σ-ufr. Then any R-disjoint closed ideal of $R[x;\sigma]$ is principal generated by a Sharma polynomial of minimal degree.*

Proof. Let I be a closed ideal of $R[x;\sigma]$. If $x \in I$ we have $I = xR[x;\sigma]$ and we are done. So we may assume that $x \notin I$. Then there exists a monic polynomial $f_I \in Q[x;\sigma]$ such that $I = f_I Q[x;\sigma] \cap R[x;\sigma]$. Also, $\sigma(f_I) = f_I$ and $f_I r = \sigma^n(r) f_I$, for all $r \in R$, where $n = \delta(f_I) = Min(I)$.

To prove the result we use the characterization corresponding to Proposition 1.2 (see Remark 1.9). Denote by U the ideal of R consisting of all the elements $b \in R$ with $bf_I \in R[x;\sigma]$. It is clear that U is a σ-ideal. By Lemma 2.1 there exists $a \in U$ such that $a = p_1 ... p_r$, where p_i is a σ-prime element for all i. We can assume that r is a smallest integer with the property that U contains a product of this type. Thus a is a normal element, Ra is a σ-ideal and $f = af_I = ax^n + a_{n-1}x^{n-1} + ... + a_0 \in I$. Also, since $\delta(f) = Min(I)$, it easily follows that f is a normal element of $R[x;\sigma]$.

We claim that f is a Sharma polynomial. To prove this we show that $U = aR$ (Proposition 1.2). Thus using Theorem 1.5 the proof is complete provided we proved the claim.

First we show that for every $1 \leq j \leq r$ there exists $0 \leq i \leq n-1$ such that $a_i \notin p_jR$. Since by Lemma 2.2 we can change the order of the primes p_i, $i = 1, ..., r$, we may assume $j = 1$. In fact, if $a_i = p_1b_i$, for $b_i \in R$, we have

$$p_1(p_2...p_r)f_I = f = p_1(p_2...p_rx^n + b_{n-1}x^{n-1} + ... + b_0).$$

Since $R[x;\sigma]$ is prime and $Rp_1 = p_1R$ is a σ-ideal we obtain that $p_2...p_rf_I \in R[x;\sigma]$, which contradicts the minimality of r.

Take $b \in U$ and put $bf_I = g \in I$. Then $ag = abf_I = wf$, where $ab = wa$. Thus there exist $s_i \in R$, $0 \leq i \leq n$, with $s_ia = wa_i$, and so $wa_i \in Rp_1$. Note that since f is normal and its leading coefficient generates a σ-ideal, any other coefficient of f also generates a σ-ideal. Furthermore we know that there exists k such that $a_k \notin Rp_1$. It follows from $wa_k \in Rp_1$ that $w \in Rp_1$. Hence $w = p_1w_1$, $w_1 \in R$, and we obtain $t_ip_2...p_r = w_1a_i$, for all i and some $t_i \in R$. It follows by the same way that $w_1 \in p_2R$ and an induction argument shows that $w \in p_1...p_rR = aR$. So $b \in aR$. Therefore $U = aR$ and the proof is complete. □

Similar result can be proved for skew polynomial rings of derivation type. Let R be a ring and d a derivation of R. A d-prime element of R is defined as a normal element such that pR is a d-prime ideal of R. We say that R is a *d-unique factorization ring* (d-ufr, for short) if R is d-prime and any non-zero d-prime ideal of R contains a d-prime element.

Using similar arguments as in the automorphism case it is not hard to prove results corresponding to Lemmas 2.1, 2.2 and Theorem 2.3. Actually, in the derivation case the computation is slightly more complicated, but we will omit here the details. So we have

Theorem 2.4. *Let R be a d-ufr. Then any R-disjoint closed ideal of $R[x;d]$ is principal generated by a Sharma polynomial of minimal degree.*

References

[1] Chatters, A.W., Jordan, D.A., *Noncommutative Unique Factorization Rings*, J. London Math. Soc. **33**(2) (1986), 22–32.

[2] Chun, J.H., Park, J.W., *Principal Ideals in Skew Polynomial Rings*, Comm. Korean Math. Soc. **14**(4) (1999), 699–706.

[3] Cisneros, E., Ferrero, M., and Gonzáles, M.I., *Prime Ideals of Skew Polynomial Rings and Skew Laurent Polynomial Rings*, Math. J. Okayama Univ. **32** (1990), 61–72.

[4] Kanemitsu, M., Yoshida, K., *Conditions for an Ideal in a Polynomial Ring to be Principal*, Comm. Alg. **19**(3) (1981), 749–766.

[5] Lam, T.Y., Leroy, A., Matczuk, J., *Primeness, Semiprimeness and Prime Radical of Ore Extensions*, Comm. Alg. **25**(8) (1997), 2459–2506.

[6] Leroy, A., Matczuk, J., *Prime Ideals of Ore Extensions*, Comm. Alg. **19**(7) (1991), 1893–1907.

[7] Leroy, A., Matczuk, J., *The Extended Centroid and X-Inner automorphisms of Ore Extensions*, J. Algebra **145**(1) (1992), 143–177.

[8] Sharma, P.K., *A note on Ideals in Polynomial Rings*, Arch. Math. **37** (1981), 325–329.

[9] Voskoglou, M.G., Sapanci, M., *Principal Ideals in Skew Polynomial Rings*, Bull. Greek Math. Soc. **36** (1994), 133–137.

Wagner Cortes
Departamento de Matemática
Universidade Estadual
Maringá
87020-900, Brazil
e-mail address: wocortes@uem.br

Miguel Ferrero
Instituto de Matemática
Universidade Federal do Rio Grande do Sul
91509-900, Porto Alegre-RS, Brazil
e-mail address: mferrero@mat.ufrgs.br

(*Received February 18, 2004*)