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Some Toda Bracket in $\pi^s{}_{26}(S^0)$

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SOME TODA BRACKET IN $\pi_{26}^S(S^0)$

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1. Introduction

Throughout this note, we work in the 2-primary components of homotopy groups of spheres. Let $\iota \in \pi_0^S(S^0)$, $\eta \in \pi_1^S(S^0)$, $\nu \in \pi_3^S(S^0)$, $\sigma \in \pi_7^S(S^0)$, ε, $\bar{\nu} \in \pi_8^S(S^0)$, $\mu \in \pi_9^S(S^0)$, $\zeta \in \pi_{11}^S(S^0)$, $\kappa \in \pi_{14}^S(S^0)$, $\rho \in \pi_{15}^S(S^0)$, ω, $\eta^* \in \pi_{16}^S(S^0)$, $\bar{\mu} \in \pi_{17}^S(S^0)$, ν^*, $\xi \in \pi_{18}^S(S^0)$, $\bar{\zeta}$, $\bar{\sigma} \in \pi_{19}^S(S^0)$ and $\bar{\kappa} \in \pi_{20}^S(S^0)$ be generators ([10], [6]). We know the following ([4], [5], [8]): $\pi_{21}^S(S^0) = \mathbf{Z}_2\{\sigma^3\} \oplus \mathbf{Z}_2\{\eta\bar{\kappa}\}$, $\pi_{22}^S(S^0) = \mathbf{Z}_2\{\nu\bar{\sigma}\} \oplus \mathbf{Z}_2\{\eta^2\bar{\kappa}\}$, $\pi_{23}^S(S^0) = \mathbf{Z}_{16}\{\bar{\rho}\} \oplus \mathbf{Z}_8\{\nu\bar{\kappa}\} \oplus \mathbf{Z}_2\{\phi\}$, $\pi_{24}^S(S^0) = \mathbf{Z}_2\{\delta\} \oplus \mathbf{Z}_2\{\bar{\mu}\sigma\}$, $\pi_{25}^S(S^0) = \mathbf{Z}_2\{\mu_{3,*}\} \oplus \mathbf{Z}_2\{\eta\bar{\mu}\sigma\}$ and $\pi_{26}^S(S^0) = \mathbf{Z}_2\{\eta\mu_{3,*}\} \oplus \mathbf{Z}_2\{\nu^2\bar{\kappa}\}$.

About a Toda bracket $\langle \sigma, 2\sigma, \zeta \rangle$, Mahowald obtained the equality $\langle \sigma, 2\sigma, \zeta \rangle = \nu^2\bar{\kappa}$ and he has the several proofs of that. The purpose of this note is to give a proof of this fact by using the calculations based on the composition methods [10].

Theorem 1. $\nu^2\bar{\kappa} = \langle \sigma, 2\sigma, \zeta \rangle = \langle \zeta, \sigma, 2\sigma \rangle = \langle \sigma, \zeta, \sigma \rangle = \langle \sigma, 4\nu^*, 2\iota \rangle = \langle \nu^*, 2\sigma, 8\iota \rangle = \langle 2\sigma, 8\iota, \nu^* \rangle = \langle \eta\phi, \eta, 2\iota \rangle = \langle \varepsilon\omega, \eta, 2\iota \rangle$.

The equality $\langle \sigma, 4\nu^*, 2\iota \rangle = \nu^2\bar{\kappa}$ is used to determine the group extension of the 2-primary component of $\pi_{41}(F_4/G_2)$ ([2]). The fact $\langle \zeta, \sigma, 2\sigma \rangle = \nu^2\bar{\kappa}$ gives an information that the element $\upsilon \in \pi_{35}(S^9)$ ([8], Part I. (8.22)) becomes stably $\nu^2\bar{\kappa}$.

The key step to the equality $\langle \sigma, 2\sigma, \zeta \rangle = \nu^2\bar{\kappa}$ is to use Oda's relation $4\Sigma^2\delta'' = \nu_9^3\bar{\kappa}_{18}$ ([7]). We use the result, the notation of [10] and the properties of Toda brackets freely.

The authors wish to thank Mahowald for giving the definite information.

2. Equalities of the Toda brackets

We denote by $SO(n)$ the rotation group and by $J : \pi_k(SO(n)) \to \pi_{n+k}(S^n)$ the J-homomorphism. In general we have

$$J(\alpha \circ \beta) = J(\alpha) \circ \Sigma^n \beta$$

and

$$J\{\alpha, \beta, \gamma\} \subset \{J(\alpha), \Sigma^n\beta, \Sigma^n\gamma\}.$$

Suppose that n is a sufficiently large integer and s, t, u, a, b, c are integers with $a = 3$ or 7. We denote by $\alpha_{s,a}(n) \in \pi_{8s+a}(SO(n)) \cong \mathbf{Z}$ ([1]) a generator and we set $J(\alpha_{s,a}(n)) = j_{s,a}(n) \in \pi_{n+8s+a}(S^n)$ and $j_{s,a} = \Sigma^\infty j_{s,a}(n) \in \pi^S_{8s+a}(S^0)$. Suppose that $\beta \in \pi_{8(s+t)+a+b}(S^{8s+a})$ and $\gamma \in \pi_{8(s+t+u)+a+b+c}(S^{8(s+t)+a+b})$ are elements such that

$$\alpha_{s,a}(n) \circ \beta = 0,\ \beta \circ \gamma = 0 \text{ with } a + b + c \equiv 2, 3, 4, 5, 6 \bmod 8$$

and that the order of γ is finite. Then a Toda bracket $\{\alpha_{s,a}(n), \beta, \gamma\}$ is trivial, because $\pi_{8(s+t+u)+a+b+c+1}(SO(n)) = 0$ or $\cong \mathbf{Z}$ ([1]) and

$$d\{\alpha_{s,a}(n), \beta, \gamma\} = -\alpha_{s,a}(n) \circ \{\beta, \gamma, d\iota_{8(s+t+u)+a+b+c}\}$$

is finite [9], where d is the order of γ.

Since $\pi_k(SO(n)) \cong 0$ if $k \equiv 2, 4, 5$ or $6 \bmod 8$ ([1]), we have the following: $\alpha_{s,a}(n) \circ \nu_{8s+a} = 0$ if $s \geq 1$ or $s = 0$ and $a = 7$; $\alpha_{1,3}(n) \circ \eta_{11} = 0$; $\alpha_{s,a}(n) \circ \sigma_{8s+a} = 0$ if $s \geq 1$; $\alpha_{0,7}(n) \circ \kappa_7 = 0$; $\alpha_{1,3}(n) \circ \zeta_{11} = 0$; $\alpha_{0,7}(n) \circ \bar{\zeta}_7 = \alpha_{0,7}(n) \circ \bar{\sigma}_7 = \alpha_{1,7}(n) \circ \zeta_{15} = 0$.

We often use the anti-commutativity of the composition of two elements of $\pi^S_*(S^0)$ ([10], (3.4)). We know that $\nu'\zeta_6 = 0$, $\nu_{11}\sigma_{14} = 0$, $\sigma_{12}\nu_{19} = 0$ and $2\sigma^2_{16} = 0$ ([10]). Hence we have the following.

Lemma 2.1. **(i):** $\nu\sigma = 0$, $\eta\zeta = 0$, $\nu\zeta = 0$, $\nu\rho = \sigma\zeta = 0$, $\sigma\kappa = 0$, $\sigma\rho = \zeta^2 = 0$, $\sigma\bar{\zeta} = \sigma\bar{\sigma} = \zeta\rho = 0$.
(ii): $0 \in \langle \sigma, \nu, 2\nu \rangle$, $0 \in \langle \nu, 2\nu, \zeta \rangle$, $0 \in \langle j_{s,a}, \nu, \sigma \rangle$ *if* $s \geq 1$, $0 \in \langle j_{s,a}, \sigma, \nu \rangle$ *if* $s = 1$ *and* $a = 7$, *or if* $s \geq 2$ *and* $0 \in \langle j_{s,a}, \sigma, 2\sigma \rangle$ *if* $s \geq 2$.

The indeterminacy of $\langle \sigma, \nu, 2\nu \rangle$ is σ^2. By Lemma 2.1.(i), the indeterminacy of $\langle \nu, 2\nu, \zeta \rangle$ is $\nu \circ \pi^S_{15}(S^0) + \pi^S_7(S^0) \circ \zeta = 0$ and that of $\langle \rho, \nu, \sigma \rangle$ is $\rho \circ \pi^S_{11}(S^0) + \pi^S_{19}(S^0) \circ \sigma = 0$. This implies the following.

Lemma 2.2. **(i):** $\langle \sigma, \nu, 2\nu \rangle \ni 0 \bmod \sigma^2$ *and* $\langle \nu, 2\nu, \zeta \rangle = 0$.
(ii): $\langle \rho, \nu, \sigma \rangle = 0$.

By the definition of ν^* and by use of (3.9).i), (3.5).ii) and (3.10) of [10], we have

$$\nu^* \in -\langle \sigma, 2\sigma, \nu \rangle = -\langle \nu, 2\sigma, \sigma \rangle = -\langle \nu, \sigma, 2\sigma \rangle = \langle \sigma, \nu, \sigma \rangle.$$

So we have

$$\sigma\nu^* \in -\sigma \circ \langle \nu, \sigma, 2\sigma \rangle = -\langle \sigma, \nu, \sigma \rangle \circ 2\sigma \ni -2\sigma\nu^* \bmod 0.$$

This implies the relation

$$\sigma\nu^* = 0.$$

We recall that $\pi_{15}^S(S^0) = \mathbf{Z}_{32}\{\rho\} \oplus \mathbf{Z}_2\{\eta\kappa\}$ and $\pi_{19}^S(S^0) = \mathbf{Z}_8\{\bar{\zeta}\} \oplus \mathbf{Z}_2\{\bar{\sigma}\}$ ([10]). From the facts $\bar{\sigma} \in \langle \nu, \sigma, \eta\sigma \rangle$, $\nu^* \in \langle \sigma, \nu, \sigma \rangle$ and $\eta\nu^* = 0$ ([10]), it follows that

$$\sigma\bar{\sigma} \in \sigma \circ \langle \nu, \sigma, \eta\sigma \rangle = -\langle \sigma, \nu, \sigma \rangle \circ \eta\sigma \ni \nu^*\eta\sigma = 0 \bmod 0.$$

So we have $\sigma\bar{\sigma} = 0$.

The indeterminacies of Toda brackets $\langle \zeta, \sigma, 2\sigma \rangle$ and $\langle \sigma, \zeta, \sigma \rangle$ are trivial because $\zeta \circ \pi_{15}^S(S^0) = 0$ and $\sigma \circ \pi_{19}^S(S^0) = 0$. Hence, by (3.10) of [10], we have

$$\langle \zeta, \sigma, 2\sigma \rangle = \langle \sigma, \zeta, \sigma \rangle.$$

By Proposition 12.20 of [10], $\sigma\mu = \eta\rho$ and $\omega \equiv \eta^* \bmod \sigma\mu$. By Theorem 14.1 of [10], $\nu\rho = 0$ and $4\nu^* = \eta^2\eta^*$. Since

$$4\nu^* = \eta^2\eta^* \equiv \eta^2\omega \bmod \eta^3\rho = 4\nu\rho = 0,$$

we have $4\nu^* = \eta^2\omega$. So, by the fact $\eta\sigma\omega = \eta\phi = \varepsilon\omega$ ([5], (6.3)), we have

$$\langle \sigma, 4\nu^*, 2\iota \rangle \supset \langle \eta\phi, \eta, 2\iota \rangle \bmod \sigma \circ \pi_{19}^S(S^0) + 2\pi_{26}^S(S^0) = 0.$$

Therefore we have $\langle \sigma, 4\nu^*, 2\iota \rangle = \langle \eta\phi, \eta, 2\iota \rangle = \langle \varepsilon\omega, \eta, 2\iota \rangle$.

Next, by the symmetry of the stable Toda barcket ((3.9).i) of [10]) and (3.10) of [10], we have $\nu^* \in \langle 2\sigma, \sigma, \nu \rangle$. By (3.9).i) and (3.5).ii) of [10], we have

$$\langle \sigma, 2\sigma, \zeta \rangle = \langle \zeta, \sigma, 2\sigma \rangle.$$

By the Jacobi identity ((3.7) of [10]) and by the fact $\nu^* \in \langle \nu, \sigma, 2\sigma \rangle$, we have

$$0 \in \langle \langle \sigma, 2\sigma, \sigma \rangle, 4\nu, 2\iota \rangle - \langle \sigma, \langle 2\sigma, \sigma, 4\nu \rangle, 2\iota \rangle \rangle + \langle \sigma, 2\sigma, \langle \sigma, 4\nu, 2\iota \rangle \rangle.$$

By the proof of Lemma 8.2 of [4], $\langle \sigma, 2\sigma, \sigma \rangle = 0$. By Lemma 9.1 of [10], we have $\zeta \in \langle \sigma, 4\nu, 2\iota \rangle$. The indeterminacies of $\langle \sigma, 4\nu^*, 2\iota \rangle$ and $\langle \sigma, 2\sigma, \zeta \rangle$ are $\sigma \circ \pi_{19}^S(S^0) + 2\pi_{26}^S(S^0) = 0$ and $\sigma \circ \pi_{19}^S(S^0) + \pi_{15}^S(S^0) \circ \zeta = 0$ respectively. So we have $\langle \sigma, 4\nu^*, 2\iota \rangle = \langle \sigma, 2\sigma, \zeta \rangle$.

By the Jacobi identity, we have

$$\begin{aligned} \langle \sigma, \nu^*, 8\iota \rangle &= \langle \sigma, \langle \nu, \sigma, 2\sigma \rangle, 8\iota \rangle \\ &\equiv \langle \langle \sigma, \nu, \sigma \rangle, 2\sigma, 8\iota \rangle + \langle \sigma, \nu, \langle \sigma, 2\sigma, 8\iota \rangle \rangle \\ &= \langle \nu^*, 2\sigma, 8\iota \rangle + \langle \sigma, \nu, \rho \rangle. \end{aligned}$$

So, by Lemma 2.2.(ii), we have $\langle \sigma, \nu^*, 8\iota \rangle = \langle \nu^*, 2\sigma, 8\iota \rangle$. Since the indeterminacy of $\langle \sigma, 4\nu^*, 2\iota \rangle$ is trivial, we have $\langle \sigma, 4\nu^*, 2\iota \rangle = \langle \sigma, \nu^*, 8\iota \rangle$.

By (3.9).i), of [10], we have

$$\begin{aligned} \langle 2\sigma, 8\iota, \nu^* \rangle &= \langle \nu^*, 8\iota, 2\sigma \rangle \\ &\subset \langle \nu^*, 8\sigma, 2\iota \rangle \\ &\supset \langle \nu^*, 2\sigma, 8\iota \rangle. \end{aligned}$$

By Lemma 12.24 of [10] and Part II. (6.3) of [8], we have $\nu^*\varepsilon = \xi\varepsilon = 0$. By Lemma 12.24 of [10], Part II. (6.3) of [8] and Lemma 2.1.(i) we have $\nu^*\bar{\nu} = \xi\bar{\nu} = \sigma\bar{\sigma} = 0$. Hence the indeterminacy of $\langle \nu^*, 8\sigma, 2\iota\rangle$ is $\nu^*\pi_8^S(S^0) + 2\pi_{26}^S(S^0) = \{\nu^*\varepsilon,\ \nu^*\bar{\nu}\} = 0$. Thus we have $\langle 2\sigma, 8\iota, \nu^*\rangle = \langle \nu^*, 2\sigma, 8\iota\rangle$. This concludes that all Toda brackets of Theorem 1 are equal. We show

Lemma 2.3. $\langle 2\sigma, 8\iota, \nu^*\rangle \ni\ 0 \bmod \nu^2\bar{\kappa}$.

Proof. By the Jacobi identity, we have

$$\begin{aligned}\langle 2\sigma, 8\iota, \nu^*\rangle &= \langle 2\sigma, 8\iota, \langle 2\sigma, \sigma, \nu\rangle\rangle \\ &\equiv \langle\langle 2\sigma, 8\iota, 2\sigma\rangle, \sigma, \nu\rangle - \langle 2\sigma, \langle 8\iota, 2\sigma, \sigma\rangle, \nu\rangle.\end{aligned}$$

We have $\langle 2\sigma, 8\iota, 2\sigma\rangle \subset \langle \sigma, 16\sigma, 2\iota\rangle = \langle \sigma, 0, 2\iota\rangle \ni 0 \bmod 2\rho$ and $\langle 2\rho, \sigma, \nu\rangle = \langle 2\sigma, \rho, \nu\rangle \ni 0 \bmod \nu^2\bar{\kappa}$. This completes the proof. □

3. Proof of the theorem

First we prepare the materials. We recall the element $\sigma_{16}^* \in \pi_{38}(S^{16})$ ([3]). By [7], there exist elements $\delta' \in \{\sigma'' \circ \sigma_{13}, \sigma_{20}, 2\sigma_{27}\}_3 \subset \pi_{35}(S^6)$ and $\delta'' \in \{\sigma' \circ \sigma_{14}, \sigma_{21}, 2\sigma_{28}\}_4 \subset \pi_{36}(S^7)$, which satisfies the relations $2\delta'' = -\Sigma\delta'$, $\Sigma^2\delta'' = 2(\sigma_9\sigma_{16}^*)$ and $2\delta' \equiv \nu_6^3\bar{\kappa}_{15} \bmod \nu_6\sigma_9\bar{\sigma}_{16}$. By Part III. Proposition 4.5.(2) of [8], $\nu_9\sigma_{12}\bar{\sigma}_{19} = 0$. So, by Part III.Theorem 3.(a) of [8], we have

$$\nu_9^3\bar{\kappa}_{18} = 4\Sigma^2\delta'' = 8(\sigma_9\sigma_{16}^*) \neq 0.$$

By (10.7) and (12.25) of [10], we know $\nu_8\zeta_{11} = 4\Sigma\sigma' \circ \sigma_{15}$ and

$$\zeta_{10}\sigma_{17} = 2\sigma_{10}\zeta_{17} = [\iota_{10}, \mu_{10}].$$

We show the following.

Lemma 3.1. $\nu_9^3\bar{\kappa}_{18} = \nu_9 \circ \Sigma\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}_5$.

Proof. We have

$$\begin{aligned}4\Sigma^2\delta'' &\in \Sigma\{4\Sigma\sigma' \circ \sigma_{15}, \sigma_{22}, 2\sigma_{29}\}_5 \\ &= \Sigma\{\nu_8\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}_5 \\ &\supset \nu_9 \circ \Sigma\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}_5 \\ &\bmod \Sigma(4\Sigma\sigma' \circ \sigma_{15}) \circ \Sigma^6\pi_{32}(S^{17}) + 2\Sigma\pi_{30}(S^8) \circ \sigma_{31}.\end{aligned}$$

We have $\Sigma(4\Sigma\sigma'\circ\sigma_{15})\circ\Sigma^6\pi_{32}(S^{17}) = 8\{\sigma_9^2\circ\rho_{23}\} = 0$ and $2\Sigma\pi_{30}(S^8)\circ\sigma_{31} = 2\{\sigma_9\rho_{16}\sigma_{23}\} = 0$ by Lemma 6.2 of [3] and [4]. This completes the proof. □

By Part I.Theorem 1.(b) of [8], we have

$$\pi_{37}(S^{11}) = \mathbf{Z}_8\{\tau'''\} \oplus \mathbf{Z}_2\{\theta' \circ \kappa_{23}\} \oplus \mathbf{Z}_2\{\nu_{11}^2\bar{\kappa}_{17}\} \oplus \mathbf{Z}_2\{\sigma_{11}\bar{\sigma}_{18}\} \oplus \mathbf{Z}_2\{\eta_{11}\mu_{3,12}\}.$$

By the proof of Part I.Proposition 4.2.(1) of [8], $\tau''' \in \{2\sigma_{11}, \nu_{18}, \rho_{21}\}_1$. Then we show

Lemma 3.2. $\tau''' \notin \{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}_1$.

Proof. By (7.21) of [10], $[\iota_{10}, \eta_{10}] = 2\sigma_{10}\nu_{17}$. So, by Proposition 2.6 of [10], we have

$$H\{2\sigma_{11}, \nu_{18}, \rho_{21}\}_1 = -\Delta(2\sigma_{10}\nu_{17}) \circ \rho_{22} = \eta_{21}\rho_{22} \neq 0.$$

On the other hand, we have

$$H\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}_1 = -\Delta(\zeta_{10}\sigma_{21}) \circ 2\sigma_{30} = 2\mu_{21}\sigma_{30} = 0.$$

This completes the proof. □

So the rest of our work is to investigate the elements $\nu_9\Sigma\theta' \circ \kappa_{24}$ and $2\nu_9 \circ \Sigma\tau'''$.

Lemma 3.3. $\nu_9\Sigma\theta' \circ \kappa_{24} \equiv 0 \bmod \eta_9\varepsilon_{10}\bar{\kappa}_{18}$.

Proof. By Lemma 7.5 of [10], $\theta' \in \{\sigma_{11}, 2\nu_{18}, \eta_{21}\}_1$. By (7.19) of [10], $\Sigma\sigma' \circ \nu_{15} = x\nu_8\sigma_{11}$ for x odd. So we have

$$\begin{aligned} \nu_9\Sigma\theta' &\in \nu_9 \circ \{\sigma_{12}, 2\nu_{19}, \eta_{22}\} \\ &\subset \{\nu_9\sigma_{12}, 2\nu_{19}, \eta_{22}\} \\ &= \{2\sigma_9\nu_{16}, 2\nu_{19}, \eta_{22}\} \\ &\supset 2\sigma_9 \circ \{\nu_{16}, 2\nu_{19}, \eta_{22}\} \\ &\bmod \nu_9\sigma_{12}\pi_{24}(S^{19}) + \pi_{23}(S^9) \circ \eta_{23}. \end{aligned}$$

Since $\{\nu_{16}, 2\nu_{19}, \eta_{22}\} \subset \pi_{24}(S^{16}) \cong \pi_8^S(S^0) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$, we have $2\sigma_9 \circ \{\nu_{16}, 2\nu_{19}, \eta_{22}\} = 0$. We have $\nu_9\sigma_{12}\pi_{24}(S^{19}) + \pi_{23}(S^9) \circ \eta_{23} = \{\sigma_9^2\eta_{23},\ \kappa_9\eta_{23}\}$ ([10]). By Proposition 7.2 of [4], $\sigma_9^2\eta_{23}\kappa_{24} = \sigma_9\eta_{16}\sigma_{17}\kappa_{24} = 0$. By Part III. Proposition 2.4.(3) of [8], $\kappa_9\eta_{23}\kappa_{24} = \eta_9\kappa_{10}^2 = \bar{\varepsilon}_9\kappa_{24} = \eta_9\varepsilon_{10}\bar{\kappa}_{18}$. This completes the proof. □

Next we show

Lemma 3.4. $2\nu_9 \circ \Sigma\tau''' = 0$.

Proof. By (7.20) of [10], we have $4\nu_9\sigma_{12} = 0$. So we have

$$\begin{aligned} 2\nu_9 \circ \Sigma\tau''' &\in 2\nu_9 \circ \{2\sigma_{12}, \nu_{19}, \rho_{22}\} \\ &\subset \{4\nu_9\sigma_{12}, \nu_{19}, \rho_{22}\} \\ &= \{0, \nu_{19}, \rho_{22}\} \\ &\bmod \pi_{23}(S^9) \circ \rho_{23}. \end{aligned}$$

By Part II. Proposition 2.1.(4) and (6) of [8], we have $\sigma_9^2\rho_{23} = 2\sigma_9\rho_{16}\sigma_{31} = \Sigma^2(\sigma'\rho_{14}\sigma_{29}) = 0$. By Part III. Proposition 2.4.(4) of [8], $\kappa_9\rho_{23} = 0$. So we have $\pi_{23}(S^9) \circ \rho_{23} = \{\sigma_9^2\rho_{23},\ \kappa_9\rho_{23}\} = 0$. This completes the proof. □

Now we show the following result implying the result $\langle\zeta, \sigma, 2\sigma\rangle = \nu^2\bar{\kappa}$.

Lemma 3.5. $\nu_{11}^2\bar{\kappa}_{17} \equiv \{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}$ mod $2\tau'''$, $\theta' \circ \kappa_{23}$, $\sigma_{11}\bar{\sigma}_{18}$, $\eta_{11}\mu_{3,12}$.

Proof. By Part I.Theorem 1 of [8] and Lemma 3.2, $\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}$ consists of elements $2\tau'''$, $\theta' \circ \kappa_{23}$, $\nu_{11}^2\bar{\kappa}_{17}$, $\sigma_{11}\bar{\sigma}_{18}$ and $\eta_{11}\mu_{3,12}$. By Lemma 3.3, $\nu_9\Sigma\theta' \circ \kappa_{24} = a\eta_9\varepsilon_{10}\bar{\kappa}_{18}$ for $a = 0$ or 1. We have $\nu_8\eta_{11}\mu_{3,12} = 0$. So $\nu_9 \circ \Sigma\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}$ consists of elements $\nu_9^3\bar{\kappa}_{18}$ and $a\eta_9\varepsilon_{10}\bar{\kappa}_{18}$. By Part III.Theorem 3.(a) of [8], $\nu_9^3\bar{\kappa}_{18} = 8(\sigma_9\sigma_{16}^*)$ and $\eta_9\varepsilon_{10}\bar{\kappa}_{18}$ are independent. Thus Lemma 3.1 leads to the assertion, completing the proof. □

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