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On the homotopy types of the orbit spaces of free T^2-actions on $S^3 \times S^5$ (II)

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ON THE HOMOTOPY TYPES OF THE ORBIT SPACES OF FREE T^2-ACTIONS ON $S^3 \times S^5$ (II)

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The purpose of this paper which is a continuation of [3] is to determine the homotopy type of the orbit spaces of free actions of the torus T^2 of rank 2 on the product $S^3 \times S^5$ of spheres of dimension 3 and 5 respectively. The main result of this paper is the following: The integral cohomology rings of the orbit spaces are rings $R = Z[x, y]/(\phi(x, y), \psi(x, y))$ of type 1 or 2 (see § 2) and there is only one (up to homotopy equivalence) simply connected finite CW complex having the given integral cohomology ring R.

1. Let L be the product space $CP(3) \times CP(3)$ of the complex projective spaces of dimension 6, and ι a generator of $H^2(CP(3); Z)$. We identify a map f and the homotopy class $[f]$ represented by f and use the braket notation [,] for Whitehead products. In the product cell decomposition of L, the 4 skelton $L^{(4)}$ is

$$(S_1^2 \vee S_2^2) \cup_{i_1 \circ h} e_1^4 \cup_{i_2 \circ h} e_2^4 \cup_{[i_1, i_2]} e_3^4$$

where i_1 and i_2 are inclusions of S^2 into S_1^2 and S_2^2 respectively, and h is the Hopf map. The dual cohomology classes of S_1^2 and S_2^2 are $x = \iota \times 1$ and $y = 1 \times \iota$ respectively. We represent the 6 skelton $L^{(6)}$ of L as follows $L^{(4)} \cup_{k_1} e_1^6 \cup_{k_2} e_2^6 \cup_{k_3} e_3^6 \cup_{k_4} e_4^6$, where the dual cohomology class of e_i^6 is $x^{4-i} y^{i-1} (i = 1, \cdots, 4)$. The first few homotopy groups of $L^{(4)}$ are as follows: $\pi_2(L^{(4)}) \cong 2Z$ with generators i_1 and i_2, $\pi_3(L^{(4)}) = \pi_4(L^{(4)}) = 0$ and $\pi_5(L^{(4)}) \cong 4Z$ with generators $k_i (i = 1, \cdots, 4)$

Lemma 1.1. *$\pi_6(L^{(4)})$ is isomorphic to $3Z + 4Z_2$ and generated by $[i_1, k_i]$, $[i_2, k_i]$ and $k_i \circ S^3(h)$ $(i = 1, \cdots, 4)$ where $S^3(h)$ denotes the three fold suspension of h. There holds the following*:

(1) $[i_1, k_2] + [i_2, k_1] = [i_1, k_4] + [i_2, k_3] = 0$.

(2) $[i_1, k_3] + [i_2, k_2] = k_2 \circ S^3(h) + k_3 \circ S^3(h)$.

(3) $[i_1, k_1] = k_1 \circ S^3(h)$ and $[i_2, k_4] = k_4 \circ S^3(h)$.

Proof. First we show the above relations modulo the torsion subgroup of $\pi_6(L^{(4)})$ by the method of J. P. Meyer [5]. Let (X_n, p_n, q_n) be a Postnikov system of $L^{(4)}$ where $p_n: X_n \longrightarrow X_{n-1}$ is a fiber map with fiber

F_n and $q_n: L^{(4)} \longrightarrow X_n$ is an n-equivalence. Then, $\pi_2(L^{(4)})$, $\pi_5(L^{(4)})$ and $\pi_6(L^{(4)})$ are identified with $\pi_2(X_5)$, $\pi_5(X_5)$ and $\pi_6(F_6)$, respectively. Let $m: F_5 \times X_5 \longrightarrow X_5$ be a map such that $m \mid F_5 \vee X_5 = i_5 \vee id.$ and $(p_3p_4p_5)\circ m = (p_3p_4p_5)\circ\pi$ where $i_5: F_5 \longrightarrow X_5$ is the inclusion and π is the projection of $F_5 \times X_5$ onto X_5. Then, by [5] we have the following formula:

(1. 2) $\Xi([\alpha, \beta]) = \tau(m_*(\Xi(\beta) \times \Xi(\alpha)))$, $(\alpha \in \pi_2(L^{(4)}),\ \beta \in \pi_5(L_{(4)}))$ where Ξ and τ are the Hurewicz homomorphism and the transgression respectively. As $X_2 = X_3 = X_4$, a simple calculation shows that $H^7(X_5; Z) \cong 3Z$ and $H^8(X_5; Z) \cong 4Z_2$, and hence $\pi_6(L^{(4)}) \cong 3Z + 4Z_2$. Let $\{\iota_i^{(5)},\ i = 1, \cdots, 4\}$ be a base of $H^5(F_5; Z)$ with $< \iota_i^{(5)}, \Xi(k_j) > = \delta_{ij}$, where F_5 is Eilenberg-MacLane space $K(4Z, 5)$. It is not difficult to show the following:

$$m^*(a_i) = 1 \times a_i + \iota_{i+1}^{(5)} \times \iota_1 - \iota_i^{(5)} \times \iota_2$$

where $a_i (i = 1, 2, 3)$ is a base of $H^7(X_5; Z)$, $q_5^*(\iota_1) = x$ and $q_5^*(\iota_2) = y$. Then we have

$$\begin{aligned} &< a_i, m_*(\Xi(k_2) \times \Xi(i_1)) > \\ &= < 1 \times a_i + \iota_{i+1}^{(5)} \times \iota_1 - \iota_i^{(5)} \times \iota_2, \Xi(k_2) \times \Xi(i_1) > = \delta_{i+1,2} \\ &= - < a_i, m_*(\Xi(k_1) \times \Xi(i_2)) >. \end{aligned}$$

Thus, $m_*(\Xi(k_2) \times \Xi(i_1)) + m_*(\Xi(k_1) \times \Xi(i_2))$ is a torsion element of $H_7(X_5; Z)$. Since Ξ and τ are isomorphisms, $[i_1, k_2] + [i_2, k_1] \equiv 0$ mod torsion by (1. 2). Similarly we can show that the other relations hold modulo the torsion subgroup.

The relation (3) follows from the fact that $[\pi_2(CP(2)), \pi_5(CP(2))] \neq 0$ [see, 1 p. 240]. Next, we shall show (1) and (2). Let $\tilde{L}$ be a subcomplex $L^{(4)} \cup_{k_2} e_2^6 \cup_{k_3} e_3^6 \cup_l e^8 (= CP(2) \times CP(2))$ of L and let $\tilde{L}^{(6)}$ be the 6 skelton of $\tilde{L}$. Then, $\pi_6(\tilde{L}^{(6)})$ is isomorphic to $2Z_2$ and generated by $k_1 \circ S^3(h)$ and $k_4 \circ S^3(h)$, and l generates a free part of $\pi_7(\tilde{L}^{(6)}) \cong Z + 2Z_2$. We consider the following part of the homotopy exact sequence of the pair $(\tilde{L}^{(6)}, L^{(4)})$:

$$\pi_7(\tilde{L}^{(6)}) \xrightarrow{j_*} \pi_7(\tilde{L}^{(6)}, L^{(4)}) \xrightarrow{\partial} \pi_6(L^{(4)}) \xrightarrow{i_*} \pi_6(\tilde{L}^{(6)}) \longrightarrow 0.$$

Let $\{\tilde{k}_2, \tilde{k}_3\}$ be the base of $\pi_6(\tilde{L}^{(6)}, L^{(4)})$ such that $\partial(\tilde{k}_i) = k_i (i = 2, 3)$. By Theorem (1. 4) in [4], we see that $\pi_7(\tilde{L}^{(6)}, L^{(4)})$ is isomorphic to $4Z + 2Z_2$ and generated by $[i_1, \tilde{k}_j]$, $[i_2, \tilde{k}_j]$ and $\tilde{k}_j \circ \tilde{h}\ (j = 2, 3)$ where $\tilde{h}$ is a generator of $\pi_7(D^6, S^5)$. Then, from the above exact sequence we see that $[i_1, k_j]$, $[i_2, k_j]$ and $k_i \circ S^3(h)\ (i = 1, \cdots, 4)$ generate $\pi_6(L^{(4)})$. Let p be the projection of $\tilde{L}^{(6)}$ onto $\tilde{L}^{(6)}/L^{(4)} = S_2^6 \vee S_3^6$. Now, from the first part of the proof we can put

$$j_*(l) = [i_1, \tilde{k}_3] + [i_2, \tilde{k}_2] + n\tilde{k}_2 \circ \tilde{h} + m\tilde{k}_3 \circ \tilde{h}$$

for some integers n and m. Then $p_*(l) = ni_2^{(6)} \circ S^4(h) + mi_3^{(6)} \circ S^4(h)$ where $i_j^{(6)}: S^6 \longrightarrow S_j^6$ $(j = 2, 3)$ is the inclusion. Let u_j be the dual cohomology class of S_j^6 $(j = 2, 3)$, and $\bar{p}: \tilde{L} \longrightarrow (S_2^6 \vee S_3^6) \cup_{p*(l)} e^8$ an extension of p. Then we have

$$< Sq^2(u_2), e^8 > = < Sq^2(\bar{p}^*(u_2), e^8 > = < Sq^2(x^2y), e^8 > = 1.$$

This implies $n = 1$, and similarly $m = 1$. Now, from $\partial j_*(l) = 0$ we obtain the relation (2). Similarly, we can prove (1).

2. Let $R = Z[x, y]/(\phi(x, y), \psi(x, y))$ be a graded ring where $\deg x = \deg y = 2$, $\phi(x, y) = \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2$ and $\psi(x, y) = \mu_1 x^3 + \mu_2 x^2 y + \mu_3 xy^2 + \mu_4 y^3$ with integers λ_i and μ_j satisfying the following equation

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & 0 \\ 0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{vmatrix} = \pm 1. \qquad (2.1)$$

If R is the integral cohomology ring of a topological space, then $R \otimes Z_2$ must be compatible with the cohomology operation Sq^2. Now, we see that any ring R satisfying the above condition has one of the following three types:

1. λ_2 is even. In this case, $R \otimes Z_2$ is isomorphic to $Z_2[x, y]/(x^2, y^3)$.
2. λ_2 is odd and $\lambda_1\lambda_3$ is even. In this case, $R \otimes Z_2$ is isomorphic to $Z_2[x, y]/(x^2 + xy, y^3)$.
3. λ_1, λ_2 and λ_3 are odd and $R \otimes Z_2$ is isomorphic to $Z_2[x, y]/(x^2 + xy + y^2, y^3)$.

In the following, we show that for any ring R having one of the above three types, there is a simply connected finite CW complex with R as its integral cohomology ring. Let $A = (a_{ij})$ be a 3×3 unimodular matrix and put $A^{-1} = (\lambda_{ij})$ where $\lambda_{i3} = \lambda_i$. We consider an another cell decomposition $(S_1^2 \vee S_2^2) \cup_{\alpha_1} e_1^4 \cup_{\alpha_2} e_2^4 \cup_{\alpha_3} e_3^4$ of the complex $L^{(4)}$ (see § 1) where $\alpha_i = a_{i1}(i_1 \circ h) + a_{i2}[i_1, i_2] + a_{i3}(i_2 \circ h)$. Then the dual cohomology class of e_i^4 is $\lambda_{1i}x^2 + \lambda_{2i}xy + \lambda_{3i}y^2$. Let $K^{(4)}$ be a subcomplex $(S_1^2 \vee S_2^2) \cup_{\alpha_1} e_1^4 \cup_{\alpha_2} e_2^4$ of $L^{(4)}$. Then, in $H^4(K^{(4)}; Z)$ we have $\phi(x, y) = 0$.

Lemma 2.2. $\pi_3(K^{(4)}) \cong Z$. *If* λ_1, λ_2 *and* λ_3 *are odd then* $\pi_4(K^{(4)}) = 0$, $\pi_5(K^{(4)}) \cong 2Z$, *otherwise* $\pi_4(K^{(4)}) \cong Z_2$, $\pi_5(K^{(4)}) \cong 2Z + Z_2$.

Proof. Let (X_n, p_n, q_n) be a Postnikov system of $K^{(4)}$. Then $H^5(X_3; Z) = 0$, and $H^6(X_3; Z)$ is isomorphic to $2Z$ or $2Z + Z_2$ according as λ_1, λ_2 and λ_3 are odd or not. If λ_1, λ_2 and λ_3 are odd then we see $\pi_4(K^{(4)}) = 0$, and hence $X_3 = X_4$ and $\pi_5(K^{(4)}) = 2Z$. Similarly we can prove the remaining case.

By (2.1), there is an 4×4 unimodular matrix $M = (m_{ij})$ such that

$$M^{-1} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \end{pmatrix}$$

We put $\kappa_i = \sum_{j=1}^{4} m_{ji} k_j$, then $\{\kappa_i,\ i = 1, \cdots, 4\}$ is a base of $\pi_5(L^{(4)})$. Let $\bar{L}$ be the complex $L^{(4)} \cup_{\kappa_1} e_1^6 \cup_{\kappa_2} e_2^6$. Then the dual cohomology classes of e_1^6 and e_2^6 are $\lambda_1 x^3 + \lambda_2 x^2 y + \lambda_3 x y^2$ and $\lambda_1 x^2 y + \lambda_2 x y^2 + \lambda_3 y^3$, respectively. Let $q: K^{(4)} \longrightarrow L^{(4)}$ and $\bar{q}: K^{(4)} \longrightarrow L$ be inclusions. We may assume that $\bar{q}$ is a fiber map with fiber F. Let $i^{(3)}: S^3 \longrightarrow F$ be a map which represents a generator of $\pi_3(F)$. Then, $i_*^{(3)}: \pi_j(S^3) \longrightarrow \pi_j(F)$ is 1—1 if $j \leqq 4$ and onto if $j \leqq 5$, since $H_4(F) = H_5(F) = 0$ from the cohomology spectral sequence of $\bar{q}$.

Now, assume R is of type 1 or 2. Then, since $\pi_4(K^{(4)}) \cong Z_2$ by Lemma 2.2 and $\pi_4(F) \cong Z_2$, from the homotopy exact sequence of $\bar{q}$ we see that $\bar{q}_*: \pi_5(K^{(4)}) \longrightarrow \pi_5(\bar{L})$ is onto. Let α and β be such that $\bar{q}_*(\alpha) = \kappa_3$ and $\bar{q}_*(\beta) = \kappa_4$. Then we have

(2.3) $$q_*(\alpha) = \kappa_3 \text{ and } q_*(\beta) = \kappa_4.$$

We put $K = K^{(4)} \cup_\beta e^6$. Then it is clear from the construction of K that $H^*(K; Z)$ is isomorphic to R.

Next, we assume R is of type 3 and show that there is a $\beta \in \pi_5(K^{(4)})$ such thct $q_*(\beta) = \kappa_4$. Putting $K = K^{(4)} \cup_\beta e^6$, we have $H^*(K; Z) \cong R$. Now, let $L_1 = \bar{L} \cup_{\kappa_3} e_3^6$ and $L_2 = \bar{L} \cup_{\kappa_4} e_4^6$ and let q_1 and q_2 be the inclusions of $K^{(4)}$ into L_1 and L_2, respectively. Then we have the following

Lemma 2.4. Coker $q_{1*} = 0$ *and* Coker $q_{2*} \cong Z_2$, *where* $q_{i*}: \pi_5(K^{(4)}) \longrightarrow \pi_5(L_i)$ $(i = 1, 2)$.

Proof. Let F_1 be a fiber of q_1. We consider the cohomology spectral sequence of the fiber space $K^{(4)}\longrightarrow L_1$. It is clear that $\{x\phi(x,y), y\phi(x,y), \psi(x,y)\}$ is a base of $H^6(L_1;Z)$. Since $q_1{}^*(\psi(x,y))=0$, there is an $a\in H^3(F_1;Z)\cong Z$ such that $d_4(a)=\phi(x,y)$. Now, we see that $d_4: E_4^{2,3}\longrightarrow E_4^{6,0}$ is injective, and hence $E_6^{6,0}$ is isomorphic to Z and generated by $\psi(x,y)$, $E_\infty^{2,3}=0$ and $H^4(F_1;Z)=0$. Thus $d_6: H^5(F_1;Z)\cong E_6^{0,5}\longrightarrow E_6^{6,0}$ is an isomorphism. We also have $H^6(F_1;Z)\cong 5Z$ and $a^2=0$. In order to show $Sq^2(a)\neq 0$, we consider the spectral sequence of the coeficients mod 2. Since R is of type 3, by (2. 1) we see $\psi(x,y)\equiv y^3$ mod $\{x\phi(x,y), y\phi(x,y)\}$ where $\phi(x,y)=x^2+xy+y^2$. Now, we have $d_6(Sq^2(a)) = Sq^2(\phi(x,y))=x^2y+xy^2\equiv y^3\equiv\psi(x,y)\neq 0$. Thus $Sq^2(a)\neq 0$.

Let $f: F_1\longrightarrow K(Z,3)$ be a map such that $f^*(\iota)=a$ where ι is a generator of $H^3(K(Z,3);Z)$. Now, from the cohomology spectral sequence of the principal fibration $G\longrightarrow F_1$ induced by f, we see that $H^4(G;Z)=0$ and $H^5(G;Z)\cong Z$. Then $\pi_4(G)\cong H_4(G;Z)=0$ by the Hurewicz isomorphism, and hence $\pi_4(F_1)=0$. Now from the homotopy exact sequence of the fiber space $F_1\longrightarrow K^{(4)}\longrightarrow L_1$, it follows that Coker $q_{1*}=0$. We can see that Coker $q_{2*}\cong Z_2$ by making use of the similar argument.

Since $\pi_4(K^{(4)})=0$ by Lemma 2. 2 and $\pi_4(F)\cong Z_2$, we obtain Coker $\bar{q}_*\cong Z_2$. Then we see by Lemma 2. 4 that there is $\beta\in\pi_5(K^{(4)})$ such that $\bar{q}_*(\beta)=\kappa_4$.

Theorem 2. 5. *Let* $R=Z[x,y]/(\phi(x,y),\psi(x,y))$ *be a ring of type* 1, 2 *or* 3. *Then there exists only one* (*up to homotopy equivalence*) *simply connected finite CW complex* K *such that* $H^*(K;Z)\cong R$.

Proof. It remains to show the uniqueness of K. It is clear that the 4 skelton $K^{(4)}$ is well determined by the cohomology ring structure of K, and hence the homotopy type of K depends only on the homotopy class of the attaching map of the 6 cell. First, let R be a ring of type 3. Then, $q^*: \pi_5(K^{(4)})\longrightarrow\pi_5(L^{(4)})$ is injective, since $\pi_5(K^{(4)})\cong 2Z$ by Lemma 2. 2. Hence the attaching map β is well determined by the cohomology ring structure of K. Thus, in this case, K is unique up to homotopy equivalence.

Now suppose that R is a ring of type 1 or 2. In this case, $\pi_5(K^{(4)})\cong 2Z+Z_2$ is generated by α, β and $\gamma=i^{(3)}\circ S(h)\circ S^2(h)$ where $i^{(3)}$ is a generator of $\pi_3(K^{(4)})$ (see (2. 3)). Thus there are two attaching maps β and $\beta+\gamma$ for the 6 cell. We show by a slight generalization of the proof of Theorem 2. 5 in [3] that there exists a homotopy equivalence f of $K^{(4)}$ such that $f_*(\beta)=\beta+\gamma$. Then this will show the uniqueness of K.

By (2. 1), we may take such a base $\{u,v\}$ of $H^4(K;Z)\cong R^4$ (R^4

the homogeneous part of R of degree 4) that u and v are the respective duals of x and y in $R\otimes Z_2$, $u\equiv xy$, $v\equiv y^2$ and $x^2\equiv 0$ (mod 2) if R is of type 1, and $u\equiv xy\equiv x^2$ and $v\equiv y^2$ (mod 2) if R is of type 2. Let $g: K^{(4)}\longrightarrow S^4$ be a map such that $g^*(\iota^{(4)})=u$ where $\iota^{(4)}$ is a generator of $H^4(S^4; Z)$, and $r: K^{(4)}\longrightarrow K^{(4)}\vee S^4$ a deformation of $id.\times g$. We put $f=(id.\vee(i^{(3)}\circ S(h)))\circ r$. Then f is a homotopy equivalence of $K^{(4)}$. Let $i^{(4)}$ be a generator of $\pi_4(S^4)$. Since, in the decomposition $\pi_5(K^{(4)}\vee S^4)$ $=\pi_5(K^{(4)})+\pi_5(S^4)+\partial\pi_6(K^{(4)}\times S^4,\ K^{(4)}\vee S^4)$, the last summand is generated by $[i_j, i^{(4)}]$ $(i=1,2)$ we have $r^*(\beta)=\beta+g_*(\beta)+n[i_1, i^{(4)}]+m[i_2, i^{(4)}]$ for some integers n and m. In the complex $S^4\cup_{g_*(\beta)}e^6$, $Sq^2(\iota^{(4)})\neq 0$ or 0 according as R is of type 1 or 2, and hence we see that $g_*(\beta)\neq 0$ or 0 according as R is of type 1 or 2. Similarly we have $n\equiv 0$, $m\equiv 1$ or $n\equiv m\equiv 1$ (mod 2) according as R is of type 1 or 2. Now, suppose that R is of type 1. Then, we have

$$f_*(\beta)=\beta+\gamma+[i_2, i^{(3)}\circ S(h)]. \tag{2.6}$$

If we put $u=\lambda_{12}x^2+\lambda_{22}xy+\lambda_{32}y^2$ and $v=\lambda_{11}x^2+\lambda_{21}xy+\lambda_{31}y^2$, then the defining matrix A of $K^{(4)}$ becomes the following: $a_{13}\equiv a_{22}\equiv a_{31}\equiv 1$ (mod 2) and the other $a_{ij}\equiv 0$ (mod 2) by the choice of u and v. Thus we have $i_1\circ h=i^{(3)}$ and $[i_1, i_2]\equiv 0$ (mod $2\pi_3(K^{(4)})$) and therefore

$$\begin{aligned}[i_2, i^{(3)}\circ S(h)] &= [i_2, i_1\circ h\circ S(h)]\\ &= [i_2, i_1]\circ S(h)\circ S^2(h) - [[i_2, i_1], i_1]\circ S^2(h)\\ &= 0.\end{aligned}$$

Hence, $f_*(\beta)=\beta+\gamma$ by (2.6). Similarly we can prove the case that R is of type 2.

3. Let $i^{(3)}$ and $i^{(5)}$ be the inclusions of S^3 and S^5 into the one point union $S^3\vee S^5$ respectively, and ζ and ξ the generators of $\pi_7(S^3)\cong Z_2$ and $\pi_7(S^5)\cong Z_2$ respectively. We put $E_i=(S^3\vee S^5)\cup_{\alpha_i}e^8$ where $\alpha_1=[i^{(3)}, i^{(5)}]+i^{(3)}\circ\zeta$, $\alpha_2=[i^{(3)}, i^{(5)}]+i^{(5)}\circ\xi$ and $\alpha_3=[i^{(3)}, i^{(5)}]+i^{(3)}\circ\zeta+i^{(5)}\circ\xi$. Then we have the following

Lemma 3.1. *Let E be a simply connected finitd CW complex. If $H^*(E; Z)\cong H^*(S^3\times S^5; Z)$, then E is homotopy equivalent to one of the following complexes:* $S^3\times S^5$, $E_i(i=1,2,3,)$ *and* $SU(3)$.

Proof. We can suppose $E=S^3\cup_\alpha e^5\cup_\beta e^8$. First, assume $Sq^2=0$, i.e. $\alpha=0$. It is well known that $\pi_7(S^3\vee S^5)\cong Z+2Z_2$ with generators $[i^{(3)}, i^{(5)}]$, $i^{(3)}\circ\zeta$ and $i^{(5)}\circ\xi$. Then, from the cohomology ring structure of

E, we see that β is either $[i^{(3)}, i^{(5)}]$ or α_i ($i=1,2,3$). It is clear that these 4 complexes have different homotopy type.

Now, suppose $Sq^2\neq 0$. Then $E=S^3\cup_{S(\eta)}e^5\cup_\beta e^8$. Since $SU(3)$ is such a complex, we have $SU(3)\simeq E^{(5)}\cup_\gamma e$ where $E^{(5)}$ is the 5 skelton of E. Since $\pi_7(SU(3))=0$, we see that $\pi_7(E^{(5)})\cong Z$ is generated by γ. Now, from the cohomology ring structure of E, we have $\beta=\gamma$, and hence E is homotopy equivalent to $SU(3)$.

It is well known that E_1 is homotopy equivalent to $SU(3)\times_{SU(2)}S^3$, where $SU(2)$ acts on S^3 via the non trivial homomorphism $SU(2)\longrightarrow SO(3)\longrightarrow SO(4)$.

Lemma 3.2. *Let E be a simply connected smooth manifold of dimension* 8. *If $H^*(E;Z)\cong H^*(S^3\times S^5;Z)$, then E is homeomorphic to $S^3\times S^5$, $SU(3)\times_{SU(2)}S^3$ or $SU(3)$.*

Proof. Let $f: S^3\longrightarrow E$ be a smooth imbedding such that $[f]$ generates $\pi_3(E)$. Since the normal bundle of S^3 is trivial, we have an extension $\bar{f}: S^3\times D^5\longrightarrow E$ of f. Let $M=E-\bar{f}(Int(S^3\times D^5))\cup_{\bar{f}}D^4\times S^4$. Then M is a homotopy sphere. Thus, $E\#(-M)$ is diffeomorphic to $S^8-g(Int(D^4\times S^4))\cup_g S^3\times D^5$ for some smooth imbedding $g: D^4\times S^4\longrightarrow S^8$. Since the imbedding $g\,|\,0\times S^4$ of S^4 into S^8 is isotopie to the standard imbedding (see, [2]), we see that $E\#(-M)$ is the orthogonal S^3 bundle over S^5. On the other hand, an orthogonal S^3 bundle over S^5 is diffeomorphic to $S^3\times S^5$, $SU(3)\times_{SU(2)}S^3$ or $SU(3)$ ([6]). These facts completes the proof.

The following theorem together with Theorem 2.5 determines completely the homotopy types of the orbit spaces of free T^2-actions on $S^3\times S^5$ and $SU(3)$.

Theorem 3.3. *Let K be a simply connected finite CW complex. Then holds the following*:

a) K is homotopy equivalent to the orbit space of a free T^2-action on $S^3\times S^5$ if and only if $H^*(K;Z)$ is a ring of type 1 *or* 2.

b) K is homotopy equivalent to the orbit space of a free T^2-action on $SU(3)$ if and only if $H^*(K;Z)$ is a ring of type 3.

Proof. The 'only if part' follows from [3]. Let $E\longrightarrow K$ be the 2-connected principal T^2 bundle. Then it follows from [3] that $H^*(E;Z)=H^*(S^3\times S^5;Z)$ and $Sq^2\neq 0$ or 0 on E according as $H^*(K;Z)$ is of type 3 or not. Now, assume $H^*(K;Z)$ is of type 3. By [7], K has

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the homotopy type of a closed smooth 6-manifold, and hence we may assume that E is a compact smooth manifold. Then E is homeomorphic to $SU(3)$ by Lemmas 3.1 and 3.2. This proves b).

Now, suppose that $H^*(K; Z)$ is of type 1 or 2. In order to show a), it is sufficient to show that $E \simeq S^3 \times S^5$. By Lemma 3.1, this follows from the fact $[\pi_3(K), \pi_5(K)] = 0$. We may suppose, by Theorem 2.5, that $K = K^{(4)} \cup_\beta e^6$ and $\pi_5(K)$ is generated by α and $\gamma = i^{(3)} \circ S(h) \circ S^2(h)$. Recall that $q_*(\alpha) = \kappa_3$ and $q_*(\beta) = \kappa_4$ (see, (2.3)). From the homotopy exact sequence of the pair $(K, K^{(4)})$ and Lemma 3.1, we have the following split exact sequence:

$$0 \longrightarrow \pi_7(K, K^{(4)}) \longrightarrow \pi_6(K^{(4)}) \longrightarrow \pi_6(K) \longrightarrow 0.$$

Thus, by [4] we see that $\pi_6(K^{(4)})$ is isomorphic to $2Z + 2Z_2 + Z_{12}$ and generated by $[i_1, \beta]$, $[i_2, \beta]$, $\alpha \circ S^3(h)$, $\beta \circ S^3(h)$ and $i^{(3)} \circ \rho$ where ρ is a generator of $\pi_6(S^3)$. Then we have

$$\begin{aligned}[i_j, \alpha] &= c_{1j}[i_1, \beta] + c_{2j}[i_2, \beta] + c_{j3}\alpha \circ S^3(h) + c_{4j}\beta \circ S^3(h) \\ &\quad + c_{5j} i^{(3)} \circ \rho \qquad (j = 1, 2)\end{aligned}$$

for some integers c_{ij}. Now, we determine c_{3j} mod 2. We may assume that $\phi(x, y) \equiv x^2$, $\psi(x, y) \equiv y^3$ (mod 2) if R is of type 1, and $\phi(x, y) \equiv x^2 + xy$, $\psi(x, y) \equiv y^3$ (mod 2) if R is of type 2. Then, we may assume that ν_i in the matrix M^{-1} cited in §2 are as follows: $\nu_1 \equiv \nu_2 \equiv \nu_4 \equiv 0$, $\nu_3 \equiv 1$ (mod 2). Then if R is of type 1, $q_*(\alpha) \equiv k_4$, $q_*(\beta) \equiv k_3$ (mod $2\pi_5(L^{(4)})$) and if R is of type 2, $q_*(\alpha) \equiv k_4$, $q_*(\beta) \equiv k_1 + k_2 + k_3$ (mod $2\pi_5(L^{(4)})$). Now, if R is of type 1, by Lemma 1.1 we have

$$\left.\begin{aligned}&q_*([i_1, \alpha]) \equiv [i_1, k_4],\ q_*([i_2, \alpha]) \equiv k_4 \circ S^3(h) \\ &q_*([i_1, \beta]) \equiv [i_1, k_3] \text{ and } q_*([i_2, \beta]) \equiv [i_1, k_4]\end{aligned}\right\} \ (\text{mod } 2\pi_6(L^{(4)})).$$

These imply $c_{31} \equiv 0$ and $c_{32} \equiv 1$ (mod 2). By the same argument we can see that the same relations hold also in case R is of type 2. Thus we have shown the following relations in $\pi_6(K)$:

$$[i_1, \alpha] = c_{51} i^{(3)} \circ \rho, \quad [i_2, \alpha] = \alpha \circ S^3(h) + c_{52} i^{(3)} \circ \rho.$$

Since $\pi_7(K) \cong 2Z_2$ by Lemma 3.1 and $i^{(3)} \equiv i_1 \circ h$, $[i_1, i_2] \equiv 0$ (mod $2\pi_3(K)$) (see, the proof of Theorem 2.5), we have

$$\begin{aligned}[i^{(3)}, \alpha] &= [i_1 \circ h, \alpha] = [i_1, \alpha] \circ S^4(h) - [[i_1, \alpha], i_1] \\ &= c_{51} i^{(3)} \circ \rho \circ S^4(h).\end{aligned}$$

On the other hand, by the Jacobi identity, we have

$$0 = [[i_1, i_2], \alpha] = [[i_2, \alpha], i_1] + [[i_1, \alpha], i_2]$$
$$= [i_1, \alpha] \circ S^4(h) = c_{51} i^{(3)} \circ \rho \circ S^4(h).$$

Thus $[i^{(3)}, \alpha] = 0$, and hence $[\pi_3(K), \pi^5(K)] = 0$. This completes the proof.

Since $[\pi_3(E_i), \pi_5(E_i)] \neq 0$ $(i = 1, 2, 3)$, we have

Corollary 3. 4. *There is no free T^2-action on the complexes E_i $(i = 1, 2, 3)$.*

4. We consider some examples of rings R treated in §2. We use a, b, c, α, β, γ and δ instead of $\lambda_1, \cdots, \mu_4$. Then (2. 1) becomes

$$\begin{aligned} &c^3\alpha^2 + ac^2\beta^2 + a^2c\gamma^2 + a^3\delta^2 - bc^2\alpha\beta + (b^2c - 2ac^2)\alpha\gamma \\ &+ (3abc - b^3)\alpha\delta - abc\beta\gamma + (ab^2 - 2a^2c)\beta\delta - a^2b\gamma\delta = \pm 1. \end{aligned} \tag{4. 1}$$

Now, we suppose $a \neq 0$ and $b^2 - 4ac \neq 0$, and put

$$X = -bc\alpha + ac\beta - a^2\delta,\ Y = (ac - b^2)\alpha + ab\beta - a^2\gamma. \tag{4. 2}$$

Then, (4. 1) becomes the following

$$aX^2 - bXY + cY^2 = \pm a^2. \tag{4. 3}$$

In the following, we consider the isomorphism classes of the rings R of type 1 such that $\phi(x, y) = 3x^2 + 20xy + 7y^2$. In this case (4. 2) and (4. 3) become

$$3X^2 - 20XY + 7Y^2 = \pm 9 \text{ and } Y \equiv 2X \pmod 9.$$

If we put $X = u + 4v$ and $Y = 2u + 17v$, then we have

$$u^2 - 79v^2 = \pm 1. \tag{4. 4}$$

Since the fundamental unit of the real quadratic field $Q(\sqrt{79})$ is $80+9\sqrt{79}$, we have $u + v\sqrt{79} = (80 + 9\sqrt{79})^n$ where n is an integer. We may assume without changing the isomorphism class of R that $\alpha = 1$ or 2 and $\beta = 0$, 1 or 2. Then the integral solutions of (4. 1) in our case are as follows:

$$\alpha = 1, \beta = 2, \gamma = -(2u + 17v + 259)/9 \text{ and } \delta = -(u + 4v + 98)/9 \tag{4. 5}$$

where u and v are given by $u + v\sqrt{79} = (80 + 9\sqrt{79})^{2l}$ or $-(80+9\sqrt{79})^{2l+1}$ and

(4. 6) $\alpha=2, \beta=0, \gamma=-(2u+17v+758)/9$ and $\delta=-(u+4v+280)/9$

where $u+v\sqrt{79}=(80+9\sqrt{79})^{2l+1}$ or $-(80+9\sqrt{79})^{2l}$ with integer l.

Proposition 4.7. *The rings R corresponding to the different values of α, β, γ and δ in* (4. 5) *and* (4. 6) *are not isomorphic.*

Proof. It is well known that linear transformations which preserve a quadratic form of 2 variables are obtained by the solutions of a Pell equation. Thus, in case $\phi(x, y)=3x^2+20xy+7y^2$, we have

$x=p\bar{x}+q\bar{y}$ and $y=r\bar{x}+s\bar{y}$

where $p=u-10v$, $q=-7v$, $r=-3v$ and $s=u+10v$ with some integers u and v satisfying (4. 4). If we put $\psi(p\bar{x}+q\bar{y}, r\bar{x}+s\bar{y})=\alpha'\bar{x}^3+\beta'\bar{x}^2\bar{y}+\cdots$, we have

$$\alpha'=(u-10v)^3\alpha-3v(u-10v)^2\beta+9v^2(u-10v)\gamma-27v^3\delta.$$

Now, sppose $\alpha'=\alpha=1$. Then we have $u^3\equiv 1 \pmod v$, and hence $u\equiv\pm 1 \pmod v$ by (4. 4). The solutions satisfying this condition are $u=\pm 1, v=0$ and $u=-80, v=\pm 9$. Since $\beta=2$, we can exclude the latter solution, and therefore we see that $x=\pm\bar{x}$, $y=\pm\bar{y}$. If $\alpha'=2$ and $\alpha=1$, we have $u\equiv\pm 2 \pmod v$. However, such a solution of (4. 4) does not exist. Similarly in case $\alpha'=\alpha=2$ we can see that $x=\pm\bar{x}$, $y=\pm\bar{y}$.

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