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1. Introduction. Let R be a semiprime torsion free ring, where a right R-module is torsion free if its singular submodule is zero. Then there is an essential direct sum of three completely unique and algebraically very different types of ideals $A \oplus B \oplus D \leqq R$ (3.5). The ideal D is discrete, that is it contains an essential direct sum of uniform right ideals, while $A \oplus B$ is continuous, contains no uniform submodules. The unique largest molecular (right) ideal of R is $A \oplus D$; the latter contains an essential direct sum of atomic right ideals (1.2). Let $\Omega = \{\tau, \rho, \ldots\}$ denote equivalence classes of similar atomic right ideals of R. Then $\Omega = \Omega^C \cup \Omega^D$, $\Omega^C \cap \Omega^D = \emptyset$, and Ω^D consists of equivalence classes of related uniform right R-modules U and V, where $U \sim V$ if and only if they have isomorphic injective hulls $E(U) \cong E(V)$. A continuous compressible right ideal is an atomic module and defines, in an appropriate way, a class in Ω^C. It is surprising that in certain contexts continuous atomic modules behave the same way as the uniform ones. The same proofs (phrased in terms of the class of molecular modules) simultaeneously cover both kinds of modules.

The sum $R_\tau = \sum\{W | W \leqq R,\ W \in \tau\}$ of all the elements of τ turns out to be an intrinsic ideal $R_\tau \triangleleft R$. Each atomic right ideal W is a prime right R-module. The latter has many consequences: R_τ is a prime right R-module; R_τ is a prime ring; the annihilator ideal $R_\tau^\perp$ of R_τ is a maximal annihilator ideal of R. Indeed, there is a plethora of prime and semiprime modules, submodules, ideals, and rings. As a ring, the injective hull $E(A \oplus D)$, is a full direct product $E(A \oplus D) = \Pi\{E(R_\tau) | \tau \in \Omega\} \triangleleft E(R)$ of ideals $E(R_\tau) \triangleleft E(R)$; in particular $E(R_\tau)E(R_\rho) = 0$ for $\tau \neq \rho \in \Omega$. (Corollary 2 to Theorem I). That $E(D) = \Pi\{E(R_\tau) | \tau \in \Omega^D\}$ had already been proved earlier in [11].

Here as much as possible is proved more generally for modules M; results for rings R are afterwards obtained as mere corollaries of the special case $M = R_R$. This applies also to the result $E(A \oplus D) = \Pi E(M_\tau)$ (Corollary 1 to Theorem I). The ordinary theory of prime and semiprime one sided ideals of an associative ring R is a special case of the more inclusive theory of prime and semiprime modules ([1], [18], [7], and [8]). Propositions A–E of section 2 extend the known theory of semiprime mod-

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ules, and are also useful in other contexts.

As one of related similar results, we also prove that a necessary and sufficient condition for a ring R to be torsion free, semiprime, molecular is that it be a right essential subdirect product $R \subseteq \Pi\{R_i | i \in \mathcal{I}\}$ of torsion free, atomic, prime rings R_i (Theorem III). This theorem gives a two fold simplification; semiprime is replaced by prime and molecular by atomic. Use of a result of Levy [24, p.66, Theorem 3.2] then shows that this subdirect product representation is unique; $\mathcal{I} \cong \Omega$ and, up to isomorphism, the R_i are the previous rings $R/R_{\tau}^{\perp}$. It is easy to construct discrete rings satisfying Theorem III; example 6.1 shows that there actually exist continuous rings which also satisfy it. Counterexamples (6.2) show that the molecular-atomic hypothesis in Theorem III may not be removed.

There is some unavoidable overlap with [13]. There some of the main results of this article were merely stated, but not proved (e.g. 4.2. Theorem II, and 5.4. Theorem III). Here, we give full proofs, as well as some results not announced in [13], in particular, 5.6. Corollary to Theorems II and III.

1. Basic concepts. Torsion free, discrete, continuous, prime and semiprime modules are defined. It is shown how to work with complement closures of modules (Lemma 1.4).

1.1. Notation. Modules M are right unital over an associative ring R ($1 \in R$); $<$ or $\leqq$ denote submodules, while $\ll$ denotes large or essential submodules. The symbol $A \not\ll B$ means that $A < B$, but that A is not large in B. If $K < M$ and $x \in M$, then $x^{\perp} = \{r \in R | \, xr = 0\} < R$; and for $x + K \in M/K$, $(x+K)^{\perp} = x^{-1}K = \{r \in R | \, xr \in K\} < R$. For any subset $Y \subset M$, set $Y^{\perp} = \{r \in R | \, yr = 0 \text{ for all } y \in Y\} = \{r | \, Yr = 0\}$. Thus $M^{\perp} = \{r | \, Mr = 0\} \triangleleft R$, where "$\triangleleft$" denotes ideals in R or any other ring.

Right R-injective hulls of right R-modules are denoted by both "$\widehat{\ }$" and "E" as $\widehat{M} = E(M) = EM$. The singular submodule $Z(M) = ZM < M$ is $ZM = \{x \in M | \, x^{\perp} \ll R\}$. A module M is *torsion free* if $ZM = 0$ (abbreviation: t.f.). A submodule $C < M$ is a *complement* submodule if C has no proper essential extension inside M, in which case C is said to be *closed* in M. A module M is *discrete* if it contains an essential direct sum of uniform submodules, and *continuous* if M contains no uniform submodules.

An ideal $I \triangleleft R$ is an *annihilator ideal* if $I = Y^{\perp}$ for some nonempty subset $Y \subseteq R$. In this case $Y \subseteq \ell(I) \equiv \{r \in R \mid rI = 0\} \triangleleft R$, and $I = \ell(I)^{\perp}$. If $\{0\}^{\perp} = R$ is the only annihilator ideal which properly contains an annihilator ideal $I \neq R$, then I is a *maximal annihilator ideal*, in which case $\ell(I) \neq 0$. Note that in a semiprime ring, the left and right annihilators of any two sided ideal coincide.

1.2. Types. The class of all torsion free modules $A, B, C, \ldots$ is quasi-ordered by defining $A \propto B$ to mean that A can be embedded in the injective hull of some direct sum of the B's, i.e. $A \subset E(\bigoplus\{B \mid J\}) = E(\bigoplus_J B)$, where J in general is an infinite index set. Then define an equivalence relation "$\sim$" on the class of torsion free modules under which $A \sim C$ provided that $A \propto C$ and also $C \propto A$. Each equivalence class $[A] = \{C \mid C \sim A\}$ is called a *type*. The set of all types $\Xi(R) = \{[A], [B], [C], \ldots\}$ becomes a partially ordered set under the order relation "$\leqq$" where $[A] \leqq [B]$ provided that $A \propto B$. Note that $[A] = [EA]$ for all A.

A t.f. module W is *atomic* if $[W]$ is an atom in the poset $\Xi(R)$ (i.e. if $0 \neq \xi \leq [W]$, then $\xi = [W]$). An atomic module is either continuous or discrete. More generally a module N is *molecular* if (i) N is torsion free and (ii) every nonzero submodule of N contains an atomic one. There are two important disjoint subclasses of molecular modules, the discrete molecular and the continuous molecular ones. In general a molecular module N contains an essential direct sum $A \oplus D \ll N$ where A is continuous and D discrete. (See Theorem I). The terms "discrete" and "discrete molecular" by definition mean the same thing. At the other extreme of the module spectrum, a t.f. continuous module B is *bottomless* if B contains no atomic submodules.

The set of atoms $\Omega = \{\tau, \rho, \ldots\} \subset \Xi(R)$ is a disjoint union $\Omega = \Omega^C \cup \Omega^D$ where $\Omega^D = \{[U] \mid U_R \text{ is t.f. uniform}\}$, while Ω^C are the types represented by t.f. continuous atomic modules. The equivalence classes of t.f. continuous compressible modules belong to Ω^C. For two t.f. uniform modules U and V, $[U] = [V]$ if and only if $\widehat{U} \cong \widehat{V}$. Thus there is a bijective correspondence between $\Omega^D = \{[U], [V], \ldots\}$ and the usual set of equivalence classes of related t.f. uniform modules. Furthermore every equivalence class $\eta \in \Xi(R)$ whatever can be represented by a torsion free right ideal $L \leqq R$ as $\eta = [L]$.

For any t.f. module M and any $\tau \in \Omega$, define a unique intrinsic submodule of M by $M_\tau = \sum\{W \mid W \leq M,\ W \in \tau\}$. Then $[M_\tau] = \tau$

([12, 3.19(ii)]). Moreover, $M_\tau = \overline{M}_\tau \leq M$ is a right complement, and $M_\tau \leq M$ is fully invariant by [12, 3.20(2)]. In particular for $M = R_R$, each $R_\tau = \overline{R}_\tau \leq R$ is a right complement, and an ideal $R_\tau \triangleleft R$.

1.3. Definition. A submodule $K < M$ is *prime* if for any $V \leqq M$ and any $A \leq R$

$$VA \subseteq K \implies \text{either } V \subseteq K, \quad \text{or } MA \subseteq K.$$

The submodule $K < M$ is *semiprime* if for any $m \in M$ and $s \in R$

$$msRs \subseteq K \implies ms \in K.$$

The module M itself is *prime* (or *semiprime*) if $(0) < M$ is a prime (or semiprime) submodule. Note that "$K < M$ is a prime submodule" and "K is a prime module" are two very different concepts. When $1 \in R$, any $L < R$ is a prime or semiprime right ideal in the usual sense if and only if $L < R$ is a prime or semiprime right R-submodule. (See [7, p.160, 1.8; and p.163, 1.16].)

1.4. Lemma. *For any module M, if $ZM \subseteq K < M$, and if $\overline{K}$ is defined by $\overline{K}/K = Z(M/K)$, then the following hold*:

(0) $\overline{K} = \{x \in M \mid x^{-1}K \ll R\} = \{x \in M \mid K \ll K + xR\}$.

(1) $K \ll \overline{K}$; *$\overline{K}$ is the unique smallest complement submodule of M containing K.*

(2) $Z(M/\overline{K}) = 0$.

(3) *If $K < M$ is fully invariant, then so is $\overline{K} < M$ also.*

(4) *In particular, if $K \triangleleft R$ is an ideal with $ZR \subseteq K$, then $\overline{K} \triangleleft R$ is an ideal.*

Proof. For (0), (1), (2), and (4), see [12, Proposition 1.3] or [7, p.165, 3.2–3.4].

(3) Let $\phi \in \mathrm{Hom}_R(M, M)$ and $\xi \in \overline{K} \setminus K$ be arbitrary. Then $(\phi\xi)\xi^{-1}K \subseteq \phi K \subseteq K$. Since $\xi^{-1}K \ll R$ by (0) above, $\phi\xi \in \overline{K}$.

1.5. Definition. For any modules $K < M$, define the *complement closure* $\overline{K}$ of K only if $ZM \subseteq K$ by $\overline{K}/K = Z(M/K)$.

2. Semiprimeness and complements. Because complement submodules are much easier to work with, this section gives methods of producing complement submodules (2.1). Propositions A and B give hypotheses under which some properties of a submodule are also inherited by its complement closure. The same proof as in [7, p.165, 3.6] gives the next lemma.

2.1. Lemma. *For a module M with $ZM \cap Y = 0$ and any subset $Y \subset M$, $Y^{\perp} < R$ is a right complement.*

The following situation occurs frequently in many contexts here and elsewhere.

2.2. Proposition A. *If $K < M$ is a right R-submodule with $ZM \subseteq K$ and $K \ll \overline{K}$ is its complement closure* (1.4), *then the following hold*:

(i) *$K < M$ is prime $\Rightarrow \overline{K} < M$ is prime.*

(ii) *$K < M$ is semiprime $\Rightarrow \overline{K} < M$ is semiprime.*

Proof. (i) Given $m \in M \setminus \overline{K}$, and $t \in R$ with $mRt \subseteq \overline{K}$, it has to be shown that $Mt \subset \overline{K}$. Since $m \notin \overline{K}$, by 1.3(0), $K \not\ll K + mR$, and hence $(ma_0 + k_0)R \cap K = 0$, where $0 \neq ma_0 + k_0$, $a_0 \in R$, $k_0 \in K$. Since $K \ll \overline{K}$, $(ma_0 + k_0)R \oplus \overline{K} \leq M$. But now

$$\left.\begin{array}{l}(ma_0 + k_0)Rt \subseteq \overline{K} \\ (ma_0 + k_0)Rt \subseteq (ma_0 + k_0)R\end{array}\right\} \Longrightarrow (ma_0 + k_0)Rt = 0.$$

In particular, $(ma_0 + k_0)Rt \subset K$ and $K < M$ prime imply that $Mt \subseteq K$.

(ii) Given $m \in M$ and $s \in R$ with $msRs \subseteq \overline{K}$, it has to be shown that $ms \in \overline{K}$. So assume $ms \notin \overline{K}$. Then exactly as before in (i) above, $\overline{K} \lneqq \overline{K} \oplus (msa_0 + k_0)R$. Let $r \in R$ be arbitrary. From $msa_0Rsa_0 \subseteq \overline{K}$, it follows that $msa_0(rsa_0) + k_0(rsa_0) \in \overline{K} \cap (msa_0 + k_0)R = 0$. Thus $msa_0rsa_0 = -k_0rsa_0 \in K$. Since $r \in R$ was arbitrary, $msa_0Rsa_0 \subseteq K$. From the semiprimeness of $K < M$, we conclude that $msa_0 \in K$, which is a contradiction.

2.3. Corollary. *If $\{W_\gamma | \gamma \in \Gamma\}$ is a family of t.f. prime modules all having the same annihilator ideal, $W_\alpha^{\perp} = W_\beta^{\perp}$ for all $\alpha, \beta \in \Gamma$, then $\Pi\{\widehat{W}_\gamma | \gamma \in \Gamma\}$ is prime.*

2.4. Proposition B. *Let R be a semiprime ring with $ZR = 0$. Then for any $K < R$, $K^{\perp} = \overline{K}^{\perp}$.*

Proof. If $\overline{K}^{\perp} \neq K^{\perp}$, then since by 2.1 both are right complements, $\overline{K}^{\perp} \oplus L \leq K^{\perp}$ for some $0 \neq L \leq R$. For any $K < R$ in a semiprime ring

R, $K^{\perp}K = 0$. Consequently $LK = 0$. If it were the case that $L\bar{K} = 0$, then $(\bar{K}L)\bar{K}L = 0$, hence $\bar{K}L = 0$, and $L \subseteq \bar{K}^{\perp}$, a contradiction. So $L\bar{K} \neq 0$. Take $0 \neq \xi \in \bar{K}$ with $L\xi \neq 0$. By 1.4(0), $\xi^{-1}K \ll R$, and thus $L\xi(\xi^{-1}K) \subseteq LK = 0$ gives the contradiction that $L\xi \subseteq ZR = 0$. Thus $\bar{K}^{\perp} = K^{\perp}$.

Not only is the following equivalent characterization of a semiprime ring new, it will also be used later.

2.5. Proposition C. *For a ring R with $ZR = 0$, the following are equivalent.*

(i) *R is semiprime.*

(ii) *For any right ideals $0 \neq A$, $0 \neq B < R$, $\widehat{A} \cong \widehat{B} \Longrightarrow BA \neq 0$.*

Proof. (ii)$\Longrightarrow$(i). Let $0 \neq x \in R$. Take $A = B = xR$. Since $(xR)xR \neq 0$ by (ii), R is semiprime.

(i)$\Longrightarrow$(ii). If ϕ: $\widehat{A} \to \widehat{B}$ is an isomorphism, take any $0 \neq a \in A$, and hence $0 \neq \phi a \in \widehat{B}$. It follows from $Z\widehat{B} = 0$ and $(\phi a)^{-1}B \ll R$ that $0 \neq (\phi a)[(\phi a)^{-1}B] = \phi[a(\phi a)^{-1}B]$. Hence there exists an $x \in a(\phi a)^{-1}B \subset A$ with $0 \neq \phi x = y \in B$. Then $x^{\perp} = y^{\perp}$ and $x^{\perp} \not\ll R$. Take any $0 \neq C \leqq R$ with $x^{\perp} \oplus C \leqq R$.

Now suppose that $BA = 0$. Then $(yR)(xR) \subseteq BA = 0$, and also $yCxC = 0$. Thus $CxC \subseteq y^{\perp} \cap C = 0$, and $CxC = 0$. Since by (i) R is semiprime and $(xC)(xC) = 0$, also $xC = 0$. But the restriction of ϕ to xR is ϕ: $xR \to yR$, $\phi xr = yr$, $r \in R$; and $0 \neq C \cong xC \cong yC$, a contradiction. Hence $BA \neq 0$.

In general, if R is a semiprime ring and $0 \neq K \triangleleft R$ is any ideal whatever, then automatically K is a semiprime right (and left) R-module and K is a semiprime ring.

2.6. Construction D. *Suppose that R is a prime (a semiprime) ring and $K \triangleleft R$ is any ideal with $ZR \subseteq K$. Then $\bar{K} \triangleleft R$ is a prime (or semiprime) ideal of R.*

Proof. By 1.4(4), $\bar{K} \triangleleft R$. Take any $P < R$ with $\bar{K} \oplus P \ll R$. If $\bar{K} \triangleleft R$ is not prime, then $xRy \subseteq \bar{K}$ for some $x, y \in R\backslash \bar{K}$. By 1.4(0), $x^{-1}K \not\ll R$. Consequently, in $x^{-1}K \subset x^{-1}(K \oplus P) \ll R$, the first inclusion is proper. Therefore for any $c \in [x^{-1}(K \oplus P)] \setminus x^{-1}K \neq \emptyset$, $0 \neq xc = x_1 + x_2$, where $x_1 \in K$ and $0 \neq x_2 \in P$. Similarly, $0 \neq yd = y_1 + y_2$ with $y_1 \in K$

and $0 \neq y_2 \in P$. Let $r \in R$ be arbitrary. Then the first three terms in $xry = x_1ry_1 + x_1ry_2 + x_2ry_1 + x_2ry_2$ belong to $\overline{K}$ because $\overline{K} \triangleleft R$. Since $xry \in \overline{K}$, also $x_2ry_2 \in \overline{K}$, and $x_2Ry_2 \subseteq \overline{K}$. But then $x_2Ry_2 \subseteq \overline{K} \cap P = 0$ contradicts the primeness of R. In the semiprime case, in the above proof take $x = y$, in which case $c = d$ and $x_2 = y_2$ automatically.

2.7. Proposition E. *Suppose that R is a torsion free semiprime ring and $K \triangleleft R$ is any ideal. Then*

(i) $K \oplus K^{\perp} \ll R$; *and*

(ii) $K^{\perp\perp} = \overline{K}$.

(iii) $\forall P = \overline{P} \triangleleft R,\ Q = \overline{Q} \triangleleft R$

$$P \oplus Q \ll R \Longrightarrow P^{\perp} = Q.$$

Proof. (i) If $K \oplus K^{\perp} \oplus D \ll R$ for $D < R$, then $DK \subseteq D \cap K = 0$. In a semiprime ring, $D \subseteq \ell(K) = \{r \mid rK = 0\} = K^{\perp}$. Hence $D = 0$ and $K \oplus K^{\perp} \ll R$.

(iii) Assume that $Q \subsetneqq P^{\perp}$. Since both are right complements, $Q \not\ll P^{\perp}$, and $W \oplus Q \ll P^{\perp}$ for some $0 \neq W < R$. Use of the modular and the fact that $P \oplus Q \ll R$ shows that J below is not zero,

$$0 \neq (W \oplus Q) \cap [P \oplus Q] = J \oplus Q \ll W \oplus Q, \quad \text{where}$$
$$0 \neq J = (W \oplus Q) \cap P$$

By semiprimeness, $P(W+Q) \subseteq PP^{\perp} = 0$ implies that also $(W+Q)P = 0$. But then $J^2 = 0$ is a contradiction. Thus $Q = P^{\perp}$.

(ii) By 2.4, $K^{\perp} = \overline{K}^{\perp}$, and hence $\overline{K}^{\perp} \oplus \overline{K} \ll R$. Then (iii) implies that $K^{\perp\perp} = \overline{K}^{\perp\perp} = \overline{K}$.

3. Fully invariant intrinsic submodules.

3.1. Observation. *If $\{A_\gamma \mid \gamma \in \Gamma\}$ is any family of modules, then every nonzero submodule of $E(\oplus\{A_\gamma \mid \gamma \in \Gamma\})$ contains an isomorphic copy of a nonzero submodule of some A_γ for some $\gamma \in \Gamma$.*

Proof. For $0 \neq \xi \in E(\oplus A_\gamma)$ chose $r_0 \in R$ such that $0 \neq \xi r_0 = a_1 + \cdots + a_n \in A_{\gamma(1)} \oplus \cdots \oplus A_{\gamma(n)}$ where all $0 \neq a_i \in A_{\gamma(i)}$, and with the length n minimal. If $z \in a_i^{\perp} \setminus a_j^{\perp}$, then $\xi r_0 z \neq 0$ has shorter length. Hence $(\xi r_0)^{\perp} = a_1^{\perp} = a_2^{\perp} = \cdots = a_n^{\perp}$. Thus $\xi r_0 R \cong a_1 R \subset A_{\gamma(1)}$.

The next observation either follows from [12, Proposition 3.14], or alternatively can be proved directly by use of 3.1 and [11, p.4, Lemma 2.4].

3.2. Observations. (1) *Consider any one of the following six classes of torsion free modules*: (a) *continuous molecular*, (b) *bottomless*, (c) *continuous*, (d) *discrete*, (e) $\tau \cup \{(0)\}$, $\tau \in \Omega$, *and* (f) *for any arbitrary* $[W] \in \Xi(R)$, *the class* $\bigcup\{[V] \mid [V] \in \Xi(R),\ [V] \leq [W]\} = \{V \mid \exists a$ *set* Γ *and an embedding* $V \subset E(\oplus\{W \mid \Gamma\})\}$. *Then this class is closed under the following five operations*: (i) *submodules*, (ii) *injective envelopes of arbitrary direct sums*, (iii) *isomorphic copies, and* (iv) *torsion free homomorphic images (i.e. quotient modules modulo complement submodules).* (v) *If* $0 \to K \to N \to Q \to 0$ *is a short exact sequence of three torsion free modules with* K *and* Q *belonging to the class, then also* N *belongs to the class.*

Actually in (v) *if any two of the terms of the sequence belong to the class in question, so does also the third. In case* (e) *above, let* $N \in \tau$ *and* $0 \neq K \lneqq N$ *be a complement submodule. Then* $[K] = [N/K] = [N] = \tau$. *Note that* (a)–(e) *are special cases of* (f).

(2) *For any* $\tau \in \Omega$ *and any t.f. module* M, $(EM)_\tau = E(M_\tau) \equiv EM_\tau$.

3.3. Theorem I. *Suppose that* M *is any torsion free unital right* R*-module over any ring* R. *Let* $A, B, C, D \leq M$ *be any submodules (which exist by Zorn's lemma) such that*

(a) A *is continuous molecular*, B *bottomless*, C *continuous, and* D *discrete.*

(b) $A = \overline{A}$, $B = \overline{B}$, $C = \overline{C}$, *and* $D = \overline{D}$ *are right complement submodules of* M.

(c) $C \oplus D \ll M$, $A \oplus B \ll C$.

Then the following hold:

(i) A, B, C, *and* D *are fully invariant in* M; $\mathrm{Hom}_R(\widehat{A}, \widehat{B}) = 0$, *and* $\mathrm{Hom}_R(\widehat{C}, \widehat{D}) = 0$.

(ii) UNIQUE: *If* $A_1 \oplus B_1 \ll C_1$ *and* $C_1 \oplus D_1 \ll M$ *satisfy* (a), (b) *and* (c), *then* $A = A_1$, $B = B_1$, $C = C_1$, *and* $D = D_1$.

(iii) $M_\tau \leq M$ *is a fully invariant complement for any* $\tau \in \Omega$. (*See* 1.2.)

(iv) $\sum\{M_\tau \mid \tau \in \Omega\} = \bigoplus\{M_\tau \mid \tau \in \Omega\} \ll A \oplus D$.

(v) $\mathrm{Hom}_R(EM_\tau, EM_\rho) = 0$ *for any* $\tau \neq \rho \in \Omega$.

Proof. (i) and (ii). By [11, p.7, 2.8(ii)] and [12, Proposition 3.14], $\widehat{M} = \widehat{C} \oplus \widehat{D}$ with $\widehat{C} = \widehat{A} \oplus \widehat{B}$, where $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and $\widehat{D}$ are unique and fully invariant right R-submodules of $\widehat{M}$. Their uniqueness and full invariance in $\widehat{M}$ implies the uniqueness and full invariance of $M \cap \widehat{A} = A$, $M \cap \widehat{B} = B$, $M \cap \widehat{C} = C$, and $M \cap \widehat{D} = D$ in M.

(iii) If W is any atomic module, then every torsion free homomorphic image of W is likewise atomic. Hence $M_\tau \leq M$ is fully invariant.

(iv) If $W_1 \in \tau$ and $W_2 \in \rho$ for $\tau \neq \rho \in \Omega$, then by the definition of $\tau \neq \rho$ together with the fact that W_1 and W_2 are atomic, it follows that W_1 does not contain a nonzero submodule isomorphic to a submodule of W_2. Now [11, p.4, Lemma 2.4(i)] shows that the sum in (iv) is direct.

(v) Conclusion (v) follows from 3.2.

Note that both A and D in the last theorem contain essential direct sums of atomic modules. The proof used in [11, p.7, Corollary 2.9] can be used to establish the next substantially more general result.

3.4. Corollary 1 to Theorem I. *For an arbitrary torsion free injective module M, let $M = A \oplus B \oplus D$ be the decomposition given by the last theorem. Then*

$$\bigoplus_{\tau \in \Omega} E(M_\tau) \ll A \oplus D = \prod_{\tau \in \Omega} E(M_\tau).$$

Consequently $A = \Pi\{E(A_\tau) | \tau \in \Omega^C\}$ and $D = \Pi\{E(D_\tau) | \tau \in \Omega^D\}$.

3.5. Corollary 2 to Theorem I. *Suppose that R is any ring with identity and $ZR = 0$. Let $A \oplus B \oplus D \ll C \oplus D \ll R$ be the right ideals given by the last theorem and let $\rho \neq \tau \in \Omega$ be arbitrary. Then the following hold:*

(i) $A, B, C, D \triangleleft R$; $CD = DC = 0$, $AB = BA = 0$; $A, B, C, D \leq R$ *are complement right ideals.*

(ii) UNIQUE: *A is unique as the largest right ideal which contains every continuous atomic* (*or molecular*) *right ideal*; *and analogously for B, C, and D.*

(iii) $R_\tau \triangleleft R$, $R_\tau = \bar{R}_\tau \leq R$.

(iv) $\sum_{\tau \in \Omega} R_\tau = \bigoplus_{\tau \in \Omega} R_\tau \ll A \oplus D$; *in particular* $R_\tau R_\rho = 0$.

(v) $R_\tau^\perp \supseteq [\bigoplus\{R_\rho | \tau \neq \rho \in \Omega\} \oplus B]^-$; $R_\tau + R_\tau^\perp \ll R$.

(vi) *Conclusions* (i)–(v) *hold for the ring $\widehat{R} = \widehat{C} \oplus \widehat{D} = \widehat{A} \oplus \widehat{B} \oplus \widehat{D}$ (if every occurence of R, A, B, C, and D is replaced by $\widehat{R}$, $\widehat{A}$, $\widehat{B}$. $\widehat{C}$, and $\widehat{D}$).*

(b) $\widehat{A} \oplus \widehat{D} = E(\bigoplus\{R_\tau | \tau \in \Omega\}) = \Pi E(R_\tau)$ *as a ring. In particular,* $E(R_\tau)$, $E(R_\rho) \triangleleft \widehat{R}$ *with* $E(R_\tau)E(R_\rho) = 0$, *and*

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(c) $\widehat{A} = E(\bigoplus\{A_\tau | \tau \in \Omega^C\}) = \Pi E(A_\tau)$ *as a ring.*

4. Applications to rings. The disjoint module theoretic developments of sections 2 and 3 are now specialized to rings and made to converge.

4.1. Lemma. *For a t.f. semiprime ring R, suppose that $0 \neq P$, $0 \neq Q \leq R$ with $P, Q \subseteq R_\tau$ for some $\tau \in \Omega$. Then $PQ \neq 0$.*

Proof. For any $R_\tau \supseteq P \leq R$, $[(0)] \neq [P] \leq [R_\tau] = \tau \in \Xi(R)$. Since τ is an atom, $[P] = \tau = [Q]$. Thus $Q \subseteq E(\bigoplus\{P | \Gamma\})$ for some index set Γ. Now 3.1 shows that for any $0 \neq \xi \in Q$, there exist $r_0 \in R$ and $a_1 \in P$ such that $0 \neq \xi r_0 R \cong a_1 R$. By 2.5, $a_1 R(\xi r_0 R) \neq 0$.

In the theorem below if the atomic right ideal $W < R$ is discrete then $\bigoplus U_\alpha \ll W \ll E(\bigoplus U_\alpha)$ where $\{U_\alpha\}$ is a family of uniform right ideals all of the same type as W. In particular, W itself could simply be a uniform right ideal of R.

4.2. Theorem II. *For a semiprime ring R with $ZR = 0$, let $A \oplus B \oplus D \ll C \oplus D \ll R$, $\tau \in \Omega$ and $0 \neq R_\tau \lhd R$ be as in 3.5, and let $W < R$ be any atomic right ideal (1.2). Then the following hold.*

(1)(a) *W is a prime right R-module.*

(b) *$W^\perp \lhd R$ is a prime ideal.*

(c) *For any two atomic right ideals V, $W < R$, if $V \cap W \neq 0$, then $\Rightarrow W^\perp = V^\perp$.*

(d) *For any $W \in \tau$, $R_\tau^\perp = W^\perp$. Hence $R_\tau^\perp \lhd R$ is a prime ideal.*

(2)(a) *$R_\tau \oplus R_\tau^\perp \ll R$.*

(b) *$R_\tau^\perp$ is a maximal annihilator ideal of R.*

(3)(a) *R_τ is a prime right R-module.*

(b) *R_τ is a prime ring.*

(c) *$A, B, C, D \lhd R$ and $R_\tau \lhd R$ are semiprime ideals.*

(d) *$A^\perp = (B \oplus D)^-$, $B^\perp = (A \oplus D)^-$, $D^\perp = (A \oplus B)^-$, $C^\perp = D$, $D^\perp = C \lhd R$ all are semiprime ideals.*

(e) *$C = (A \oplus B)^-$; and $C^\perp = A^\perp \cap B^\perp = D$, $B^\perp \cap D^\perp = A$, $D^\perp \cap A^\perp = B$.*

Proof. (1)(a) If W is not prime, then $w_0 R t = 0$ for some $0 \neq w_0 \in W$ and some $t \in R \setminus W^\perp$. Thus $w_1 t \neq 0$ for some $0 \neq w_1 \in W$. For some index set $\mathcal{I}$, there is an embedding $w_0 R \subset E(\bigoplus\{w_1 t R | \mathcal{I}\})$ because W is atomic. Now use 3.1 to show that $(w_0 r_0)^\perp = (w_1 t r_1)^\perp$ for some

$r_0, r_1 \in R$ with $w_1 t r_1 \neq 0$. Then $w_0 r_0 R w_1 t r_1 \subseteq w_0 R t r_1 = 0$, and hence $R w_1 t r_1 \subseteq (w_0 r_0)^\perp = (w_1 t r_1)^\perp$. But then $(w_1 t r_1) R w_1 t r_1 = 0$ contradicts the semiprimeness of R.

(1)(b) The annihilator ideal of any prime R-module is always a prime ideal of R ([D4, p.160, 1.7(2)]).

(1)(c) A module W is prime if and only if for any nonzero submodule, such as $0 \neq V \cap W < W$, $(V \cap W)^\perp = W^\perp$, ([7, p.159, 1.31(ii)]). By symmetry, $(V \cap W)^\perp = V^\perp$.

(1)(d) Let $V, W < R$ with $V, W \in \tau$. In view of 3.1, there exist $0 \neq P < V$, $0 \neq Q < W$ with $P \cong Q$. Then $V^\perp = P^\perp = Q^\perp = W^\perp$ by (1)(c) above. From $R_\tau = \sum\{W | W \leq R,\ W \in \tau\}$ it follows that $R_\tau^\perp = \bigcap\{W^\perp | W \leq R, W \in \tau\} = W^\perp$ is prime by (1)(b).

(2)(a) By either 2.7(i) or 3.5(v), $R_\tau^\perp \oplus R_\tau \ll R$.

(2)(b) If not, then $R_\tau^\perp \lneqq I \triangleleft R$, where $I = \ell(I)^\perp \neq R$ as in 1.1. Therefore $\ell(I) \subseteq \ell(R_\tau^\perp) = (R_\tau^\perp)^\perp = \bar{R}_\tau = R_\tau$ by 2.7 and 3.5(iii). If $\ell(I) = R_\tau$, then $R_\tau^\perp = \ell(I)^\perp = I$; so $0 \neq \ell(I) \lneqq R_\tau$. From $R_\tau^\perp \oplus R_\tau \ll R$ it follows that $R_\tau^\perp \oplus (I \cap R_\tau) \ll I$. But both $R_\tau^\perp \leqq I$ are right complements by 2.1. Hence $R_\tau^\perp \not\ll I$. Consequently, $I \cap R_\tau \neq 0$. But now 4.1 applied to the nonzero atomic right ideals $0 \neq I \cap R_\tau$, $\ell(I) \subset R_\tau$ yields the contradiction that $\ell(I)(I \cap R_\tau) \neq 0$. Therefore $R_\tau^\perp$ is a maximal annihilator ideal of R.

(3)(b) It suffices to show that for any $0 \neq \alpha, \beta \in R_\tau$, always $\alpha R_\tau \beta \neq 0$. Since $ZR = 0$ and $R_\tau^\perp \oplus R_\tau \ll R$, $0 \neq \alpha(R_\tau^\perp \oplus R_\tau) = \alpha R_\tau$ and similarly, $\beta R_\tau \neq 0$. By 4.1, $\alpha R_\tau \beta R_\tau \neq 0$.

(3)(c) In 2.6, take $K = \bar{K} = A, B, C, D$, and R_τ.

(3)(d) Since any submodule of a semiprime module is likewise semiprime, it follows that A, B, C, and D are semiprime modules. Consequently $A^\perp, B^\perp, C^\perp, D^\perp \triangleleft R$ are semiprime ideals. Since $A \oplus (B \oplus D)^- \ll R$, use of 2.7 gives that $A^\perp = (B \oplus D)^-$. The other cases are verbatim the same.

(3)(e) Since $A \oplus B \ll C = \bar{C}$, $C^\perp = (A \oplus B)^\perp = A^\perp \cap B^\perp$. Cyclic permutation in $C^\perp = D = A^\perp \cap B^\perp$ gives the rest.

The next corollary is the first step towards the solution of the still open problem of finding the real algebraic differences between the discrete and continuous rings of the type R_τ. It fails for some continuous rings R_τ.

4.3. Corollary 1 to Theorem II. *If R is a torsion free semiprime ring and $\tau \in \Omega^D$, then for any $0 \neq I \triangleleft R_\tau$, there exists a $0 \neq J \triangleleft R$ with*

$J \subseteq I$ such that $J \subset R_T$ is large as a right R_T-module (and hence also as a right R-module).

Proof. Use of 4.1 shows that for any $V < R_T$, $V \ll R_T$ if and only if $V \subset R_T$ is large as a right R_T-module. Let $0 \neq \xi \in I \triangleleft R_T$. Since R_T is a prime ring, $0 \neq a\xi b \in (\xi R_T)^3$ for some $a, b \in R_T$. There exists an indexed family of uniform right R-ideals $\{U_\alpha\}$ such that $\bigoplus U_\alpha \ll R_T$. By 4.1, $U_\alpha a\xi bR \neq 0$ for all α. Hence $\bigoplus U_\alpha a\xi bR \ll R_T$. Set $J = Ra\xi bR$. Then $J \subseteq I$, $0 \neq J \triangleleft R$, and $\bigoplus U_\alpha a\xi bR \ll J$. Thus $J \ll R_T$.

5. Subdirect products. First, some known or easily provable facts are arranged in a form in which they later will be used ([15], [16], [24], and [25]). Under the hypothesis that $\{R_i | i \in I\}$ are nonsingular rings, the inequivalent and very different concepts of an "essential subdirect product $R \subset \Pi R_i$" defined by Goodearl ([16, p.115]) and Loonstra ([25, p.91]) fortunately coincide. This happens if and only if all of the conditions (1)(a)–(d) hold below.

5.1. Lemma. *Let $R \subset \Pi\{R_i | i \in \mathcal{I}\} \equiv T$ be a subdirect product of any family of right nonsingular rings R_i with identity, and let "$\leq_e$" denote essential right ideals in the rings R_i and T.*

(1) *Then the following four conditions are equivalent*

(a) $R \ll T_R$.

(b) $\forall i,\ R_i \cap R \leq_e R_i$.

(c) $\bigoplus(R_i \cap R) \ll T_R$.

(d) $\exists J \leq_e T,\ J \subseteq R$.

Now assume that the above (1)(a)–(d) *hold. Then*

(2) $ZR = 0$.

(3) *R is an irredundant subdirect product of the R_i.*

In view of 5.1 (1)(b) and (c), the next two corollaries follow. For the sake of completeness, we include condition 5.2(ii) below, which is one of the hypotheses used in [15, p.251, Theorem 2] to guarantee irredundancy.

5.2. Corollary 1. *In addition to* (1)(a)–(d), *assume that each R_i is semiprime. Then also*

(i) *R is semiprime; and*

(ii) $\{r_i | r_i \in R_i,\ r_i(R_i \cap R) = 0\} = \{r_i | r_i \in R_i,\ (R_i \cap R)r_i = 0\} = 0$.

5.3. Corollary 2. *Assume* (1)(a)–(d) *and that each* R_i *is molecular as a right* R_i*-module. Then* R *and* T *are right molecular* R*-modules.*

5.4. Theorem III. (1) *A ring* R *is* (H1) *right nonsingular, i.e. with* $ZR = 0$, (H2) *semiprime, and* (H3) *right molecular* $\Leftrightarrow$ R *is a subdirect product* $R \subseteq \Pi\{R_i | i \in \mathcal{I}\} \equiv T$ *of rings* R_i *which are* (h1) *right nonsingular,* (h2) *prime,* (h3) *right atomic, with* (h4) $R \ll T_R$.

(2) *Now suppose that* R *satisfies* (H1), (H2), *and* (H3) *above, and let* $R_\tau^\perp \oplus R_\tau \ll R$, $\tau \in \Omega$, *be as in* 4.2. *Then*

(a) $\bigcap\{R_\tau^\perp | \tau \in \Omega\} = 0$; $R \cong \tilde{R}$ *as a ring and as a right* R*-module, where* $\tilde{R}$ *is the following irredundant subdirect product of the prime right nonsingular molecular rings* $R/R_\tau^\perp$,

$$R \overset{\cong}{\rightarrowtail\!\!\!\twoheadrightarrow} \tilde{R} \subset \prod_{\tau \in \Omega} R/R_\tau^\perp \equiv T.$$

Moreover, $\tilde{R}$ *is an essential right* R*-submodule of* T, *such that*

(b) $\bigoplus_{\tau\in\Omega} R_\tau \cong \bigoplus(R_\tau^\perp + R_\tau)/R_\tau^\perp \ll \tilde{R} \ll T_R$.

(3) UNIQUENESS: *Suppose that* $R \subset \Pi\{R_i | i \in \mathcal{I}\}$ *is any subdirect representation of* R *satisfying* (h1)–(h4). *Then there exists a bijection* $f\colon I \to \Omega$ *and a ring isomorphism* $g\colon \Pi R_i \to \Pi R/R_\tau^\perp$ *with* $gR_i = R/R_{fi}^\perp$ *for all* $i \in \mathcal{I}$.

Proof. (1)$\Longleftarrow$: this was shown in 5.1, 5.2, and 5.3.

(1)$\Longrightarrow$: For this, it suffices to prove (2).

(2)(a) Now $B = 0$ and $\bigoplus\{R_\tau | \tau \in \Omega\} \ll R$. If $z \in \bigcap R_\tau^\perp$, then $Z(\oplus R_\tau) = 0$ implies that $z \in ZR = 0$. Hence $R \cong \tilde{R}$ as a ring. Identify $\tilde{R}$ as a right R-module with $\tilde{R}$ as a right $\tilde{R}$-module. Then under the canonical ring isomorphism $R \to \tilde{R}$, the essential right R-submodule $\bigoplus R_\tau \ll R$ is mapped into $\bigoplus(R_\tau^\perp + R_\tau)/R_\tau \ll \tilde{R}$.

(2)(b) By 5.1(a) and (c), now $\tilde{R} \ll T_R$.

(3) By [24, p.66, Theorem 3.2] and 4.2(2)(b), it suffices to show that every proper annihilator ideal $I \triangleleft R$ is contained in some $R_\tau^\perp$. If not, then $b_\tau I \neq 0$ for some $0 \neq b_\tau \in R_\tau$ for every $\tau \in \Omega$. As in 1.1, $I = \ell(I)^\perp$. Since $\bigoplus R_\tau \ll R$, 3.1 shows that there exists $0 \neq \eta = a_1 + \cdots + a_n \in \ell(I) \cap [R_{\tau(1)} \oplus \cdots \oplus R_{\tau(n)}]$, $0 \neq a_i \in R_{\tau(i)}$ such that $\eta^\perp = a_1^\perp = \cdots = a_n^\perp$. Then $\eta R \cong a_1 R$, and $(\eta R)^\perp = (a_1 R)^\perp$. Thus since $\eta R \subset \ell(I)$, $b_{\tau(1)} I \subset I = \ell(I)^\perp \subseteq (\eta R)^\perp = (a_1 R)^\perp$ and hence $(a_1 R) b_{\tau(1)} I = 0$. But $0 \neq a_i R$, $0 \neq b_{\tau(1)} I \subseteq R_{\tau(1)}$ contradicts 4.1.

Note that if (h1) and (h4) above hold, then $\bigoplus\{(R \cap R_i)| \, i \in I\} \ll R \ll T_R$.

5.6. Corollary to Theorems II and III. (1) *A ring R is* (H1) *right nonsingular i.e. with* $ZR = 0$ *and* (H2) *semiprime* $\Leftrightarrow$ *R is a subdirect product* $R \subseteq R_1 \times R_2 \times R_3 \equiv T$ *of rings* R_i *which are* (h1) *right nonsingular,* (h2) *semiprime, where* (h3) R_1 *is continuous molecular,* R_2 *is bottomless, and* R_3 *is discrete, with* (h4) $R \ll T_R$.

(2) *Assume R satisfies* (H1) *and* (H2) *and let* $A \oplus B \oplus D \ll R$ *be as in* 4.2. *Define* $R_1 = R/A^\perp$, $R_2 = R/B^\perp$, $R_3 = R/D^\perp$, $\tilde{A} = [A \oplus A^\perp]/A^\perp \triangleleft R_1$, *and similarly* $\tilde{B} \triangleleft R_2$, $\tilde{D} \triangleleft R_3$. *Then*

(a) $A^\perp \cap B^\perp \cap D^\perp = 0$; $R \cong \tilde{R}$ *under the canonical map* $R \rightarrowtail\!\!\!\!\rightarrow \tilde{R} \subseteq R_1 \times R_2 \times R_3 = T$; $\tilde{R}$ *is an irredundant subdirect product of the* R_i. *The* R_i *satisfy* (1)(h1)–(h3) *above. Furthermore* $\tilde{R}$ *is a large R-submodule of T, as follows*

(b) $A \oplus B \oplus D \cong \tilde{A} \oplus \tilde{B} \oplus \tilde{D} \ll \tilde{R} \ll T_R$.

(3) *View* $R/C^\perp = R/[A^\perp \cap B^\perp] \subset (R/A^\perp) \times (R/B^\perp)$ *as a subring and right R-submodule via* $r + C^\perp \to (r + A^\perp, r + B^\perp)$ *for* $r \in R$. *Then* $\tilde{R}$ *in* (2) *is a subdirect product of the continuous t.f. ring* $R/C^\perp$ *and the t.f. discrete ring* $R/D^\perp$ *with*

$$C \oplus D \cong \tilde{C} \oplus \tilde{D} \ll \tilde{R} \ll \frac{R}{C^\perp} \times \frac{R}{D^\perp} \ll T_R.$$

The analogous result of (1) *also holds for* $R \cong \tilde{R} \subseteq (R/C^\perp) \times (R/D^\perp)$.

Proof. As before for (1), it is enough to prove (2). (2)(a) and (b), recall from Theorem II that $A^\perp = (B \oplus D)^-$. Since $A \oplus A^\perp \ll R$ and $A^\perp < R$ is a complement, $A \cong \tilde{A} \ll R/A^\perp = R_1$, and R_1 is t.f. and continuous molecular. Similarly $B \cong \tilde{B} \ll R_2$, $D \cong \tilde{D} \ll R_3$; and hence R_2 is bottomless and R_3 is discrete, both being t.f. By a total of four applications of the modular law, we conclude first that $A \oplus B \oplus D \oplus [A^\perp \cap B^\perp \cap D^\perp] \ll R$. Thus $A^\perp \cap B^\perp \cap D^\perp = 0$, and $R \cong \tilde{R}$ under $r \to \tilde{r} = (r + A^\perp, r + B^\perp, r + D^\perp)$ for $r \in R$. As before, identify $\tilde{R}$ as a right R-module with $\tilde{R}$ as a right $\tilde{R}$-module. Under the isomorphism $R \cong \tilde{R}$ large right R-submodules are preserved, and hence $\tilde{A} \oplus \tilde{B} \oplus \tilde{D} \ll \tilde{R}$. Since again 5.1(c) holds, $\tilde{R} \ll T_R$. The proof of (3) is similar, and is omitted.

6. Examples. Some examples illustrating Theorems II and III are given. At the same time they show what typical continuous molecular and bottomless rings look like.

6.1. Example. Let $F = K\{y, z\}$ be the free algebra over any field K in two noncommuting indeterminates y, z. Let $\langle y\rangle = FyF \triangleleft F$ denote the ideal generated by any single element $y \in F$. Define R to be the subring

$$R = \{(\gamma+\alpha, \gamma+\beta, \gamma) \mid \alpha \in \langle y\rangle, \beta \in \langle z\rangle, \gamma \in F\} \\ \subset F \times F \times F = T.$$

Then R is a t.f. continuous molecular ring that is semiprime, but not prime. The poset $\Xi(R)$ is the eight element Boolean lattice of all subsets of $\Omega = \Omega^C = \{\tau(1), \tau(2), \tau(3)\}$. Then $R_{\tau(1)} = \langle y\rangle \times \{0\} \times \{0\}$, $R_{\tau(2)} = \{0\} \times \langle z\rangle \times \{0\}$, $R_{\tau(3)} = \{0\} \times \{0\} \times \{\langle y\rangle \cap \langle z\rangle\} \triangleleft R$, and hence $R \ll T_R$. Some long computations show that $R/R^{\perp}_{\tau(i)} \cong F$ for all i. Yet, R is not a direct product of the F's.

The next examples show that Theorem III becomes false if the molecular atomic hypothesis (i.e. 5.4(1); (H3) and (h3)) is omitted. The counterexamples below are commutative, t.f., semiprime, bottomless rings R, which have the further property that every nonzero ideal of R contains a proper zero divisor. Suppose that any one of these rings R was a subdirect product of prime rings R_i with $\bigoplus_I(R_i \cap R) \ll R \ll \prod_i R_i = T$ (as in 5.4(1)). Then the R_i are nonzero prime domains. This is a contradiction, because $0 \neq R_i \cap R \triangleleft R$.

6.2. Counterexamples. (a) Let K be any commutative domain with $1 \in K$, and $R = \prod_1^\infty K / \bigoplus_1^\infty K$.

(b) For any infinite set X, be $\mathcal{P}(X)$ be the Boolean ring of all subsets of X with $V \cdot W = V \cap W$ and $V + W = (V \cup W) \setminus (V \cap W)$ for $V, W \subseteq X$. Let $\mathcal{F}(X) \triangleleft \mathcal{P}(X)$ be the ideal consisting of all finite subsets of X. Then set $R = \mathcal{P}(X)/\mathcal{F}(X)$.

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