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Abstract

In our previous paper [12], we considered the unicity problem of the maximal extension of a given Hilbert algebra, and established the: most fundamental property of a maximal Hilbert algebra ([12; Theorem 2J). We argued also the decomposition of maximal Hilbert algebras with respect to their centres, and, on doing it, we noticed that there exist two different types of them, i.e., the simple ones and the purely non-simple ones. The decomposition theorem to these types was given in [12; Theorem 5J with a sketch of the proof, and we announced that. further arguments concerning the decomposition would be given in some other paper. The chief aim of this paper is to give it. In §1 a short cut of the known results is given, and §2 is devoted tothe more detailed exposition of the decomposition of a given Hilbert. algebra into the simple components and the purely non-simple component. A simple Hilbert algebra is one for which the algebras of left and right multiplication constitute a couple of factors in the sense of F. J. Murray and J. von Neumann ([4J), and we are led naturally to make use of their theory. The main problem here is how the dimensionality functional can be expressed by means of the terms of the Hilbert algebra. These are discussed in §3. The reduction theory of a. purely non-simple Hilbert algebra into simple ones is given in §4. This idea, though here only applied to the separable case, can be applied in the non-separable case. But in the most general case we do not yet succeed in proving simplicity character and that will be a future problem.

ON THE STRUCTURE OF MAXIMAL HILBERT ALGEBRAS

OSAMU TAKENOUCHI

In our previous paper [12][1], we considered the unicity problem of the maximal extension of a given Hilbert algebra, and established the most fundamental property of a maximal Hilbert algebra ([12; Theorem 2]). We argued also the decomposition of maximal Hilbert algebras with respect to their centres, and, on doing it, we noticed that there exist two different types of them, i.e., the simple ones and the purely non-simple ones. The decomposition theorem to these types was given in [12; Theorem 5] with a sketch of the proof, and we announced that further arguments concerning the decomposition would be given in some other paper. The chief aim of this paper is to give it.

In §1 a short cut of the known results is given, and §2 is devoted to the more detailed exposition of the decomposition of a given Hilbert algebra into the simple components and the purely non-simple component. A simple Hilbert algebra is one for which the algebras of left and right multiplication constitute a couple of factors in the sense of F. J. Murray and J. von Neumann ([4]), and we are led naturally to make use of their theory. The main problem here is how the dimensionality functional can be expressed by means of the terms of the Hilbert algebra. These are discussed in §3. The reduction theory of a purely non-simple Hilbert algebra into simple ones is given in §4. This idea, though here only applied to the separable case, can be applied in the non-separable case. But in the most general case we do not yet succeed in proving simplicity character and that will be a future problem.

§1. Sketches of the known results and a lemma.

A *Hilbert algebra* $\mathfrak{A}$ in a Hilbert space $\mathfrak{H}$ is defined as follows ([6]):

(i) It is a dense linear manifold in $\mathfrak{H}$.

(ii) Between the elements of $\mathfrak{A}$, a multiplication law is defined such that, with respect to the linear operation defined in $\mathfrak{H}$ and this

1) Numbers in brackets denote the numbers of literatures at the end of the paper.

multiplication operation, $\mathfrak{A}$ constitutes an algebra over the field of complex numbers.

(iii) To each $a \in \mathfrak{A}$, an element a^* in $\mathfrak{A}$ is corresponded so that, for any $b, c \in \mathfrak{A}$,

$$(ab, c) = (b, a^*c), \qquad (ba, c) = (b, ca^*).$$

The operation $a \to a^*$ is called the adjoint operation.

(iv) The operator T_a^0 associated with $a \in \mathfrak{A}$, which is defined on $\mathfrak{A}$ and makes correspond ax to every $x \in \mathfrak{A}$, is bounded:

$$\| T_a^0 x \| \leqslant \gamma_a \| x \| \qquad (\gamma_a > 0, \ x \in \mathfrak{A}).$$

Thus this operator has a continuous extension T_a over the whole space $\mathfrak{H}$. Thus extended operator T_a is called the operator of left multiplication.

(v) An element $f \in \mathfrak{H}$ satisfies $T_x f = 0$ for any $x \in \mathfrak{A}$, if and only if $f = 0$.

From these assumptions follow immediately ([6]), that the adjoint element a^* is uniquely determined for $a \in \mathfrak{A}$ and, $(a, b) = (b^*, a^*)$, $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$, $(ab)^* = b^* a^*$, etc., and that the right multiplication operation S_a is also bounded and satisfies the analogous property of (v).

An element $u \in \mathfrak{A}$ is called a unit if $u^2 = u$, $u^* = u$, for which the associated operators T_u, S_u are both projection operators.

A Hilbert algebra would have generally many extensions other than itself, but among them the maximal one exists (and, of course, is uniquely determined). This extension is consisted of such elements f of $\mathfrak{H}$ for which either of the two conditions

(1.1) $\| S_x f \| \leqslant \gamma_f \| x \|$ (for any $x \in \mathfrak{A}$, $\gamma_f > 0$ is fixed),

(1.1′) $\| T_x f \| \leqslant \gamma'_f \| x \|$ (for any $x \in \mathfrak{A}$, $\gamma'_f > 0$ is fixed)

is satisfied ([12, Theorem 1]). The operator, which assignes to every $x \in \mathfrak{A}$ the element $S_x f$, will be denoted as T_f^0, and similarly S_f^0 is defined. Thus (1.1) and (1.1′) can be restated that T_f^0 or S_f^0 is bounded.

Thus to give the complete definition of a maximal Hilbert algebra, we have to add one more axiom: "

(vi) If T_f^0 or S_f^0 is bounded for an $f \in \mathfrak{H}$, then f must be in $\mathfrak{A}$,"

to the axioms (i)…(v) listed above ([12; Theorem 2]).

We shall use later the following

Lemma 1.1: $\mathfrak{H}_1$, $\mathfrak{H}_2$ *be Hilbert spaces, and in each of them a Hilbert algebra* $\mathfrak{A}_i$ $(i = 1, 2)$ *be given. Suppose that they are isomorphic in the following sense:*

(i) *Between Hilbert spaces* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$, *a linear isometric mapping exists which maps whole* $\mathfrak{H}_1$ *onto whole* $\mathfrak{H}_2$.

(ii) *Under this mapping,* $\mathfrak{A}_1$ *corresponds to* $\mathfrak{A}_2$ *isomorphically as Hilbert algebras.*

Then, denoting the respective maximal extensions as $\mathfrak{A}_1^0$, $\mathfrak{A}_2^0$, $\mathfrak{A}_2^0$ *corresponds to* $\mathfrak{A}_1^0$ *under this mapping isomorphically as Hilbert algebras.*

This will be observed at once from the method of construction of maximal extension.

We used the notation **S** and **T** to denote the whole set of operators of right and left multiplication and **M** and **M′** their respective commutator algebras of operators[1]. These **M** and **M′** are commutator algebras to each other, and was called the algebras of left and right multiplication resp., and $\mathbf{Z} = \mathbf{M} \cap \mathbf{M}'$ the centre of $\mathfrak{A}$. Moreover, for any $A \in \mathbf{M}$ or $\mathbf{M}'$, $A\mathfrak{A} = \{Aa;\ a \in \mathfrak{A}\} \subset \mathfrak{A}$ ([12; Theorem 3, 4, Definition 3]).

§2. Central decomposition of a maximal Hilbert algebra.

First we mention a more detailed proposition than [6; Theorem 4, 7].

Lemma 2.1: *For an arbitrary projection operator* $P (\neq 0) \in \mathbf{M}$ *or* $\mathbf{M}'$, *the subset* $P\mathfrak{A}$ *of* $\mathfrak{A}$ *contains a unit* u.

Proof: We assume that $P \in \mathbf{M}$. As $P \neq 0$, $P\mathfrak{A} \neq (0)$[2], so we can take an a, $a \neq 0$, $a \in P\mathfrak{A}$. The element $h = aa^*$ is a self-adjoint element, and $\neq 0$[3] and also belongs to $P\mathfrak{A}$. If $P = T_h$, h is a unit, and this meets our desired condition. In the general case, let $T_h = \int_0^\infty \lambda dE_\lambda$ be the

1) i.e. the set of all those bounded linear operators with domain $\mathfrak{H}$ which commute with all A and A^* where $A \in \mathbf{T}$ or **S**. Here and in what follows, an algebra of operators means that it is not only an algebra in the algebraic sense but also it is closed with respect to the adjoint operation (of operators) and closed with respect to the weak topology of operators. Cf. [5; II].

2) If $P\mathfrak{A}$ is consisted of 0 only, we see for an arbitrary $a \in \mathfrak{A}$, $Pa = 0$, $(1-P)a = a$, thus $(1-P)\mathfrak{A} = \mathfrak{A}$. But if P were not 0, this induces a contradiction, since $\mathfrak{A}$ is dense in $\mathfrak{H}$.

3) As $T_h = T_{aa^*} = T_{a^*}^* T_{a^*}$, we have, for any $f \in \mathfrak{H}$, $(T_h f, f) = \| T_{a^*} f \|^2$. Thus $h = 0$ implies $T_{a^*} = 0$, or $a^* = 0$, so $a = 0$.

spectral form of T_h[1]. Then [6; Theorem 4, 7] shows that for a suitably chosen $\varepsilon > 0$, there exists a unit u_ε such that $T_{u_\varepsilon} = 1 - E_\varepsilon \neq 0$. Thus for an $x \in \mathfrak{H}$ with $Px = 0$, $T_h x$ is also 0, or what is the same $E_0 x = x$, $PT_h = T_{Ph}$[2]$= T_h$, and a fortiori $1 - P \leqslant E_0$ or $1 - E_0 \leqslant P$. Therefore we have for an arbitrary $\lambda > 0$, $1 - E_\lambda \leqslant P$ so that $PT_{u_\varepsilon} = P(1 - E_\varepsilon) = 1 - E_\varepsilon = T_{u_\varepsilon}$. From this, it follows $Pu_\varepsilon = u_\varepsilon$, and this u_ε is the desired one.

Lemma 2.2: *The subspace $\mathfrak{M}$ of $\mathfrak{H}$ invariant under all the operators of left multiplication, has as its projection operator what belongs to* $\mathbf{M}'$: $P_{\mathfrak{M}} \in \mathbf{M}'$ (*The converse statement that for a projection operator $P \in \mathbf{M}'$, its corresponding subspace is invariant under all T_a $(a \in \mathfrak{A})$ is evident*). *In the same way, for the subspace $\mathfrak{M}$ invariant under all S_a $(a \in \mathfrak{A})$, we have $P_{\mathfrak{M}} \in \mathbf{M}$.*

Proof: As $T_a\mathfrak{M} \subset \mathfrak{M}$ $(a \in \mathfrak{A})$ means $P_{\mathfrak{M}}$ is commutative with all T_a $(a \in \mathfrak{A})$, our proposition is clear.

Remark: This subspace $\mathfrak{M}$ invariant under $\mathbf{T}$ was defined by W. Ambrose as the left-ideal of an H-system ([2; §4]). Thus the 2-sided ideal in his sense is the subspace $\mathfrak{M}$ whose corresponding projection operator $P_{\mathfrak{M}}$ belongs to $\mathbf{Z}$. As to the ideal defined by H. Nakano ([6; §5]) we shall call our attention in Lemma 2.6.

Lemma 2.3: *Let $\mathfrak{U}$ be the set of all units in $\mathfrak{A}$. Then for any $P \in \mathbf{M}$ (or $\in \mathbf{M}'$)*

(2.1) $\cup \{T_u;\ u \in \mathfrak{U},\ T_u \leqslant P\}$[3]$= P$ (*or* $\cup \{S_u;\ u \in \mathfrak{U},\ S_u \leqslant P\} = P$).

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1) We choose the resolution of the identity $E_\lambda\ (-\infty < \lambda < \infty)$ always to be continuous to the right.

2) If an operator A belongs to either $\mathbf{M}$ or $\mathbf{M}'$, we have for an arbitrary $a \in \mathfrak{A}$, $AT_a = T_{Aa}$ or $AS_a = S_{Aa}$ resp. This is a simple corollary to [12; Theorem 5].

3) Take a family of projection operators $\{P_\lambda\}_{\lambda \in \Lambda}$ and their corresponding family of subspaces $\mathfrak{M}_\lambda = P_\lambda\mathfrak{H}$. Let $\overline{\mathfrak{M}}$ be the smallest subspace comprising all the subspaces $\mathfrak{M}_\lambda$ $(\lambda \in \Lambda)$, then its corresponding projection operator $\overline{P} = P_{\overline{\mathfrak{M}}}$ has the following properties:

(i) $P_\lambda \leqslant \overline{P}$ $(\lambda \in \Lambda)$, and

(ii) If P is a projection operator such that $P_\lambda \leqslant P\ (\lambda \in \Lambda)$, then necessarily

$$\overline{P} \leqslant P.$$

Thus in accordance with the usual notation in the theory of lattices, we can write this projection operetor $\overline{P}$ as $\cup_{\lambda \in \Lambda} P_\lambda$.

Let all the P_λ's be contained in some algebra of operators $\mathbf{M}$. Then, as $\mathfrak{M}_\lambda$'s are all invariant under any transformation belonging to $\mathbf{M}'$, $\overline{\mathfrak{M}}$ is too, and the above-mentioned property (ii) shows $\overline{P}$ is a projection operator smaller than the maximal projection operator (Haupteinheit, see [7; Definition 4]) of $\mathbf{M}$. Thus

$$\overline{P} = \cup_{\lambda \in \Lambda} P \in \mathbf{M}.$$

(by [7; Satz 5]).

More precisely, the set of u in (2.1) *may be limited to any maximal family of mutually orthogonal units satisfying the mentioned condition.*

Proof: Take a maximal family of mutually orthogonal units satisfying the condition $u_\lambda \in \mathfrak{U}$, $T_{u_\lambda} \leqslant P$. Let $Q = \bigcup_{\lambda \in \Lambda} T_{u_\lambda}$, then $Q \in \mathbf{M}$. Supposing that $P \neq Q$, $P - Q \in \mathbf{M}$ and $\neq 0$. Thus by Lemma 2.1, we know the existence of a unit u_0 which is contained in $(P-Q)\mathfrak{A}$. But this clearly induces a contradiction, since $u_0 \in (P-Q)\mathfrak{A} \subset P\mathfrak{A}$, and $u_\lambda u_0 = 0$ $(\lambda \in \Lambda)$ from $0 = Q(P-Q) \geqslant T_{u_\lambda} T_{u_0} = T_{u_\lambda u_0}$, contrary to that $\{u_\lambda\}_{\lambda \in \Lambda}$ was the maximal set of mutually orthogonal units in $P\mathfrak{A}$. The rest of the lemma can be shown in the same way.

Lemma 2.4: *Take arbitrarily a projection operator* $P\,(\neq 0)$ *belonging to* $\mathbf{Z}$, *then* $P\mathfrak{A}\,(\subset \mathfrak{A})$ *is again a maximal Hilbert algebra in* $\mathfrak{M} = P\mathfrak{H}$ *(defining the multiplication and the adjoint operation are the same as in* $\mathfrak{A}$*). We write this algebra as* $\mathfrak{A}_P$, *when we are considering in* $\mathfrak{M}$. *Its corresponding algebras of left and right multiplication* $\mathbf{M}_P$ *and* $\mathbf{M}'_P$ *are identified with the algebra of operators* $\mathbf{M}_{(P)}$ *and* $\mathbf{M}'_{(P)}$ *in* $\mathfrak{M}$ *resp. which are obtained by contracting all the operators belonging to* $\mathbf{M}$ *or* $\mathbf{M}'$ *to the subspace* $\mathfrak{M}$. *Thus, in particular, the centre* $\mathbf{Z}_P$ *of* $\mathfrak{A}_P$ *is what we obtain by contracting each operator of* $\mathbf{Z}$ *to* $\mathfrak{M}$.

Proof: It is easily seen by verifying the conditions in the definition that $\mathfrak{A}_P$ is a Hilbert algebra. To prove that this is a maximal one it only needs to show that for an $f \in \mathfrak{H}$ which satisfies

$$\| T_{Px} Pf \| \leqslant r \, \| Px \| \qquad \text{(for every } x \in \mathfrak{A}\text{)},$$

we must have $Pf \in P\mathfrak{A}$. But, as $T_{Px} = PT_x = T_x P$, $T_{Px}Pf = T_x Pf$, and from the assumed inequalities,

$$\| T_x Pf \| \leqslant r \, \| Px \| \leqslant r \, \| x \| \qquad \text{(for every } x \in \mathfrak{A}\text{)}.$$

This shows that the operator S^0_{Pf} is bounded on $\mathfrak{A}$ but this is the same to $Pf \in \mathfrak{A}$. Thus $Pf = PPf \in P\mathfrak{A}$, which was to be proved.

Next we consider the algebra of left multiplication $\mathbf{M}_P$. For this sake, we note first that if a is in $\mathfrak{A}_P$ then its associated operator of left multiplication T^P_a (a notation used only here) is the very one that is obtained by contracting T_a (in $\mathfrak{H}$) to $\mathfrak{M}$: $T^P_a = (T_a)_{(P)}$. The same holds for its operator of right multiplication. As in the above, $S_{Px} = PS_x = S_x P$, we can assert for an arbitrary $A \in \mathbf{M}$ that $(AP)S_{Px} = (PA)(S_x P) = PS_x AP = S_{Px}(AP)$ thus the contracted operator $A_{(P)}$ of A to $\mathfrak{M}$ belongs to $\mathbf{M}_P$. Conversely if we take arbitrarily $A_P \in \mathbf{M}_P$ the operator A in $\mathfrak{H}$

defined as $A = A_P P$ belongs to **M**. We can show this as follows. From $AS_x = A_P PPS_x = A_P PS_{Px}P$, $AS_x = (AS_x)_{(P)}P = (A_P(S_{Px})_{(P)})P$. But $(S_{Px})_{(P)} = S^P_{Px}$ as noted above and $A_P \in \mathbf{M}_P$ or $(S_{Ix})_{(P)} \in \mathbf{S}_P$, $A_P \in \mathbf{S}'_P$, we have

$$AS_x = (A_P(S_xP)_{(P)})P = ((S_xP)_{(P)}A_P)P$$
$$= (S_{Ix})_{(P)}PA_PP = S_{Px}A = S_xPA = S_xA.$$

And a fortiori $A \in \mathbf{S}'(=\mathbf{M})$ as we wanted to show. Moreover $A_P = A_{(P)}$. Therefore we have shown that $\mathbf{M}_P = \mathbf{M}_{(P)}$. In the same way $\mathbf{M}'_P = \mathbf{M}'_{(P)}$, and as their intersection $\mathbf{Z} = \mathbf{Z}_{(P)}$, as was asserted.

In the above proof we distinguished in two ways the operators in $\mathfrak{M}$. The operator suffixed with P simply denotes the operator in $\mathfrak{M}$, that suffixed with (P) denotes the operator which is contracted to $\mathfrak{M}$, that is for an operator A in $\mathfrak{H}$, $A_{(P)}$ means the contracted operator of PAP to $\mathfrak{M}$.

Now we can decompose the maximal Hilbert algebra into a direct sum of algebras of two types. For its sake, first we put the following

Definition 2.1: *A projection operator $P \neq 0$ belonging to the centre of $\mathfrak{A}$ is called* minimal *if*

$$Q \in \mathbf{Z},\ 0 \leqq Q \leqq P \ \textit{mean}\ Q = 0 \ \textit{or}\ P.$$

Lemma 2.5: *Take all of the minimal projection operators in* **Z**, *and let them be $\{P_\lambda\}_{\lambda \in \Lambda}$. Then by denoting $P_0 = 1 - \bigcup_{\lambda \in \Lambda} P_\lambda$, we see that*

(i) *P_0, $P_\lambda (\lambda \in \Lambda)$ are pairwise orthogonal projection operators.*

(ii) *The centre of $P_0\mathfrak{A}$ does not contain any minimal projection operators, when considered $P_0\mathfrak{A}$ to be the maximal Hilbert algebra in $P_0\mathfrak{H}$, as was done in* Lemma 2.4.

(iii) *Each $P_\lambda\mathfrak{A} (\lambda \in \Lambda)$ has the centre composed of only the constant multiples of the identity operator $1_{P_\lambda}: \mathbf{Z}_{P_\lambda} = \{\alpha \cdot 1_{P_\lambda}\}$. Thus the algebra of left multiplication $\mathbf{M}'_{P_\lambda}$ of $P_\lambda\mathfrak{A}$ constitute a couple of factors in the sense of* F. J. Murray *and* J. von Neumann [4 ; Definition 3.13].

Proof: (i) As the projection operators $P_\lambda (\lambda \in \Lambda)$ are mutually commutative, their binary products $P_\lambda P_\mu (\lambda, \mu \in \Lambda)$ are also projection operators which are smaller than both of the factors, and $\in \mathbf{Z}$. Thus if $\lambda \neq \mu$, $P_\lambda P_\mu \lneqq P_\lambda$ or P_μ which means from the minimality of P_λ and P_μ that $P_\lambda P_\mu = 0$. Thus P_λ's$(\lambda \in \Lambda)$ are pairwise orthogonal. As to P_0, all the P_λ's$(\lambda \in \Lambda)$ belong to **Z** and **Z** is an algebra of operators, $1 - P_0 = \bigcup_{\lambda \in \Lambda} P_\lambda$ is also in **Z** by footnote 3 in p. 4 and so $P_0 \in \mathbf{Z}$. Of course it is doubtless to say that P_0 and $P_\lambda(\lambda \in \Lambda)$ are orthogonal. And there exist no minimal projection operator $\leqslant P_0$.

(ii) Owing to Lemma 2.4, the centre of $P_0\mathfrak{A}$ is $\mathbf{Z}_{(P_0)}$. Take arbitrarily a projection operator $P_{P_0} \neq 0$ in $\mathbf{Z}_{(P_0)}$. Then $P_{P_0}P_0$ is a projection operator in $\mathfrak{H}$, $\in \mathbf{Z}$, $\leqslant P_0$, $\neq 0$. But from what was done in the above (i), there exists no minimal projection operator in $\mathbf{Z}$ which is smaller than P_0, a fortiori $P_{P_0}P_0$ is not minimal. Therefore the existence of a projection operator $Q \in \mathbf{Z}$, $0 \lneqq Q \lneqq P_{P_0}P_0$ is sure, and for such a Q, $Q = Q_{(P_0)}P_0$, so $0 \lneqq Q_{(P_0)} \lneqq P_{P_0}$, $Q_{(P_0)} \in \mathbf{Z}_{(P_0)}$. Thus P_{P_0} is not a minimal projection operator, and a fortiori $\mathbf{Z}_{(P_0)}$ does not contain any minimal projection operator.

(iii) The centre of $P_\lambda\mathfrak{A}$ is $\mathbf{Z}_{(P_\lambda)}$, and if $\mathbf{Z}_{(P_\lambda)} \ni P_{P_\lambda}$, $P_{P_\lambda}P_\lambda$ is a projection operator which belongs to $\mathbf{Z}$, and $\leqslant P_{P_\lambda}$. Therefore $P_{P_\lambda}P_\lambda = 0$ or P_λ. But as $(P_{P_\lambda}P_\lambda)_{(P_\lambda)} = P_{P_\lambda}$, P_{P_λ} itself is 0 or 1_{P_λ}. Therefore $\mathbf{Z}_{(P_\lambda)} = \{\alpha \cdot 1_{P_\lambda}\}$, where 1_{P_λ} is the identity operator in $\mathfrak{H}_{P_\lambda}$.

Lemma 2.6: *Let $\mathfrak{p}$ be a linear manifold in $\mathfrak{A}$, which satisfies the conditions*

(i) *for arbitrary* $x, y \in \mathfrak{A}$, $x\mathfrak{p}y (= T_xS_y\mathfrak{p}) \subset \mathfrak{p}$, and

(ii) $\mathfrak{p}$ *is closed in* $\mathfrak{A}$,

then there exists a projection operator P in $\mathbf{Z}$, *such that* $\mathfrak{p} = P\mathfrak{A}$. *Thus, by taking the multiplication and the adjoint operation as the same as in* $\mathfrak{A}$, $\mathfrak{p}$ *itself is a (maximal) Hilbert algebra in the minimal subspace* $[\mathfrak{p}]$ *that contains* $\mathfrak{p}$.

Proof: That $\mathfrak{p}$ is closed in $\mathfrak{A}$ is the same as $\mathfrak{p} = [\mathfrak{p}] \cap \mathfrak{A}$.

We show first that $T_x\mathfrak{p} \subset \mathfrak{p}$, $S_x\mathfrak{p} \subset \mathfrak{p}$ for the $x \in \mathfrak{A}$. Take an arbitrary $a \in \mathfrak{p}$, then $xay = S_yxa \in \mathfrak{p}$. Thus for any $u \in \mathfrak{A}$, we have $S_uxa \in \mathfrak{p}$ also. But if we take a maximal family of pairwise orthogonal units $\{u_\lambda\}_{\lambda \in \Lambda}$, then as is shown in Lemma 2.3, $\underset{\lambda \in \Lambda}{\in} S_{u_\lambda} = 1$, therefore from $xa = (\underset{\lambda \in \Lambda}{\cup} S_{u_\lambda})xa$, xa is the limit element of sums of $S_{u_\lambda}xa \in (\mathfrak{p})$. Thus $xa \in [\mathfrak{p}]$ and as $xa \in \mathfrak{A}$, $xa \in [\mathfrak{p}] \cap \mathfrak{A} = \mathfrak{p}$, which means $T_xa \in \mathfrak{p}$. But a was an arbitrary element of $\mathfrak{p}$, $T_x\mathfrak{p} \subset \mathfrak{p}$. That $S_x\mathfrak{p} \subset \mathfrak{p}$ is shown in the same way.

As T_x and S_x are bounded, we see, from what has been shown, $T_x[\mathfrak{p}] \subset [\mathfrak{p}]$, $S_x[\mathfrak{p}] \subset [\mathfrak{p}]$. According to Lemma 2.2, this means that $P_{[\mathfrak{p}]}$ (= the projection operator on $[\mathfrak{p}]$) $\in \mathbf{Z}$. Thus $P_{[\mathfrak{p}]}\mathfrak{A} \subset \mathfrak{A}$, while $P_{[\mathfrak{p}]}\mathfrak{A} \subset [\mathfrak{p}]$, so $P_{[\mathfrak{p}]}\mathfrak{A} \subset \mathfrak{A} \cap [\mathfrak{p}] = \mathfrak{p}$. On the other hand $\mathfrak{p} \subset \mathfrak{A}$ imples $\mathfrak{p} = P_{[\mathfrak{p}]}\mathfrak{p} \subset P_{[\mathfrak{p}]}\mathfrak{A}$, and a fortiori $P_{[\mathfrak{p}]}\mathfrak{A} = \mathfrak{p}$. The rest part of this Lemma is obvious by referring to Lemma 2.4.

Remark: The set $\mathfrak{p}$ considered in the above was defined as the *ideal* in [6; §5], added with a further restriction that this is closed under the adjoint operation. In our case this last condition of self-adjointness is unnecessary, because of the maximal property of the Hilbert algebra in consideration. By the way it may be mentioned that we have shown that from $x\mathfrak{p}y\subset\mathfrak{p}$ $(x, y\in\mathfrak{A})$ follows $x\mathfrak{p}\subset\mathfrak{p}$, $\mathfrak{p}x\subset\mathfrak{p}$, though there may be no identity element e: $ex = xe = x$ (for all $x\in\mathfrak{A}$) in $\mathfrak{A}$. Moreover, the subspace $\mathfrak{M}$ satisfying $T_x\mathfrak{M}\subset\mathfrak{M}$, $S_x\mathfrak{M}\subset\mathfrak{M}$ for every $x\in\mathfrak{A}$ was defined as the 2-sided ideal by W. Ambrose as was remarked already, while for the linear manifold $\mathfrak{p}$ of $\mathfrak{A}$ considered in this Lemma, $[\mathfrak{p}]$ satisfies $T_xS_y[\mathfrak{p}]\subset[\mathfrak{p}]$ for every pair $x, y\in\mathfrak{A}$ and from this we obtain $T_x[\mathfrak{p}]\subset[\mathfrak{p}]$, $S_x[\mathfrak{p}]\subset[\mathfrak{p}]$ for any $x\in\mathfrak{A}$ as is seen from the above proof, therefore this $[\mathfrak{p}]$ is the 2-sided ideal in the W. Ambrose's sense. Conversely, given a 2-sided ideal $\mathfrak{M}$ in the W. Ambrose's sense, then, $\mathfrak{p} = \mathfrak{M}\cap\mathfrak{A}$ satisfies the conditions of the Lemma, and $[\mathfrak{p}] = \mathfrak{M}$ as is readily seen. These mean that the both definitions completely correspond to each other in this sense.

Upon this base we introduce the notion of the ideal as the following by imitating the definition of H. Nakano [6]:

Definition 2.2: *A linear manifold* $\mathfrak{p}$ *in* $\mathfrak{A}$ *is called the* ideal, *if the followings are the case*:

(i) *for arbitrary* $x, y\in\mathfrak{A}$, $x\mathfrak{p}y\,(=T_xS_y\mathfrak{p})\subset\mathfrak{p}$,

(ii) $\mathfrak{p}$ *is closed in* $\mathfrak{A}$.

Then the résumé of the above obtained results reads:

Theorem 2.1: *Let* $\mathfrak{A}$ *be a maximal Hilbert algebra in* $\mathfrak{H}$. *Then there exists a family of pairwise orthogonal projection operators* $\{P_\lambda\}_{\lambda\in\Lambda}$ *in the centre* $\mathbf{Z}$ *of* $\mathfrak{A}$, *and* $\mathfrak{A}$ (*and* $\mathfrak{H}$) *is decomposed as follows*:

(i) *each* $P_\lambda\mathfrak{A}$ *is an ideal of* $\mathfrak{A}$,

(ii) $P_\lambda\mathfrak{A}$ *is the maximal Hilbert algebra in* $P_\lambda\mathfrak{H}$,

(iii) $P_\lambda\mathfrak{A}$ *belongs to each of the following two types*:

Type (S): $\mathfrak{A}$ *is simple, that is to say* $\mathfrak{A}$ *contains no ideal other than* $\{0\}$ *and* $\mathfrak{A}$. *This coindides with that the algebras of left and right multiplication* $\mathbf{M}$, $\mathbf{M}'$ *resp. form a couple of factors.*

Type (0) $\mathfrak{A}$ *is purely non-simple, i.e.* $\mathfrak{A}$ *does not involve any simple ideal*; *namely, for an arbitrary choice of the ideal* $\mathfrak{p}$ *of* $\mathfrak{A}$, *when considered* $\mathfrak{p}$ *itself as a maximal Hilbert algebra in* $[\mathfrak{p}]$, *it is never simple.*

Remark: This has the same content as [6; Theorem 6.4].

We know ultimately that the investigation of the structure of Hilbert algebra is reduced to that of type (S) and that of type (0). We shall discuss the Hilbert algebra of type (S) in the following § and that of type (0) in §4.

Before going to the next §, we state here several more lemmas which will be useful later.

Lemma 2.7 : *For an Hilbert algebra $\mathfrak{A}$, its maximality is necessary for that* $\mathbf{T} = \{T_a;\ a \in \mathfrak{A}\}$ *and* $\mathbf{S} = \{S_a;\ a \in \mathfrak{A}\}$ *are the commutator algebras of one another, and then $\mathfrak{A}$ involves the identity element e:*

$$ea = ae = a\,(a \in \mathfrak{A}) \qquad \text{or equivalently } T_e = S_e = 1.^{1)}$$

Conversely if $\mathfrak{A}$ contains an identity element e, this condition is also sufficient. And if this is true, then $\mathbf{T}$ *and* $\mathbf{S}$ *are the algebras of left and right multiplication resp., i.e.* $\mathbf{M} = \mathbf{T}(= \mathbf{S}')$, $\mathbf{M}' = \mathbf{S}(= \mathbf{T}')$.

Proof: We first note that if $\mathbf{T}$ and $\mathbf{S}$ are the commutator algebras of one another, then $\mathfrak{A}$ contains an identity element e. Such an e is the element such that $T_e = 1(\in \mathbf{T}' = \mathbf{S})$[1]. We show under this condition that $\mathfrak{A}$ is maximal. For this purpose we only need to show that if T_f^0 or S_f^0 is bounded for some $f \in \mathfrak{H}$, then $f \in \mathfrak{A}$. The proof is similar anyhow, we shall assume that S_f^0 is bounded, that is for any $x \in \mathfrak{A}$, there exists $\gamma > 0$ such that

$$\| S_f^0 x \| = \| T_x f \| \leqslant \gamma \| x \| .$$

Let then extend continuously this S_f^0 so as to have the whole $\mathfrak{H}$ as its domain, and write it as $\widetilde{S_f^0}$. Then for an arbitrary $x, y \in \mathfrak{A}$,

$$T_x S_f^0 y = T_x T_y f = T_{T_x y} f = S_f^0 T_x y,$$

thus $\widetilde{S}_f^0$ commutes with any $T_x (x \in \mathfrak{A})$, and also, as $T_x^* = T_{x^*}$, with their adjoint operators: $\widetilde{S}_f^0 T_x^* = T_x^* \widetilde{S}_f^0$, therefore it is contained in $\mathbf{T}' = \mathbf{S}$. Thus it has the form S_a, $a \in \mathfrak{A}$. But then $f = a \in \mathfrak{A}$, which is what we wanted to prove.

Next, assume $\mathfrak{A}$ to be maximal, and contain an identity element. Then for an arbitrary $A \in \mathbf{M}$, $Ax \in \mathfrak{A}\,(x \in \mathfrak{A})$, thus if we put $x = e$, a

1) For a unit $e \in \mathfrak{A}$, conditions $ex = xe = x$ for any $x \in \mathfrak{A}$ and $T_e = S_e = 1$ are equivalent: If $ex = x\,(x \in \mathfrak{A})$, then $T_e = 1$. If S_e were not equal to 1, then, as $S_e \in \mathbf{S} \subset \mathbf{M}'$, $(1 - S_e)\mathfrak{A}$ would contain such a unit u (Lemma 2.1) as $S_e S_u = S_u S_e = 0$. But this implies $eu = 0$ or $u = 0$, which is a contradiction. If $S_e = 1$, then $S_e x = x$, and $xe = x$ for any $x \in \mathfrak{A}$. If $xe = x$ for any $x \in \mathfrak{A}$, then also $x^* e = x^*$ or $ex = x$ for any $x \in \mathfrak{A}$. Thus all these conditions are equivalent.

$= Ae \in \mathfrak{A}$, and for an arbitrary $x \in \mathfrak{A}$, $Ax = A(ex) = (Ae)x = ax = T_a x$. As A and T_a are both bounded, it follows from this $A = T_a$. Thus every $A \in \mathbf{M}$ has the form T_a. Thus $\mathbf{M} \subset \{T_a ;\ a \in \mathrm{A}\} = \mathbf{T}$. But as $\mathbf{M}$ was $\mathbf{R}$ ($\mathbf{T}$, 1), $\mathbf{M} \supset \mathbf{T}$, therefore $\mathbf{M}$ is identical with $\mathbf{T}$. The same is true for $\mathbf{M}'$ and $\mathbf{S}$, and because $\mathbf{M}$ and $\mathbf{M}'$ are the commutator algebras of one another, $\mathbf{T}$ and $\mathbf{S}$ are so.

Thus the proof is completed.

Lemma 2.8: *Let $\mathfrak{A}$ be a maximal Hilbert algebra which contains an identity element e. Then for an operator A belonging to the centre $\mathbf{Z}$, there exists an element $a \in \mathfrak{A}$, and*

$$A = T_a = S_a .$$

And the element having this property is an arbitrary one in the centre of $\mathfrak{A}$ (in the algebraical sense).

Proof: Put $a = Ae$. Then $T_a = T_{Ae} = AT_e = A$, and $S_a = S_{Ae} = AS_e = A$. Thus $A = T_a = S_a$. But $T_a = S_a$ means $T_a x = S_a x$ for any $x \in \mathfrak{A}$, and writing this algebraically $ax = xa$, we know that a may be any element of the (algebraical) centre of $\mathfrak{A}$. q.e.d.

Take an arbitrary unit e, then T_e, S_e are projection operators in $\mathbf{M}$, $\mathbf{M}'$ resp., and putting $T_e \mathfrak{H} = \mathfrak{M}$, $S_e \mathfrak{H} = \mathfrak{M}'$, $\mathfrak{M} \cap \mathfrak{M}' = T_e S_e \mathfrak{H}$ and $\mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A} = T_e S_e \mathfrak{A} \subset \mathfrak{M} \cap \mathfrak{M}'$. If A is an operator $\in \mathbf{M}$ ($\in \mathbf{M}'$) such that $AT_e = T_e A = A$ ($AS_e = S_e A = A$), we denote the contracted operator of this A to $\mathfrak{M} \cap \mathfrak{M}'$ as $A_{(\mathfrak{M} \cap \mathfrak{M}')}$, and the total set of them as $\mathbf{M}_{(\mathfrak{M} \cap \mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M} \cap \mathfrak{M}')}$ (cf. [4; Definition 11.3.1. Lemma 11.4.1]). Then

Lemma 2.9:

$$\mathbf{M}_{(\mathfrak{M} \cap \mathfrak{M}')} = \{T_{a(\mathfrak{M} \cap \mathfrak{M}')} ;\ a \in \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A}\},$$
$$\mathbf{M}'_{(\mathfrak{M} \cap \mathfrak{M}')} = \{S_{a(\mathfrak{M} \cap \mathfrak{M}')} ;\ a \in \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A}\}.$$

Proof: We prove the former one.

Take arbitrarily an $A_{(\mathfrak{M} \cap \mathfrak{M}')} \in \mathbf{M}_{(\mathfrak{M} \cap \mathfrak{M}')}$, then this is the contraction of an $A \in \mathbf{M}$ with $AT_e = T_e A = A$. Put now $a = Ae$, then $a \in \mathfrak{A}$ and

$$T_e S_e a = T_e S_e Ae = T_e AS_e e = Ae = a,$$

and a fortiori $a \in \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A}$. Therefore also

$$T_a x = T_{Ae} x = AT_e x = Ax \qquad (x \in A),$$

thus $A = T_a$ and $A_{(\mathfrak{M} \cap \mathfrak{M}')} = T_{a(\mathfrak{M} \cap \mathfrak{M}')}$.

Conversely take arbitrarily an $a \in \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A}$, then $T_e T_a = T_{ea} = T_a = T_{ae} = T_a T_e$. Therefore we can consider the contracted operator $T_{a(\mathfrak{M} \cap \mathfrak{M}')}$. Thus we have shown that $\mathbf{M}_{(\mathfrak{M} \cap \mathfrak{M}')}$ is precisely composed of all the operators of the form $T_{a(\mathfrak{M} \cap \mathfrak{M}')}$.

Lemma 2.10: *For an arbitrary unit* e, $T_e S_e \mathfrak{A} = \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A}$ *is again a maximal Hilbert algebra in* $T_e S_e \mathfrak{H} = \mathfrak{M} \cap \mathfrak{M}'$, *when we introduce in it the multiplication and the adjoint operation as it is in* $\mathfrak{A}$. *According to the notation in* Lemma 2.4, *we can write this* $\mathfrak{A}_{\mathfrak{M} \cap \mathfrak{M}'}$ (*or* $\mathfrak{A}_{T_e S_e}$). *The algebras of left and right multiplication of this* $\mathfrak{A}_{\mathfrak{M} \cap \mathfrak{M}'}$ *is* $\mathbf{M}_{(\mathfrak{M} \cap \mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M} \cap \mathfrak{M}')}$ *and the centre of it is* $\mathbf{Z}_{(\mathfrak{M} \cap \mathfrak{M}')}$.

Proof: We must first examine that $\mathfrak{A}_{\mathfrak{M} \cap \mathfrak{M}'}$ constitutes a Hilbert algebra in $\mathfrak{M} \cap \mathfrak{M}'$. From the form $T_e S_e \mathfrak{A}$, it is seen that this is a dense linear manifold in $T_e S_e \mathfrak{H} = \mathfrak{M} \cap \mathfrak{M}'$. And other conditions as for (ii), (iii), (iv) in the definition, it suffices for us to show that the result of multiplication of two elements of $T_e S_e \mathfrak{A}$ also belongs to $T_e S_e \mathfrak{A}$ and the same holds for the adjoint operation. These follows from

$$(T_e S_e a)\ (T_e S_e b) = T_e\,((S_e a)\ (S_e T_e b)) = T_e S_e\,((S_e a)\ (T_e b)),$$
$$(T_e S_e a)^* = T_e\,(S_e a)^* = T_e S_e a^*.$$

The last condition of the definition is rather troublesome. Let for an arbitrary $a \in \mathfrak{A}$ (especially for e), $T_{T_e S_e a}\,(T_e S_e f) = 0$ be valid. We take the maximal family of units $\{e_\lambda\}_{\lambda \in \Lambda}$ which are orthogonal to each other, and contains e, then one sees easily by Lemma 2.3, $\cup_{\lambda \in \Lambda} T_{e_\lambda} = 1$. For each $e_\lambda\ (\lambda \in \Lambda)$, $T_e T_{e_\lambda} = T_{e_\lambda} T_e = T_e T_{e_\lambda} T_e = T_{ee_\lambda} = T_e$ or $= 0$. Thus if the former holds

$$T_{e_\lambda} T_e S_e f = T_e T_{e_\lambda} T_e T_e S_e f = T_{T_e S_e e_\lambda} T_e S_e f = 0,$$

and if the latter holds, clearly

$$T_{e_\lambda} T_e S_e f = 0.$$

Therefore $\| T_e S_e f \|^2 = \sum_{\lambda \in \Lambda} \| T_{e_\lambda} T_e S_e f \|^2 = 0$, and this shows $T_e S_e f = 0$.

Next consider $\mathbf{T}_{\mathfrak{M} \cap \mathfrak{M}'}$ and $\mathbf{S}_{\mathfrak{M} \cap \mathfrak{M}'}$, the total sets of operators of left and right multiplication. The operation of left multiplication of $\mathfrak{A}_{\mathfrak{M} \cap \mathfrak{M}'}$ is the operator $T_{a(\mathfrak{M} \cap \mathfrak{M}')}$ $(a \in \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A})$, and the whole of them is precisely $\mathbf{M}_{(\mathfrak{M} \cap \mathfrak{M}')}$ as is in Lemma 2.9. Therefore

$$\mathbf{T}_{\mathfrak{M}\cap\mathfrak{M}'} = \mathbf{M}_{(\mathfrak{M}\cap\mathfrak{M}')}.$$

Similarly

$$\mathbf{S}_{\mathfrak{M}\cap\mathfrak{M}'} = \mathbf{M}'_{(\mathfrak{M}\cap\mathfrak{M}')}.$$

But the theory of F. J. Murray and J. von Neumann shows (cf. their [4; Lemma 11.3.2]) that $\mathbf{M}_{(\mathfrak{M}\cap\mathfrak{M}')}$ and $\mathbf{M}'_{(\mathfrak{M}\cap\mathfrak{M}')}$ are the commutator algebras of one another, $\mathbf{T}_{\mathfrak{M}\cap\mathfrak{M}'}$ and $\mathbf{S}_{\mathfrak{M}\cap\mathfrak{M}'}$ are so. Thus by Lemma 2.7, $\mathfrak{A}_{\mathfrak{M}\cap\mathfrak{M}'}$ is maximal.

Now our proof is completed.

§3.0. Simple Hilbert algebras (with the identity element).

Let $\mathfrak{A}$ be a simple Hilbert algebra which contains the identity element e:

$$ex = xe = x \qquad \text{for all } x \in \mathfrak{A}, \text{ or equivalently} \qquad T_e = S_e = 1.$$

(see p. 9 footnote 1)

Then, as we have shown already (Lemma 2.7), every $A \in \mathbf{M}$ has the form $A = T_a$ $(a = A_e \in \mathfrak{A})$, and in the same way $A = S_a$ $(a = Ae \in \mathfrak{A})$ if $A \in \mathbf{M}'$. Moreover, for any A: $A \in \mathbf{M}$ or $A \in \mathbf{M}'$,

$$(3.0.1) \qquad (Ae)^* = A^*e.$$

because, $A \in \mathbf{M}$ implies $A^* \in \mathbf{M}$, and

$$((Ae)x, y) = (A(ex), y) = (ex, A^*(ey)) = (x, (A^*e)y),$$

for arbitrary $x, y \in \mathfrak{A}$, while we have by definition

$$((Ae)x, y) = (x, (Ae)^*y),$$

and so $(Ae)^* = A^*e$.

In this § and also in the followings the theory and notations in [4] shall be used with only suggesting references. There the space is restricted to be separable, but as we know easily, the most parts of their results are valid independently with the dimensionality of the space. (The existence and the uniqueness of the relative dimension function, the contraction to subspaces, etc.)

Lemma 3.0.1: *Let $\mathfrak{A}$ be a Hilbert algebra in $\mathfrak{H}$ (, we do not assume it to be simple, nor have an identity element). If*

$$\mathbf{T} = \{T_a \ ; \ a \in \mathfrak{A}\}, \qquad \mathbf{S} = \{S_a \ ; \ a \in \mathfrak{A}\}$$

constitute a couple of factors: $\mathbf{T}' = \mathbf{S}$, $\mathbf{S}' = \mathbf{T}$, $\mathbf{T} \cap \mathbf{T}' = \{\alpha \cdot 1\}$, *then* $\mathfrak{A}$ *is a maximal, simple Hilbert algebra which contains an identity element.*

Proof: We have already shown that $\mathfrak{A}$ is a maximal Hilbert algebra and contains the identity element e (Lemma 2.7). So we must show that it is simple. But this is clear from

$$\mathbf{Z} = \mathbf{M} \cap \mathbf{M}' = \mathbf{S}' \cap \mathbf{T}' = \mathbf{T} \cap \mathbf{T}' = \{\alpha \cdot 1\} \qquad \text{q.e.d.}$$

In what follows we assume that $\mathfrak{A}$ is simple and has an identity element e.

In this case, $\mathbf{M}$ is a factor, and there is introduced the notion of equivalence of two projection operators (cf. [4; Definition 6.1.1]). Concerning this,

Lemma 3.0.2: *Let* u_1, u_2 *be units in* $\mathfrak{A}$, *and if* T_{u_1} *and* T_{u_2} *are equivalent projection operators with respect to* $\mathbf{M}$, *then* $\|u_1\| = \|u_2\|$, *and the same for the case of* S_{u_1} *and* S_{u_2}.

Proof: $T_{u_1} \sim T_{u_2} (\cdots \mathbf{M})$ means that there exists a partially isometric operator W in $\mathbf{M}$ such that $W^*W = T_{u_1}$, $WW^* = T_{u_2}$. Therefore

$$\begin{aligned} \|u_1\| &= \|T_{u_1}e\| = \|WT_{u_1}e\| = \|T_{u_1}W^*e\| = \|W^*T_{u_2}e\| \\ &= \|T_{u_2}e\| = \|u_2\|, \end{aligned}$$

as was to be proved.

Lemma 3.0.3: *For any projection operator* P *belonging to* $\mathbf{M}(\mathbf{M}')$ *there exists a unit* u *in* $\mathfrak{A}$ *such that* $T_u(S_u) = P$, *and for such we put*

$$\mathrm{D}(P) = \|u\|^2 \qquad (\mathrm{D}'(P) = \|u\|^2).$$

Then these have the following properties. We write them only in the case of $\mathbf{M}$.

(i) $0 \leqslant \mathrm{D}(P) \leqslant \|e\|^2$, $\mathrm{D}(1) = \|e\|^2 \neq 0$,

(ii) $P_1P_2 = 0$ implies $\mathrm{D}(P_1 + P_2) = \mathrm{D}(P_1) + \mathrm{D}(P_2)$,

(iii) $P_1 \sim P_2 (\cdots \mathbf{M})$ implies $\mathrm{D}(P_1) = \mathrm{D}(P_2)$.

According to this lemma, the functionals of a projection operator $\mathrm{D}(P)$, $\mathrm{D}'(P)$ are the relative dimension functions with respect to $\mathbf{M}$, $\mathbf{M}'$ and as $\mathrm{D}(1) = \mathrm{D}'(1) < \infty$, $\mathbf{M}$, $\mathbf{M}'$ is in the finite case, and

$$\mathfrak{M}_e^{\mathbf{M}} = [Ae \ ; \ A \in \mathbf{M}] = [\mathfrak{A}] = \mathfrak{H}, \quad \mathfrak{M}_e^{\mathbf{M}'} = [A'e \ ; \ A' \in \mathbf{M}'] = [\mathfrak{A}] = \mathfrak{H},$$

thus by $D(1) = D'(1) < \infty$ we know that the constant C for the couple of factors $\mathbf{M}$, $\mathbf{M}'$ (cf. [4; Theorem X, p. 182]) is equal to 1 in their standard normalizations. Therefore this is just the case considered in [5; Chap. IV] (where only the case that $\mathfrak{H}$ is separable is treated, but as is easily seen the assumption concerning the dimensionality of the space is quite unnecessary. Thus if the space is finite dimensional we are considering the case where the factor $\mathbf{M}$ is in the case (I). And that their theory is valid in such a case is also remarked in [8; §1.6 (A)]).

Accordingly the algebra $\mathfrak{A}$ in the last line of [5; p. 240] is the maximal Hilbert algebra in $\mathfrak{H}$. And the above argument shows that the simple Hilbert algebra which contains an identity element is exhausted with such things.

Now that $\mathbf{M}$ is in the case (I) means that $\mathbf{M}$ contains minimal projection operators and replacing the statement to the Hilbert algebra $\mathfrak{A}$, that $\mathfrak{A}$ contains minimal units. This case having been already treated algebraically in detail (see [6; Theorem 5.2]), we don't think it necessary to repeat it in this paper, but we can include it in the general considerations which we are going to make hereafter. The fact which is fundamental in the argument of this case is the following lemma ([6; Theorem 4.8]):

Lemma 3.0.4: *If $\mathfrak{A}$ does not contain any unit other than e, then $\mathfrak{A} = \mathfrak{H} = \{\alpha e\}$. That is, $\mathfrak{H}$ is the one-dimensional linear space generated by e.*

Proof: As $\mathbf{M}$ is an algebra of operators, $\mathbf{M}$ is generated by all the projection operators in it: $\mathbf{M} = \mathbf{R}(\mathbf{M}^{(P)})$ (cf. [7; Satz 2, p. 399]). By Lemma 2.7, $\mathbf{M} = \mathbf{T} = \{T_a;\ a \in \mathfrak{A}\}$ as $\mathfrak{A}$ contains e as its identity element, and the above assumption means $\mathbf{M}^{(P)} = \mathbf{T}^{(P)} = \{0, T_e\} = \{0, 1\}$. Thus $\mathbf{M} = \mathbf{R}(\mathbf{M}^{(P)}) = \mathbf{R}(0, 1) = \{\alpha \cdot 1\}$. Therefore if $a \in \mathfrak{A}$ there exists a constant α such that $T_a = \alpha \cdot 1$, but $\alpha \cdot 1 = \alpha T_e = T_{\alpha e}$, and a fortiori $a = \alpha e$. As $\mathfrak{A}$ was a linear manifold, $\mathfrak{A} = \{\alpha e\}$ and so, $\mathfrak{A}$ is finite dimensional, and it is closed too: $\mathfrak{A} = [\mathfrak{A}]$. Thus we have shown $\mathfrak{H} = [\mathfrak{A}] = \mathfrak{A} = \{\alpha e\}$.

§3. Simple Hilbert algebras.

In this section we shall give the investigations of general simple Hilbert algebras and deduce the most vizual forms of them.

Let $\mathfrak{A}$ be a simple Hilbert algebra in $\mathfrak{H}$, then its algebra of left multiplication $\mathbf{M}$ and that of right multiplication $\mathbf{M}'$ constitute a couple of factors. The dimensionality functionals for them are introduced by F. J. Murray and J. von Neumann ([4]) and we denote ones of them as $\mathrm{D}_{\mathbf{M}}(\cdot)$, $\mathrm{D}_{\mathbf{M}'}(\cdot)$ resp.

Take arbitrarily a unit e in $\mathfrak{A}$, then T_e, S_e are the projection operators contained in $\mathbf{M}$, $\mathbf{M}'$ resp. Put $T_e\mathfrak{H} = \mathfrak{M}$, $S_e\mathfrak{H} = \mathfrak{M}'$, then $\mathfrak{M} \cap \mathfrak{M}' = T_e S_e \mathfrak{H}$, $\mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A} = T_e S_e \mathfrak{A}$. We have shown already that $\mathfrak{A}_{\mathfrak{M}\cap\mathfrak{M}'} = \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A}$ is again a maximal Hilbert algebra in $\mathfrak{M} \cap \mathfrak{M}'$ when we introduce in it the multiplication and the adjoint operation as it is in $\mathfrak{A}$. The algebras of left and right multiplication is $\mathbf{M}_{(\mathfrak{M}\cap\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cap\mathfrak{M}')}$ and the centre of it is $\mathbf{Z}_{(\mathfrak{M}\cap\mathfrak{M}')}$ (Lemma 2.1). But as $\mathbf{Z} = \{\alpha \cdot 1\}$ by assumption,

Lemma 3.1: *$\mathfrak{A}_{\mathfrak{M}\cap\mathfrak{M}'}$ is a simple Hilbert algebra in $\mathfrak{M} \cap \mathfrak{M}'$, which contains e as its identity element.*

Lemma 3.2: *Corresponding to an arbitrary unit e in $\mathfrak{A}$, there exist finite, positive constants α_e, β_e such that for any unit $u \leqslant e$,*

$$\mathrm{D}_{\mathbf{M}}(T_u) = \alpha_e \| u \|^2, \qquad \mathrm{D}_{\mathbf{M}}(S_u) = \beta_e \| u \|^2.$$

Proof: We shall show the existence of α_e.

If $u \leqslant e$, then $u \in \mathfrak{M} \cap \mathfrak{M}' \cap \mathfrak{A}$ and as an element of $\mathfrak{A}_{\mathfrak{M}\cap\mathfrak{M}'}$, it is also a unit. Conversely if u is a unit in $\mathfrak{A}_{\mathfrak{M}\cap\mathfrak{M}'}$, then u is when considered as an element of $\mathfrak{A}$, a unit in A such that $u \leqslant e$.

Put $\mathrm{D}_{\mathbf{M}}^{(\mathfrak{M}\cap\mathfrak{M}')}(T_{u(\mathfrak{M}\cap\mathfrak{M}')}) = \mathrm{D}_{\mathbf{M}}(T_u)$ (for $u \leqslant e$), then $\mathrm{D}_{\mathbf{M}}^{(\mathfrak{M}\cap\mathfrak{M}')}(\cdot)$ is a relative dimensionality functional for $\mathbf{M}_{(\mathfrak{M}\cap\mathfrak{M}')}$ (cf. [4; Lemma 11.4.2] and also Lemma 2.7 above). But as was considered in §3.0, $\mathrm{D}^{(\mathfrak{M}\cap\mathfrak{M}')}(T_{u(\mathfrak{M}\cap\mathfrak{M}')}) = \| u \|^2$ is also a relative dimensionality functional for $\mathbf{M}_{(\mathfrak{M}\cap\mathfrak{M}')}$ (Lemma 3.0.4).

Thus owing to the uniqueness of relative dimensionality functional for factors ([4; Lemma 8.2.3]), there exists a finite constant $\alpha_e > 0$ such that

$$\mathrm{D}_{\mathbf{M}}^{(\mathfrak{M}\cap\mathfrak{M}')}(\cdot) = \alpha_e \mathrm{D}^{(\mathfrak{M}\cap\mathfrak{M}')}(\cdot).$$

Thus if $u \leqslant e$, $\mathrm{D}_{\mathbf{M}}(T_u) = \mathrm{D}_{\mathbf{M}}^{(\mathfrak{M}\cap\mathfrak{M}')}(T_{u(\mathfrak{M}\cap\mathfrak{M}')}) = \alpha_e \mathrm{D}^{(\mathfrak{M}\cap\mathfrak{M}')}(T_{u(\mathfrak{M}\cap\mathfrak{M}')}) = \alpha_e \| u \|^2$, as was to be proved.

Concerning these α_e, β_e we have

Lemma 3.3: *For any two units e_1 and e_2, if either*

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(i) $e_1 \geqslant e_2$, *or*
(ii) $e_1 e_2 = 0$
is valid then $\alpha_{e_1} = \alpha_{e_2}$, $\beta_{e_1} = \beta_{e_2}$.

Proof: Ad (i). $D_M(T_{e_2}) = \alpha_{e_1} \| e_2 \|^2 = \alpha_{e_2} \| e_2 \|^2$. Therefore $\alpha_{e_1} = \alpha_{e_2}$. In the same way $\beta_{e_1} = \beta_{e_2}$.

Ad (ii): Put $e = e_1 + e_2$. Then e is also a unit and $e \geqslant e_1, e_2$. Thus we have $\alpha_{e_1} = \alpha_e = \alpha_{e_2}$, $\beta_{e_1} = \beta_e = \beta_{e_2}$.

We want to show that the α_e's and the β_e's are the same constants in their respective sets. First we prove the following two Lemmas simultaneously.

Lemma 3.4: *For any projection operator* P *belonging to* **M** (*or* **M'**) *we take the maximal family of units* $\{e_\lambda\}_{\lambda \in \Lambda}$ *such that*

(3.1) $$e_\lambda e_\mu = 0 \quad (\lambda \neq \mu), \qquad T_{e_\lambda} \leqslant P \quad (or \quad S_{e_\lambda} \leqslant P).$$

Then

(3.2) $$P = \bigcup_{\lambda \in \Lambda} T e_\lambda \qquad (or \quad P = \bigcup_{\lambda \in \Lambda} S_{e_\lambda}).$$

α_e*'s or* β_e*'s are equal to a constant* α_0 *or* β_0 *not depending on* λ *and the formula*

(3.3) $$D_M(P) = \alpha_0 \sum_{\lambda \in \Lambda}{}' \| e_\lambda \|^2 \;{}^{1)} (or \quad D_{M'}(P) = \beta_0 \sum_{\lambda \in \Lambda}{}' \| e_\lambda \|^2)$$

holds. (The meaning of this is that if the infinite sum standing in the right side is finite, then P is a finite projection operator with respect to **M** and its dimensionality which is finite is equal to it, while if that is infinite, then P is an infinite projection operator with respect to **M**, and, its dimensionality being infinite, both sides are equal in this sense.)

Lemma 3.5: *A projection operator* $P \in$ **M** (*or* $\in$ **M'**) *is finite with respect to* **M** *or* **M'**, *if and only if* P *has the form* T_e (*or* S_e) *with a suitable unit* $e \in \mathfrak{A}$.

Proof: We only consider the case that $P \in$ **M**, and $P \neq 0$.

As was stated in Lemma 2.3, we have (3.2) for any maximal family $\{e_\lambda\}_{\lambda \in \Lambda}$ satisfying (3.1).

---

1) Infinite sum of positive numbers $\alpha_\lambda (\lambda \in \Lambda)$, which is denoted as $\sum'_{\lambda \in \Lambda} \alpha_\lambda$, is, by definition, equal to the least upper bound of all finite sums of α_λ: $\sum'_{\lambda \in \Lambda} \alpha_\lambda = \sup_{\lambda_1, \lambda_2, \cdots, \lambda_\kappa \in \Lambda} \sum_{\nu=1}^{\kappa} \alpha_{\lambda_\nu}$. If this is finite, the members α_λ which are not 0 are at most countable.

The second condition of (3.1) applies to give that $\alpha_{e_\lambda} = \alpha_{e_\mu}$ $(\lambda, \mu \in \Lambda)$ according to Lemma 3.3. Thus α_{e_λ} is a fixed constant not depending on $\lambda(\in \Lambda)$ and writing it as α_0 we have $\mathrm{D}_{\mathbf{M}}(T_{e_\lambda}) = \alpha_0 \| e_\lambda \|^2$ $(\lambda \in \Lambda)$. Take arbitrarily a finite subset $\{\lambda_1, \lambda_2, \cdots, \lambda_\kappa\}$ of Λ, then

$$T_{e\lambda_\nu} T_{e\lambda_\mu} = 0 \quad (\nu \neq \mu), \qquad \sum_{\nu=1}^{\kappa} T_{e\lambda_\nu} = \bigcup_{\nu=1}^{\kappa} T_{e\lambda_\nu} \leqslant \bigcup_{\lambda\in\Lambda} T_{e\lambda} = P.$$

Consequently

$$\mathrm{D}_{\mathbf{M}}(P) \geqslant \mathrm{D}_{\mathbf{M}}\Big(\sum_{\nu=1}^{\kappa} T_{e\lambda_\nu}\Big) = \sum_{\nu=1}^{\kappa} \mathrm{D}_{\mathbf{M}}(T_{e\lambda_\nu}) = \sum_{\nu=1}^{\kappa} \alpha_0 \| e_{\lambda_\nu} \|^2 = \alpha_0 \sum_{\nu=1}^{\kappa} \| e_{\lambda_\nu} \|^2$$

and from the definition of the infinite sum $\sum_{\lambda\in\Lambda} \| e_\lambda \|^2$ we have

$$\mathrm{D}_{\mathbf{M}}(P) \geqslant \alpha_0 \sum_{\lambda\in\Lambda} \| e_\lambda \|^2. \tag{3.4}$$

Assume now that $\sum_{\lambda\in\Lambda} \| e_\lambda \|^2 < +\infty$. Then as e_λ was to be $\neq 0$, the set Λ is at most countable (see p. 16 footnote 1), thus we can write them as $\{e_1, e_2, \cdots\}$ instead. By the assumption that $e_1, e_2, \cdots$ are mutually orthogonal,

$$\Big\| \sum_{\nu=1}^{\kappa} e_\nu \Big\|^2 = \sum_{\nu=1}^{\kappa} \| e_\nu \|^2,$$

and moreover $\sum_{\nu=1}^{\infty} \| e_\nu \|^2$ was finite, $\{\sum_{\nu=1}^{\mu} e_\nu;\ \mu = 1, 2 \cdots\}$ constitutes a fundamental sequence whose elements are exclusively units of $\mathfrak{A}$, and therefore converges to a certain $e \in \mathfrak{H}$. But $T_{\sum_{\nu=1}^{\mu} e_\nu}$ $(\mu = 1, 2, \cdots)$ are all projection operators, their bounds are equal to 1. That is: The sequence of elements of $\mathfrak{A}$ $\{\sum_{\nu=1}^{\mu} e_\nu\}_{\mu=1,2,\ldots}$ converges to e and the bounds of $T_{\sum_{\nu=1}^{\mu} e_\nu}$ are bounded. Consequently e belongs to $\mathfrak{A}$ (cf. [6; §2] and [12; Corollary to Theorem 2]), and

$$T_e = \lim_{\mu\to\infty} T_{\sum_{\nu=1}^{\mu} e_\nu}$$

is a projection operator, moreover

$$T_e = \lim_{\mu\to\infty} T_{\sum_{\nu=1}^{\mu} e_\nu} = \lim_{\mu\to\infty} \sum_{\nu=1}^{\mu} T_{e_\nu} = \lim_{\mu\to\infty} \bigcup_{\nu=1}^{\mu} T_{e_\nu} = \bigcup_{\nu=1}^{\infty} T_{e_\nu} = \bigcup_{\lambda\in\Lambda} T_{e_\lambda} = P.$$

As $e \geqslant e_\nu$ $(\nu = 1, 2, \cdots)$, $\alpha_e = \alpha_{e_\nu} = \alpha_0$ $(\nu = 1, 2, \cdots)$, and

$$\mathrm{D}_{\mathbf{M}}(P) = \mathrm{D}_{\mathbf{M}}(T_e) = \alpha_0 \| e \|^2 = \alpha_0 \sum_{\lambda\in\Lambda} \| e_\lambda \|^2 < +\infty,$$

therefore P is a projection operator finite with respect to $\mathbf{M}$ ([4; Definition 8.2.1]).

Thus we know that

(i) if the infinite sum $\sum_{\lambda \in \Lambda} \| e_\lambda \|^2$ is finite, P is a projection operator finite with respect to **M** and then (3.3) is valid. Moreover there exists an element in $\mathfrak{A}$ symbolically to be denoted as $\sum_{\lambda \in \Lambda} e_\lambda$ which is a unit and $T_{\sum_{\lambda \in \Lambda} e_\lambda} = P$.

(ii) if the infinite sum $\sum_{\lambda \in \Lambda} \| e_\lambda \|^2$ diverges to the infinity, then $\mathrm{D}_{\mathbf{M}}(P) = \infty$ by (3.4), and P is a projection operator infinite with respect to **M**. In that case also the formula (3.4) is true.

Conversely Lemma 3.2 states that for any unit e, the projection operator T_e belonging to **M** is finite.

Thus the Lemmas 3.4 and 3.5 are thoroughly proven.

Lemma 3.6: *For any two units e_1 and $e_2 \in \mathfrak{A}$ there exists a unit $e \in \mathfrak{A}$ such that $e_1 \leqslant e$, $e_2 \leqslant e$.*

Proof: We have said that T_{e_1} and T_{e_2} are both projection operators finite with respect to **M**, and $T_{e_1} \cup T_{e_2}$ is also finite ([4; Lemma 7.3.3], see also [3; Lemma 1.6]). Therefore there exists a unit e so that $T_e = T_{e_1} \cup T_{e_2}$ by Lemma 3.5, and such is by definition $e \geqslant e_1$, $e \geqslant e_2$ as $T_e \geqslant T_{e_1}$, $T_e \geqslant T_{e_2}$.

Lemma 3.7: *The constants α_e β_e are the same for any units $e \in \mathfrak{A}$.*

Proof: Take arbitrarily two units e_1, $e_2 \in \mathfrak{A}$, and then a unit e_0 such that $e_0 \geqslant e_1$, e_2 which exists due to Lemma 3.6. Then Lemma 3.3 assures $\alpha_{e_1} = \alpha_{e_2} (= \alpha_{e_0})$, $\beta_{e_1} = \beta_{e_2}$. Therefore α_e's and β_e's are the constants in their sets.

We write these constants as α and β, then putting

$$\mathrm{D}(\cdot) = \frac{1}{\alpha} \mathrm{D}_{\mathbf{M}}(\cdot), \qquad \mathrm{D}'(\cdot) = \frac{1}{\beta} \mathrm{D}'_{\mathbf{M}}(\cdot),$$

$\mathrm{D}(\cdot)$ and $\mathrm{D}'(\cdot)$ are another relative dimensionality functionals with respect to **M** and **M**$'$ resp. and they satisfy

$$\mathrm{D}(T_e) = \| e \|^2, \qquad \mathrm{D}'(S_e) = \| e \|^2.$$

Consider now the constant C defined in [4; Theorem X, p. 182]. As this constant equals to 1 owing to [4; Lemma 11.4.2] and Lemma 3.0.4 above when we contract **M**, **M**$'$ to such a subspace $\mathfrak{M} \cap \mathfrak{M}' = T_e S_e \mathfrak{H}$ as $\mathrm{D}(T_e) = \mathrm{D}'(S_e)$, we must have $C = 1$ by Lemma 11.4.3 in [4].

Thus we have shown the following

Theorem 3.1 : *Let* $\mathfrak{A}$ *be a simple Hilbert algebra in a Hilbert space* $\mathfrak{H}$. *Then its algebras of left and right multiplication* **M**, **M′** *resp. form a couple of factors. Concerning these,*

1) *A projection operator* P *in* **M** (*or* **M′**) *is finite with respect to* **M** (*or* **M′**) *if and only if there exists a unit* e *in* $\mathfrak{A}$ *such that* $P = T_e$ (*or* $= S_e$).

2) *Put*

$$\mathrm{D}(T_e) = \| e \|^2, \qquad \mathrm{D}'(S_e) = \| e \|^2$$

for any arbitrary unit e *in* $\mathfrak{A}$, *and for any projection operators* P, P' *infinite with respect to* **M**, **M′** *resp.*

$$\mathrm{D}(P) = \infty, \qquad \mathrm{D}'(P) = \infty.$$

Then $\mathrm{D}(\cdot)$ *and* $\mathrm{D}'(\cdot)$ *is one of the relative dimensionality functionals with respect to* **M** *and* **M′** *respectively.*

3) *The constant* C *is equal to* 1 *when we take the relative dimensionality functionals with respect to* **M**, **M′** *as above. And clearly*

$$\mathrm{D}(T_e) = \mathrm{D}'(S_e).$$

Now factors are classified into three types (I), (II), (III) according to the nature of their respective relative dimensionality functionals. But in our case **M** contains necessarily a finite projection operator, i.e. T_e for an arbitrary unit $e \in \mathfrak{A}$ and so the purely infinite case (III) cannot occur. Therefore only types (I), (II) are the case.

The algebra to which the factor of type (I) corresponds has a rather simple structure, but the case where the factor of type (II) appears is not so simple. We shall give here a method of reducing the problem to the simpler case where there exists the identity element. Namely we show that they are constructed as a matrix algebra with elements in some simple Hilbert algebra containing an identity element, after the prototype of the case where arises the factors of type (I). The detailed proof is rather longsome, and so we shall be contented to follow the thread of the argument.

Let $\mathfrak{H}_0$ be a Hilbert space, in which a maximal Hilbert algebra $\mathfrak{A}_0$ be given. Λ be a set of indices, and $\bar{\mathfrak{H}}$ be the set whose elements are consisted of all the elements of $\mathfrak{H}$ doubly indexed by the elements of Λ namely the matrices $<f_{\lambda,\mu}>_{\lambda,\mu\in\Lambda}$ with $\sum_{\lambda,\mu\in\Lambda}\|f_{\lambda,\mu}\|^2<+\infty$. Introduce now in this $\bar{\mathfrak{H}}$, the

linear operation as $$\alpha<f_{\lambda,\mu}>_{\lambda,\mu\in\Lambda}+\beta<g_{\lambda,\mu}>_{\lambda,\mu\in\Lambda}=<\alpha f_{\lambda,\mu}+\beta g_{\lambda,\mu}>_{\lambda,\mu\in\Lambda},$$

and the

inner product as $$(<f_{\lambda,\mu}>_{\lambda,\mu\in\Lambda},<g_{\lambda,\mu}>_{\lambda,\mu\in\Lambda})=\sum_{\lambda,\mu\in\Lambda}(f_{\lambda,\mu},g_{\lambda,\mu}).$$

Now $\bar{\mathfrak{A}}$ be the set of all elements $\bar{a}=<a_{\lambda,\mu}>_{\lambda,\mu\in\Lambda}$ of $\bar{\mathfrak{H}}$ where $a_{\lambda,\mu}\in\mathfrak{A}(\lambda,\mu\in\Lambda)$ and which satisfies for any choice of finite λ's: $\lambda_1,\lambda_2,\cdots,\lambda_\kappa\in\Lambda$, the inequality

$$\sum_{\nu=1}^{\kappa}\|\sum_{\sigma=1}^{\kappa}a_{\lambda_\nu\lambda_\sigma}x_\sigma\|^2\leqslant\gamma^2\sum_{\nu=1}^{\kappa}\|x_\nu\|^2 \tag{3.5}$$

for any $x_1,x_2,\cdots,x_\kappa\in\mathfrak{A}_0$ and a fixed $\gamma>0$ (: a constant depending only on $\bar{a}$). Introduce now in this $\bar{\mathfrak{A}}$ the

multiplication as $$<a_{\lambda,\mu}>_{\lambda,\mu\in\Lambda}<b_{\lambda,\mu}>_{\lambda,\mu\in\Lambda}=<\sum_{\rho\in\Lambda}a_{\lambda,\rho}b_{\rho,\mu}>_{\lambda,\mu\in\Lambda},$$

and the

adjoint operation as $$<a_{\lambda,\mu}>^*_{\lambda,\mu\in\Lambda}=<a^*_{\mu,\lambda}>_{\lambda,\mu\in\Lambda}.$$

These definitions are always possible and we are able to show the

Theorem 3.2: *By the above definitions $\bar{\mathfrak{H}}$ is a Hilbert space, in which $\bar{\mathfrak{A}}$ constitutes a maximal Hilbert algebra. If the original $\mathfrak{A}_0$ is simple then $\bar{\mathfrak{A}}$ is too.*

The converse of this statement holds. Namely,

Theorem 3.3: *Any simple Hilbert algebra can be brought in the form defined above, where $\mathfrak{A}_0$ can be made into a simple Hilbert algebra with an identity element. Thus we can say that any simple Hilbert algebra is the total matrix algebra of some dimensions over a suitably chosen simple Hilblert algebra with an identity element.*

Theorem 3.4: *If $\mathfrak{A}$ is a simple Hilbert algebra whose algebras of operators of left and right multiplication are factors of type* (I), *it is the total matrix algebra over the field of complex numbers of some dimension.*

This follows at once from Theorem 3.3 and Lemma 3.0.4, and is the case treated by W. Ambrose [1] and H. Nakano [6].

§4. Purely non-simple Hilbert algebras.

In this section, we consider the purely non-simple Hilbert algebra $\mathfrak{A}$ in the Hilbert space $\mathfrak{H}$ and its decomposition into direct sum (in some sense—integral sum) of simple ones. The method used here, rather near to that of F. Riesz [10] than to that of J. von Neumann [9], essentially depends on the theory of integration and differentiation, and the countability conditions are necessary. Thus, we shall assume in this section that the space $\mathfrak{H}$ is separable, and, the finite dimensional case being excluded[1], is of infinite dimension.

Lemma 4.1: *There exists a countable set consisting of units in $\mathfrak{A}$ which satisfies*

(4.1) $$e_1 \leqslant e_2 \leqslant \cdots,$$

(4.2) $$\cup T_{e_\nu} = \cup S_{e_\nu} = 1.$$

Proof: Take a maximal family of mutually orthogonal units $\{u_\lambda\}_{\lambda \in \Lambda}$ then by Lemma 2.3 we have

$$\bigcup_{\lambda \in \Lambda} T_{u_\lambda} = \bigcup_{\lambda \in \Lambda} S_{u_\lambda} = 1.$$

But as such an orthogonal set must be countable, we can enumerate them as $u_1, u_2, \cdots$. Put now

$$e_1 = u_1, \quad e_2 = e_1 + u_2, \ \cdots, \quad e_\nu = e_{\nu-1} + u_\nu, \ \cdots.$$

Then this meets the desired condition.

Lemma 4.2: *There exists a dense countable set $\mathfrak{A}_{(0)}$ in $\mathfrak{H}$ which satisfies the following conditions:*

(i) *$\mathfrak{A}_{(0)}$ contains the set $\{e_\nu\}_{\nu=1,2,\ldots}$ mentioned in Lemma 4.1.*

(ii) *$x, y \in \mathfrak{A}_{(0)}$ imply $r_1 x + r_2 y$, xy, $x^* \in \mathfrak{A}_{(0)}$, where r_1, r_2 denote arbitrary complex numbers whose real and imaginary parts are rational.*

---

1) Let e be an arbitrary unit $\in \mathfrak{A}$. Then, for any natural number $n = 1, 2, \cdots$, we can take in $\mathfrak{A}$ a maximal orthogonal set of units e_λ such that $e_\lambda \leqslant e$, $\| e_\lambda \| \leqslant \frac{1}{n} \| e \|$. As, for such a set, $\| e \|^2 = \sum_{\lambda \in \Lambda} \| e_\lambda \|^2$ holds, the cardinal numbers of Λ cannot be smaller than n^2. Since n was arbitrary, we must have that the dimensionality of the space must be infinite.

Form now, for this $\mathfrak{A}_{(0)}$ its enveloping linear manifold. Let it be $\mathfrak{A}^{(0)}$. Then

$$\mathfrak{A}_{(0)} \subset \mathfrak{A}^{(0)} \subset \mathfrak{A},$$

and $\mathfrak{A}^{(0)}$ is again dense in $\mathfrak{H}$.

Lemma 4.3: *The centre* Z *of* $\mathfrak{A}$ *comprises a family of projection operators* $\{E(t);\ 0 \leqslant t \leqslant 1\}$ *which has the following properties*

(i) $\{E(t);\ 0 \leqslant t \leqslant 1\}$ *spans* Z, i.e. *the least algebra of operators containing this family is precisely* Z,

(ii) $0 \leqslant t_1 < t_2 \leqslant 1$ *implies* $E(t_1) \lneqq E(t_2)$,

(iii) $\lim_{t \to t_0,\ 0 \leqslant t \leqslant 1} E(t) = E(t_0)$,

(iv) $E(0) = 0, \quad E(1) = 1$,

(v) *for an arbitrary* $f \in \mathfrak{H}$ *the continuous function* $\| E(t)f \|^2$ *of* t *is absolutely continuous with respect to the ordinary Lebesgue measure on* [0, 1].

Proof: There is a particular element f_0 in $\mathfrak{H}$ which has the property that

(4.3) For a projection operator $E \in Z$, $Ef = 0$ can occour if and only if $E = 0$.

This is easily seen by taking notice to that, for any $f \in \mathfrak{H}$, there exists an $E \in Z$, such that $F \leqslant E$, $F \in Z$, and $Ff = 0$ imply $F = 0$, while $EF = 0$, $F \in Z$ imply $Ff = 0$, and by noting that the maximal orthogonal family of such paires (E, f) can contain at most countable many elements.

An abelian algebra of operators in a separable Hilbert space is always generated by a resolution of unity $E(t)$ $0 \leqslant t \leqslant 1$ for which we may assume that

$$t = \| E(t) f_0 \|^2 \qquad (0 \leqslant t \leqslant 1). \tag{4.4}$$

Then, thus parametrized $E(t)$ has the desired property.

Now, under these preparations, we shall consider the decomposition of $\mathfrak{H}$, $\mathfrak{A}$ with respect to the centre Z of $\mathfrak{A}$.

I. The family of projection operators $E(t)$ $(0 \leqslant t \leqslant 1)$ being taken as in Lemma 4.3, the function $\| E(t)f \|^2$ $(f \in \mathfrak{H})$ of t: $0 \leqslant t \leqslant 1$ has the derivatives at almost everywhere in [0, 1]. We shall denote the set of t which must be excluded for f (: namely, the set of t_0

such that $\frac{d}{dt}\| E(t) f \|^2$ does not exist for $t = t_0$, or exists and $= \pm\infty$) as $N_1(f)$. Then meas. $N_1(f) = 0$ and $t_0 \in N_1(f)'$ [1] implies the existence and finiteness of $\left(\frac{d}{dt}\| E(t) f \|^2\right)_{t=t_0}$ which we shall denote as

$$\frac{d}{dt}\| E(t_0) f \|^2$$

symbolically.

The set

$$N_1 = \bigcup_{a \in \mathfrak{A}_{(0)}} N_1(a) \tag{4.5}$$

($\mathfrak{A}_{(0)}$ denotes the countable set described in Lemma 4.2), is also of Lebesgue measure 0, and, for any $t_0 \in N_1'$, $a \in \mathfrak{A}_{(0)}$, $\frac{d}{dt}\| E(t_0) a \|^2$ exists and is finite.

II. $\frac{d}{dt}(E(t_0)a, E(t_0)b)$ exists for any $a, b \in \mathfrak{A}^{(0)}$ and $t \in N_1'$.

We shall prove this step by step.

(i) $\frac{d}{dt}(E(t_0)a, E(t_0)b)$ exists for any $a, b \in \mathfrak{A}_{(0)}$ because of the formula

$$\begin{aligned}&(E(t)a, E(t)b)\\ &= \frac{1}{2}\Big\{\| E(t)(a+b) \|^2 - \| E(t)a \|^2 - \| E(t)b \|^2 + \| E(t)(a+ib) \|^2\\ &\quad - \| E(t)a \|^2 - \| E(t)b \|^2\Big\},\end{aligned}$$

where the elements appearing on the right hand side, i.e. $a + b$, $a + ib$ belong to $A_{(0)}$ and so have derivatives at $t = t_0 \in N_1'$.

(ii) For an arbitrary $f \in \mathfrak{H}$, the elements g of $\mathfrak{H}$ for which $\frac{d}{dt}(E(t_0)f, E(t_0)g)$ exists for every $t_0 \in N_1'$ consist a linear manifold in $\mathfrak{H}$.

(iii) Thus, fixing an element a of $\mathfrak{A}_{(0)}$, the elements f of $\mathfrak{H}$ for which $\frac{d}{dt}(E(t_0)a, E(t_0)f)$ exists for any $t_0 \in N_1'$ form a linear manifold containing $\mathfrak{A}_{(0)}$ by (i), and so $\mathfrak{A}^{(0)}$ too.

(iv) For any $a \in \mathfrak{A}^{(0)}$, the elements f of $\mathfrak{H}$ for which

1) In what follows, for any set contained in the interval [0, 1], its accented notation means the complementation in [0, 1].

$\frac{d}{dt}(E(t_0)a,\ E(t_0)f)$ exists for every $t_0 \in N_1'$ form a linear manifold which contains $\mathfrak{A}_{(0)}$ by (iii) and so $\mathfrak{A}^{(0)}$ too.

Thus the desired proposition is proved.

Corollary: For any $a \in \mathfrak{A}^{(0)}$,

$$\frac{d}{dt}\ \| E(t_0)a \|^2 \qquad (t_0 \in N_1')$$

exists.

We shall now proceed to define the elements of decomposition.

III. Fixing a $t_0 \in N_1'$ we consider an ideal element $a'(t_0)$ corresponding to each $a \in \mathfrak{A}^{(0)}$, and set

$$(a'(t_0),\ b'(t_0))' = \frac{d}{dt}(E(t_0)a,\ (E(t_0)b),$$

$$\| a'(t_0) \|'^2 = \frac{d}{dt}\ \| E(t_0)a \|^2.$$

Clearly we have

$$((\alpha a + \beta b)'(t_0),\ c'(t_0))' = \alpha(a'(t_0),\ c'(t_0))' + \beta(b'(t_0),\ c'(t_0))' \tag{4.6}$$

$$(a'(t_0),\ b'(t_0))' = \overline{(b'(t_0),\ a'(t_0))'}, \tag{4.7}$$

$$(a'(t_0),\ a'(t_0))' = \| a'(t_0) \|'^2 \geqslant 0, \tag{4.8}$$

$$\| (\alpha a)'(t_0) \|' = | \alpha |\ \| a'(t_0) \|', \tag{4.9}$$

and, moreover, the Schwarz' inequality and the triangular relation hold:

$$| (a'(t_0),\ b'(t_0))' | \leqslant \| a'(t_0) \|'\ \| b'(t_0) \|', \tag{4.10}$$

$$\| (a + b)'(t_0) \|' \leqslant \| a'(t_0) \|' + \| b'(t_0) \|'. \tag{4.11}$$

These latter formulas can be deduced easily from formulas (4.6) ⋯ (4.9) by the method usually adopted.

IV. We shall classify these $a'(t_0)$ $(a \in \mathfrak{A}^{(0)})$, by considering two elements $a_1'(t_0)$, $a_2'(t_0)$ $(a_1,\ a_2 \in \mathfrak{A}^{(0)})$ to be equivalent if and only if

$$\| (a_1 - a_2)'(t_0) \|' = 0$$

holds. The class which contains $a'(t_0)$ under the classification by this equivalence relation defined above shall be denoted as $a(t_0)$, and the totality of these elements as $\mathfrak{A}^{(0)}(t_0)$.

The above-defined equivalence relation has the following properties:

If $a_1'(t_0) \sim a_2'(t_0)$, $b_1'(t_0) \sim b_2'(t_0)$, then

$$(a_1 + b_1)'(t_0) = (a_2 + b_2)'(t_0)$$

$$(\alpha a_1)'(t_0) = (\alpha a_2)'(t_0) \quad (\alpha : \text{ any complex number})$$

$$(a_1'(t_0),\ b_1'(t_0))' = (a_2'(t_0),\ b_2'(t_0))'.$$

And so we can make the following difinitions.

For any $a(t_0)$, $b(t_0) \in \mathfrak{A}^{(0)}(t_0)$ and complex number α, define

(+) $\quad a(t_0) + b(t_0) =$ the class which contains $(a+b)'(t_0)$ for $a'(t_0) \in a(t_0)$, $b'(t_0) \in b(t_0)$,

(·) $\quad \alpha a(t_0) =$ the class which contains $(\alpha a)'(t_0)$ for $a'(t_0) \in a(t_0)$,

(,) $\quad (a(t_0),\ b(t_0)) = (a'(t_0),\ b'(t_0))'$ for $a'(t_0) \in a(t_0)$, $b'(t_0) \in b(t_0)$.

Then we can establish.

Lemma 4.4: $\mathfrak{A}^{(0)}(t_0)$ *is a linear space with inner product under the operations defined in* (+), (·), (,).

The completion of $\mathfrak{A}^{(0)}(t_0)$ is a (separable) Hilbert space, which shall be denoted as $\mathfrak{H}(t_0)$.

Now we shall define $\mathfrak{A}^{(0)}(t_0)$ as a Hilbert algebra in $\mathfrak{H}(t_0)$.

V. $\mathfrak{A}^{(0)}$ being a subalgebra of $\mathfrak{A}$, $a, b \in \mathfrak{A}^{(0)}$ implies $ab \in \mathfrak{A}^{(0)}$ and so $(ab)'(t_0)$ is defined. The fact which we show below that

(4.12) $\quad a_1'(t_0) \sim a_2'(t_0)$, $b_1'(t_0) \sim b_2'(t_0)$ implies $(a_1 b_1)'(t_0) \sim (a_2 b_2)'(t_0)$

allows us to make the following definition.

(×) $\quad a(t_0)\, b(t_0) =$ the class which contains $(ab)'(t_0)$ for $a'(t_0) \in a(t_0)$, $b'(t_0) \in b(t_0)$.

Proof of (4.12).

$$\| (a_1 b_1 - a_2 b_2)'(t_0) \|' = \| (\{a_1 - a_2\} b_1 + a_2 \{b_1 - b_2\})'(t_0) \|'$$
$$\leqslant \| (\{a_1 - a_2\} b_1)'(t_0) \|' + \| (a_2 \{b_1 - b_2\})'(t_0) \|'$$

and so, for the proof of the formula, the proof of

$$a'(t_0) \sim 0'(t_0) \quad \text{or} \quad b'(t_0) \sim 0'(t_0) \qquad \text{implies} \qquad \| (ab)'(t_0) \|' = 0$$

suffices. But this is clear from

$$\| E(t)(ab) \| \leqslant ||| T_a ||| \ \| E(t)b \| \quad \text{or} \quad ||| S_a ||| \ \| E(t)a \|.$$

VI. As is shown below,

(4.13) $$a'_1(t_0) \sim a'_2(t_0) \quad \text{implies} \quad (a_1^*)'(t_0) \sim (a_2^*)'(t_0)$$

is true, we can make the following definition:

(*) $$a(t_0)^* = \text{the class which contains } (a^*)'(t_0) \quad \text{for } a'(t_0) \in a(t_0).$$

Then

(4.14)
$$(a(t_0)b(t_0), c(t_0)) = (b(t_0), a(t_0)^* c(t_0)),$$
$$(b(t_0)a(t_0), c(t_0)) = (b(t_0), c(t_0)a(t_0)^*).$$

Proof of (4.13) *and* (4.14). We note first that for any $a \in \mathfrak{A}$ and $A \in \mathbf{Z}$, $(Aa)^* = A^*a^*$ which follows from

$$T_{(Aa)^*} = T_{Aa}^* = (AT_a)^* = (T_aA)^* = A^*T_a^* = A^*T_{a^*} = T_{A^*a^*}$$

Therefore, from

$$\| E(t)(a_1^* - a_2^*) \|^2 = \| (E(t)(a_1 - a_2))^* \|^2 = \| E(t)(a_1 - a_2) \|^2$$

it is clear that, if $a'_1(t_0) \sim a'_2(t_0)$, then $a_1^{*\prime}(t_0) \sim a_2^{*\prime}(t_0)$. Also (4.14) follows from

$$(E(t)(ab), c) = (a(E(t)b), c) = (E(t)b, a^*(E(t)c)) = (E(t)b, E(t)(a^*c)),$$
$$(E(t)(ba), c) = (E(t)b, E(t)(ca^*)).$$

VII. Define now the operator $T_{a(t_0)}^{[t_0]}$ corresponding to an $a(t_0) \in \mathfrak{A}^{(0)}(t_0)$ as

$$T_{a(t_0)}^{[t_0]}: \quad T_{a(t_0)}^{[t_0]} x(t_0) = a(t_0)x(t_0) \qquad x(t_0) \in \mathfrak{A}^{(0)}(t_0).$$

Then

$$\| T_{a(t_0)}^{[t_0]} x(t_0) \| \leqslant ||| T_a ||| \ \| x(t_0) \|$$

for any element $a\ (\in \mathfrak{A}^{(0)})$ to which $a'(t_0) \in a(t_0)$ is corresponded.

Proof: Let $a'(t_0) \in a(t_0)$, $x'(t_0) \in x(t_0)$, then

$$\begin{aligned} \| T_{a(t_0)}^{[t_0]} x(t_0) \|^2 &= \| a(t_0)\ x(t_0) \|^2 = \| (ax)'(t_0) \|'^2 \\ &= \frac{d}{dt} \| E(t_0)(ax) \|^2 = \frac{d}{dt} \| a(E(t_0)x) \|^2 \\ &\leqslant ||| T_a |||^2 \frac{d}{dt} \| E(t_0)x \|^2 = ||| T_a |||^2 \ \| x(t_0) \|^2 \end{aligned}$$

We can now state the following

Lemma 4.6 : $\mathfrak{A}^{(0)}(t_0)$ *defined in the Hilbert space* $\mathfrak{H}(t)$ *for* $t \in N_1'$ *is a Hilbert algebra with respect to the operations defined in* $(+)$, $(\cdot)$, $(\prime)$, $(\times)$, $(*)$, *if we disregard a set of measure* 0 N_2 *further.*

Proof: By the definition of $\mathfrak{H}(t)$, $\mathfrak{A}^{(0)}(t)$ is a dense linear manifold in $\mathfrak{H}(t)$, Moreover, the multiplication and the adjoint operation defined in $(\times)$, $(*)$ satisfies the axioms for the Hilbert algebra except the last one:

(4.15) $\quad T^{(t)}_{x(t)} f(t) = 0 \quad$ (for any $x(t) \in \mathfrak{A}^{(0)}(t)$) implies $f(t) = 0$.

We shall, in what follows, prove this statement.

$\mathfrak{A}^{(0)}$ contains the sequence of units $\{e_\nu\}_{\nu=1,2,\dots}$ which satisfies the conditions (4.1) and (4.2). Therefore, for any $f \in \mathfrak{H}$,

$$\| T_{e_1} f \|^2 \leqslant \| T_{e_2} f \|^2 \leqslant \cdots, \qquad \lim_{\nu \to \infty} \| T_{e_\nu} f \|^2 = \| f \|^2.$$

Define now a function of an interval $\sigma(t_1, t_2; f)$ for $0 \leqslant t_1 < t_2 \leqslant 1$ as

$$\sigma(t_1, t_2; f) = \| (E(t_2) - E(t_1)) f \|^2 = \| E(t_2) f \|^2 - \| E(t_1) f \|^2.$$

This is clearly an additive function of bounded variation of an interval. By (4.1) $\sigma(t_1, t_2; T_{e_\nu} f)$ $(\nu = 1, 2, \cdots)$ form a monotone increasing sequence, thus a theorem concerning the derivates of such a sequence (e.g. [11; p. 116, Theorem 5.7]) teaches us that there exists a set $N_2(f)$ of measure 0, so that

$$\lim_{\nu \to \infty} \frac{d}{dt} \| E(t_0) T_{e_\nu} f \|^2 = \frac{d}{dt} \| E(t_0) f \|^2 \qquad (t_0 \in N_2(f)').$$

Put now $N_2 = \cup_{a \in \mathfrak{A}^{(0)}} N_2(a)$, then N_2 also is a set of measure 0, and $t_0 \in N_2'$ implies

(4.16) $$\lim_{\nu \to \infty} \frac{d}{dt} \| E(t_0) T_{e_\nu} a \|^2 = \frac{d}{dt} \| E(t_0) a \|^2 \qquad (\text{for each } a \in \mathfrak{A}^{(0)}).$$

But here is included the case that both sides are equal to $+\infty$ and if we consider (4.16) only for $t_0 \in (N_1 + N_2)'$ this case is excluded. When we limit ourselves to this case, the same method used in article II applies, and the validity of

$$\lim_{\nu \to \infty} \frac{d}{dt} (E(t_0) T_{e_\nu} a, E(t_0) T_{e_\nu} b) = \frac{d}{dt} (E(t_0) a, E(t_0) b)$$

for any $a, b \in \mathfrak{A}^{(0)}$ follows. This is rewritten as

$$\lim_{\nu\to\infty} \frac{d}{dt}(E(t_0)(e_\nu a),\ E(t_0)(e_\nu b)) = \frac{d}{dt}(E(t_0)a,\ E(t_0)b)$$

or

$$\lim_{\nu\to\infty} (e_\nu(t)\,a(t),\ e_\nu(t)\,b(t)) = (a(t),\ b(t)) \qquad (t \in (N_1 \dotplus N_2)')$$

Put here $a = b$, then

$$\lim_{\nu\to\infty} \| e_\nu(t)\,a(t)\|^2 = \| a(t)\|^2 \qquad (t \in (N_1 \dotplus N_2)'). \tag{4.17}$$

As the definitions $(\times)$, $(*)$ shows, $e_\nu(t)$'s are again units in $\mathfrak{H}(t)$ $(t \in N_1')$ and moreover

$$e_1(t) \leqslant e_2(t) \leqslant \cdots \qquad t \in N_1'.$$

Thus, for the sequence of projection operators in $\mathfrak{H}(t)$: $T^{[t]}_{e_\nu(t)}$ $(\nu = 1, 2, \cdots)$ the existence of $\lim_{\nu\to\infty} T^{[t]}_{e_\nu(t)}$ is assured. Put this projection operator as $P^{[t]}$, then, if $t \in (N_1 \dotplus N_2)'$, (4.17) implies $\| P^{[t]} a(t)\|^2 = \| a(t)\|^2$ for any $a(t) \in \mathfrak{A}^{(0)}(t)$, and so, $\mathfrak{A}^{(0)}(t)$ being dense in $\mathfrak{H}(t)$, for any $a(t) \in \mathfrak{H}(t)$. This means that $P^{[t]}$ must be the identity operator in $\mathfrak{H}(t)$ for $t \in (N_1 \dotplus N_2)'$.

Now make the assumption in (4.15). Then as $e_\nu(t) \in \mathfrak{A}^{(0)}(t)$ $(\nu = 1, 2, \cdots)$, $T^{[t]}_{e_\nu(t)} f(t) = 0$. Therefore $\lim_{\nu\to\infty} T^{[t]}_{e_\nu(t)} f(t) = f(t)$ is also equal to 0, which was to be proved in (4.15).

The maximal extension of $\mathfrak{A}^{(0)}(t)$ in $\mathfrak{H}(t)$ will be denoted as $\mathfrak{A}(t)$.

Finally we consider the relations between $\mathfrak{H}$ and $\mathfrak{H}(t)$.

Lemma 4.7: *The functions $f(t)$, which are defined on $(N_1 \dotplus N_2)'$, have values in $\mathfrak{H}(t)$ and satisfy*

$$(4.18)\quad \begin{cases} (f(t),\ a(t)) \text{ are measurable functions of } t \text{ for } a(t) \in \mathfrak{A}^{(0)}(t) \\ \text{corresponding to any } a \in \mathfrak{A}^{(0)}, \end{cases}$$

have norms $\| f(t)\|$ measurable as functions of $t \in [0, 1]$.

$\mathfrak{H}^*$ be the space consisted of the whole of these functions which have norms whose squares are integrable on $[0, 1]$. Of course two functions which are different only on a set of measure 0 are considered as identical. Then we can make this a Hilbert space isomorphic to $\mathfrak{H}$, and under this isomorphism $a(t)$ and $a \in \mathfrak{A}^{(0)}$ correspond isomorphically in the sense of Hilbert algebra.

Proof: We can define the linear operation in a usual obvious manner. As

$$\begin{aligned}\| f(t) \| &= \sup_{g(t)\in\mathfrak{H}(t),\ \|g(t)\|\leq 1} |(f(t), g(t))| \\ &= \sup_{g(t)\in\mathfrak{A}^{(0)}(t),\ \|g(t)\|\leq 1} |(f(t), a(t))| \\ &= \sup_{g(t)\in\mathfrak{A}_{(0)}(t),\ \|g(t)\|\leq 1} |(f(t), a(t))|,\end{aligned}$$

$\| f(t) \|$ is measurable.

The assumption that $\| f(t) \|^2$ is integrable for any $f(\cdot)\in\mathfrak{H}^*$ allows us to introduce the inner product in $\mathfrak{H}^*$ by the formula

$$\begin{aligned}(f(\cdot), g(\cdot)) &= \int_0^1 (f(t), g(t))\,dt \\ &= \frac{1}{2}\int_0^1 \{ \| f(t) + g(t) \|^2 + \| f(t) + ig(t) \|^2 \\ &\quad - 2(\| f(t) \|^2 + \| g(t) \|^2)\}\,dt\end{aligned}$$

and by this definition of the inner product and the definition of the norm attendant on it, the space $\mathfrak{H}^*$ turns to be a Hilbert space. The $a(\cdot)$'s for $a\in\mathfrak{A}^{(0)}$ are obviously contained in it, and form a linear manifold $\mathfrak{A}^{(0)*}$. If it were not dense in $\mathfrak{H}^*$, there would exist an element $f(\cdot) \neq 0(\cdot)$ in $\mathfrak{H}^*$ which is orthogonal to any of $\mathfrak{A}^{(0)*}$. But this implies $f(\cdot) = 0(\cdot)$, which is impossible. Therefore $\mathfrak{A}^{(0)*}$ is a dense linear manifold in $\mathfrak{H}^*$. The correspondences

$$\alpha a + \beta b \to \alpha a(\cdot) + \beta b(\cdot), \qquad (a, b) = (a(\cdot), b(\cdot))$$

establish the isomorphism of $\mathfrak{A}^{(0)}$ and $\mathfrak{A}^{(0)*}$ as linear spaces with inner product, and we can extend it to the isomorphism of $\mathfrak{H}$ and $\mathfrak{H}^*$ as Hilbert spaces. We can define the multiplication operation and the adjoint operation in $\mathfrak{A}^{(0)*}$ by this isomorphism, but this coincides with the method of defining them by element-wise way. Anyhow, $\mathfrak{A}^{(0)*}$ can be made into a Hilbert algebra which is isomorphic to $\mathfrak{A}^{(0)}$ under this isomorphism.

Lemma 4.8: *The elements of $\mathfrak{A}$ corresponds to such $a(\cdot)$'s $\in\mathfrak{H}^*$, for which $a(t)\in\mathfrak{A}(t)$ (for each t), and $|||T^{(t)}_{a(t)}|||$ are uniformly bounded with respect to t.*

Proof: First, we note that, by setting for $f(\cdot)\in\mathfrak{H}^*$

$$f_s(\cdot): \quad f_s(t) = f(t) \quad (t \leqslant s), \quad = 0 \quad (t > s),$$

$f_s(\cdot)$ corresponds to $E(s)f(,\ f(\cdot) \leftrightarrow f \in \mathfrak{H})$, because

$$(E(s)f, a) = \int_0^s (f(t), a(t))\,dt^{1)} = \int_0^1 (f_s(t), a(t))\,dt = (f_s(\cdot), a(\cdot))$$

for any $a \in \mathfrak{A}^{(0)}$. Especially, as $E(t)a \in \mathfrak{A}$ for $a \in \mathfrak{A}^{(0)}$, we have $a_t(\cdot)$ as the image of $E(t)a$.

Now we shall proceed to the proof of the lemma. As $a(\cdot)$'s for $a \in \mathfrak{A}^{(0)}$ satisfy the statements of the lemma (e.g. by VII), all the $a_t(\cdot)$'s ($a \in \mathfrak{A}^{(0)}$) do too, and these elements correspond to those in $\mathfrak{A}$. Thus Theorem 1 in our previous paper ([12]) shows that, by denoting the operator of right multiplication in $\mathfrak{H}^*$ as $S^*_{a(\cdot)}$, those and only those elements $f(\cdot)$ of $\mathfrak{H}^*$, for which there exists a $\gamma > 0$ such that

(4.19) $$\| S^*_{a_s(\cdot)} f(\cdot) \| \leqslant \gamma \, \| a_s(\cdot) \| \qquad (a \in \mathfrak{A}^{(0)}),$$

correspond to the elements of $\mathfrak{A}$. But (4.19) is rewritten as

$$\int_0^s \| S^{[t]}_{a(t)} f(t) \|^2 dt \leqslant \gamma^2 \int_0^s \| a(t) \|^2 dt,$$

and so with possible exception of a set of measure 0,

(4.20) $$\| S^{[t]}_{a(t)} f(t) \|^2 \leqslant \gamma^2 \, \| a(t) \|^2 \qquad \text{for all } a(t) \in A_{(0)}(t).$$

Therefore $f(t)$ belongs to $A(t)$ as (4.20) is obviously extended to $a(t) \in \mathfrak{A}^{(0)}(t)$, and

$$||| T^{[t]}_{f(t)} ||| \leqslant \gamma$$

except for a set of measure 0, which is of no importance.

Conversely, if, for $f(\cdot) \in \mathfrak{H}^*$, $f(t) \in \mathfrak{A}(t)$ and $||| T^{[t]}_{f(t)} ||| \leqslant \gamma$ for every $t \in (N_1 + N_2)'$, then, for an arbitrary $a(\cdot) \in \mathfrak{A}^*$,

$$\| S^*_{a(\cdot)} f(\cdot) \|^2 = \int_0^1 \| S^{[t]}_{a(t)} f(t) \|^2 dt = \int_0^1 \| T^{[t]}_{f(t)} a(t) \|^2 dt$$
$$\leqslant \gamma^2 \int_0^1 \| a(t) \|^2 dt = \gamma^2 \, \| a(\cdot) \|^2.$$

---

1) We can easily show, as $\mathfrak{A}$ being dense in $\mathfrak{H}$, that if $f \longleftrightarrow f(\cdot)$, $g \longleftrightarrow g(\cdot)$, then

$$(E(s)f, g) = \int_0^s (f(t), g(t))\,dt.$$

and so $f(\cdot)$ corresponds to an element of $\mathfrak{A}$.

Thus all the statements of the lemma is proved.

Before stating these results as a theorem, it will be convenient to make a definition.

Definition 4.1 : *J. von Neumann* [9] *has given the definition of the integral sum* (generalised direct sum according to his terminology) *of Hilbert spaces. We shall adopt this terminology as it is. Let* $\mathfrak{H}$ *be isomorphic to the* integral sum *of* $\mathfrak{H}(t)$, *and, in* $\mathfrak{H}$ *and in each* $\mathfrak{H}(t)$, *maximal Hilbert algebras* $\mathfrak{A}$ *and* $\mathfrak{A}(t)$ *resp. are given. Then we shall term* $\mathfrak{A}$ *to be isomorphic to the* integral sum *of* $\mathfrak{A}(t)$, *under the isomorphism of* $\mathfrak{H}$ *and the integral sum of* $\mathfrak{H}(t)$, *if the elements of* $\mathfrak{A}$ *correspond to those and only those elements* $a(\cdot)$ *of the integral sum of* $\mathfrak{H}(t)$ *such that*

$$a(t) \in \mathfrak{A}(t), \quad |||T^{t}_{a(t)}||| \leqslant r \qquad \textit{for each } t.$$

Then the results obtained in this section can be stated as follows.

Theorem 4.1 : *Let be given a separable Hilbert space* $\mathfrak{H}$ *and a maximal purely non-simple Hilbert algebra* $\mathfrak{A}$ *in it. Then we can construct for each* $t \in [0, 1]$ *a Hilbert space* $\mathfrak{H}(t)$ *and a maximal Hilbert algebra* $\mathfrak{A}(t)$ *in it such that* $\mathfrak{H}$ *is isomorphic to the integral sum of the Hilbert spaces of* $\mathfrak{H}(t)$ *and, under this isomorphism, the Hilbert algebra* $\mathfrak{A}$ *is isomorphic to the integral sum of the Hilbert algebras* $\mathfrak{A}(t)$.

The von Neumann's reduction theorem now shows that

Theorem 4.2 : *In the preceding theorem, each* $\mathfrak{A}(t)$ *is a simple Hilbert algebra in* $\mathfrak{H}(t)$.

Thus in the separable case all the problems are reduced to the case of simple algebras. In the non-separable case, the like-wise integral sum representation on a suitable measure space can be obtained, but we do not yet succeed to prove the simplicity character.

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