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On two theorems of A. Abian

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ON TWO THEOREMS OF A. ABIAN

Dedicated to Professor Kiiti Morita on his 60th birthday

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A (non-zero) ring without non-zero nilpotent elements is called a reduced ring. Recently, in his papers [1] and [2], A. Abian proved the following :

(I) A commutative reduced ring is a direct product of fields if and only if it is orthogonally complete and hyperatomic.

(II) A commutative reduced ring is a direct product of integral domains if it is orthogonally complete and superatomic.

In this paper, we shall prove that both these are still true for non-commutative reduced rings, more precisely,

Theorem 1. *The following conditions are equivalent* :

(1) *R is a reduced ring which is orthogonally complete and hyperatomic.*

(2) *R is a direct product of division rings.*

Theorem 2. *The following conditions are equivalent* :

(1) *R is a reduced ring which is orthogonally complete and superatomic.*

(2) *R is a direct product of integral domains and the annihilators of those integral domains exhaust the proper prime ideals of R.*

Although Theorem 1 has been obtained in [4] and our proofs of Theorems 1 and 2 are very similar to those of (I) and (II) in [1] and [2] respectively, we are much more skilful in performing those.

1. Definitions and lemmas. In a reduced ring, as is well-known, the intersection of prime ideals equals 0, namely, every non-zero element is excluded by some prime ideal (see for instance [5, p. 56]), and every idempotent is central. In what follows, R will represent always a reduced ring.

Lemma 1. *Let r and s be elements of a reduced ring R.*

(a) *If $rs=0$, then $sr=0$, and for every prime ideal P of R either r or s is contained in P.*

(b) *If $r^2s=r$, then there exists one and only one element r' such that $rr'=r'r$, $r^2r'=r$ and $rr'^2=r'$.* (r' will be called the *semi-inverse* *of* r.)

Proof. (a) $sr=0$ is clear by $(sr)^2=0$. Moreover, $srR=0$ yields $rRs=0$. Hence, $r \in P$ or $s \in P$.

(b) This is only a consequence of [3, Lemma 1 and Theorem 1]. However, for the sake of completeness, we give here the proof. Evidently, $(rsr-r)^2=0$, and so $rsr=r$. By making use of this, we have $(sr^2-r)^2=0$, which implies $sr^2=r$ and $sr=sr^2s=rs$. Then, $r'=rs^2$ satisfies the relations requested. Now, let $rr''=r''r$, $r^2r''=r$ and $rr''^2=r''$. Then, $r^2r'=r=r''r^2$ implies $r''r=r''r^2r'=rr'$. Hence, $r''=r''^2r=r''rr'=rr'^2=r'$.

Now, for $x, y \in R$ we define $x \leqq y$ if and only if $xy=x^2$ (and $yx=x^2$ by Lemma 1 (a)). Then, the relation $\leqq$ is a partial order in R. In fact, the reflexibility and the antisymmetry are easy, and the transitivity can be seen as follows: If $xy=x^2$ and $yz=y^2$ then $x^2z=xyz=xy^2=x^2y=x^3$, i. e., $x(xz-x^2)=0$, which implies $(xz-x^2)x=0$ (Lemma 1 (a)). Hence, $(xz-x^2)^2=0$, and eventually $xz=x^2$.

Following [1], R is defined to be *orthogonally complete* if for every orthogonal subset T of R (i. e., a subset T such that $st=0$ for every different $s, t \in T$) there exists $\sup T$ with respect to $\leqq$ mentioned above. A non-zero element $a \in R$ is called an *atom* of R if $x \leqq a$ implies $x=0$ or $x=a$. An atom a is called a *hyperatom* if $ax \neq 0$ $(x \in R)$ implies always $axs=a$ for some $s \in R$, and H will denote the set of all hyperatoms of R. R is defined to be *hyperatomic* if for every non-zero element $r \in R$ there exists $a \in H$ such that $a \leqq r$. Finally, an element $a \in R$ is called a *superatom* if a is contained in every proper prime ideal except exactly one $P(a)$, and S will denote the set of all superatoms of R. (In [2], a superatom in our sense is called an atom.) Obviously, if a is in S then a is non-zero and $-a$ is a superatom with $P(-a)=P(a)$. R is defined to be *superatomic* if for every proper prime ideal P and every element $r \in R \backslash P$ there exists $a \in S$ such that $a \in R \backslash P$ and $a \leqq r$.

Lemma 2. *If a is a hyperatom of a reduced ring R then aa' is an idempotent hyperatom, where a' is the semi-inverse of a.*

Proof. Since $a \in H$ and $aa \neq 0$, by Lemma 1 (b) a has the semi-inverse a' and $e=aa'$ is a central idempotent. If $0 \neq er = aa'r$ $(r \in R)$ then $a(a'r)t=a$ with some $t \in R$, and so $(er)(ta')=e$. It remains therfore to show that e is an atom. Assume that $x \leqq e$, i. e., $ex=x^2$. Then, $ex^2=e(ex)=x^2$, which yields $(ex-x)^2=0$. Hence,

$x=ex=x^2$. Recalling that x is then central, we have $xa\leqq a$. Accordingly, $(xa)a'=xe=x$ is either 0 or $aa'=e$. This proves that e is an atom.

Now, let $E=\{e_\gamma | \gamma\in\Gamma\}$ be the set of all idempotent hyperatoms of R. We claim that $A_\gamma=e_\gamma R$ is a division ring. In fact, e_γ is the identity of A_γ, and for every non-zero element $e_\gamma r (r\in R)$ there exists an element $s\in R$ such that $e_\gamma=e_\gamma rs=(e_\gamma r)(e_\gamma s)$ (Lemma 2).

Lemma 3. *If R is a hyperatomic reduced ring, then for every non-zero element $r\in R$ there holds $rE\neq 0$.*

Proof. By hypothesis, $ra=a^2$ with some $a\in H$. Then, by Lemma 2, $aa'\in E$ and $raa'=a^2a'=a\neq 0$, where a' is the semi-inverse of a.

Lemma 4. *Let R be a reduced ring.*

(a) *Let $a\in S$, and $r\in R$. If $ar\neq 0$ then ar, $ra\in S$ and $P(ar)=P(ra)=P(a)$.*

(b) *Let a, $b\in S$. Then, $ab\neq 0$ if and only if $P(a)=P(b)$.*

(c) *Let $a, b\in S$. If $ab\neq 0$ and $a-b\neq 0$ then $a-b\in S$ and $P(a-b)=P(a)$.*

Proof. (a) Immediately, ar is contained in every proper prime ideal different from $P(a)$. On the other hand, there exists a proper prime ideal excluding ar. Hence, $ar\in S$ and $P(ar)=P(a)$. Similarly, by Lemma 1 (a) we see that $ra\in S$ and $P(ra)=P(a)$.

(b), (c) If $ab\neq 0$ then $P(a)=P(ab)=P(b)$ by (a), and $a-b$ is contained in every prime ideal different from $P(a)$. Hence, in case $a-b\neq 0$, $a-b\in S$ and $P(a-b)=P(a)$. Conversely, assume that $P(a)=P(b)$. If $ab=0$ then by Lemma 1 (a) $a\in P(a)$ or $b\in P(b)$, a contradiction.

Corollary 1. *In a reduced ring R, every superatom a is an atom.*

Proof. Assume that $x\leqq a$ and $x\neq 0$. By Lemma 4 (a), $xa=x^2\in S$ and $P(xa)=P(a)$, whence it follows $x\notin P(a)$. Hence, $x(a-x)=0$ implies $a-x\in P(a)$ (Lemma 1 (a)), and so $a(a-x)\in P(a)$. On the other hand, $a(a-x)$ is contained in every prime ideal different from $P(a)$. We obtain therefore $a(a-x)=0$. Combining this with $x(a-x)=0$, we readily obtain $(x-a)^2=0$, and hence $x=a$.

In virtue of Lemma 4 (b), we can define an equivalence relation $\sim$ in S, where $a\sim b$ if and only if $ab\neq 0$, or equivalently, $P(a)=P(b)$. Let

$S=\bigcup_{\lambda\in\Lambda} S_\lambda$ be the partition of S into the equivalence classes with respect to $\sim$. Obviously, $S_\lambda \longmapsto P_\lambda = P(a)$ $(a\in S_\lambda)$ is well-defined, and $S_\lambda \cap P_\lambda = \emptyset$. As a direct consequence of Lemma 4, we see that $B_\lambda = S_\lambda \cup \{0\}$ is an ideal of R which is an integral domain and $B_\lambda P_\lambda = 0$.

In the rest of this section, we assume further that R is superatomic. Then, $S_\lambda \longmapsto P_\lambda$ gives a $1-1$ correspondence between $\{S_\lambda \mid \lambda\in\Lambda\}$ and the set of all proper prime ideals of R. If r is in $R\backslash P_\lambda$ then by hypothesis there exists some $r_\lambda\in S_\lambda$ with $r_\lambda \leqq r$. We claim here that such r_λ is unique. In fact, if $\bar r_\lambda \leqq r$ and $\bar r_\lambda\in S_\lambda$ then $(r-r_\lambda)r_\lambda = 0 = (r-\bar r_\lambda)\bar r_\lambda$ implies $r-r_\lambda$, $r-\bar r_\lambda \in P_\lambda$ (Lemma 1 (a)). Hence, $\bar r_\lambda - r_\lambda\in P_\lambda\cap B_\lambda = 0$, namely, $\bar r_\lambda = r_\lambda$. On the other hand, if r is in P_λ then there is no $r_\lambda\in S_\lambda$ with $r_\lambda\leqq r$. We define here the map $g_\lambda : R\longrightarrow B_\lambda$ by

$$g_\lambda(r) = \begin{cases} r_\lambda & \text{if } r\notin P_\lambda. \\ 0 & \text{if } r\in P_\lambda. \end{cases}$$

Lemma 5. *Let R be a superatomic reduced ring. Then, g_λ is a ring homomorphism leaving every element of the integral domain B_λ invariant and* Ker $g_\lambda = P_\lambda$. *Accordingly, $R = P_\lambda \oplus B_\lambda$ and P_λ coincides with the annihilator of B_λ.*

Proof. Since R is a reduced ring, the (right and left) annihilator of B_λ has the intersection 0 with B_λ. It remains therefore to prove (i) $g_\lambda(r+s) = g_\lambda(r)+g_\lambda(s)$ and (ii) $g_\lambda(rs) = g_\lambda(r)g_\lambda(s)$ $(r,\ s\in R)$. First, we consider the case $r\in P_\lambda$ and $s\notin P_\lambda$. Since $s_\lambda r = 0$, we obtain $s_\lambda(r+s) = s_\lambda s = s_\lambda^2$, which means $(r+s)_\lambda = s_\lambda$. Hence, we have (i), and readily (ii). Next, we consider the case $r\notin P_\lambda$ and $s\notin P_\lambda$. We claim that $rs_\lambda = r_\lambda s_\lambda$ and $sr_\lambda = s_\lambda r_\lambda$. In fact, by $r_\lambda\leqq r$ and $s_\lambda \leqq s$ it follows $r_\lambda rs_\lambda = r_\lambda^2 s_\lambda$. Since B_λ is an integral domain, we have $rs_\lambda = r_\lambda s_\lambda$, and similarly $sr_\lambda = s_\lambda r_\lambda$. Hence, we have $(rs)(r_\lambda s_\lambda) = rs_\lambda r_\lambda s_\lambda = (r_\lambda s_\lambda)^2$, namely , $(rs)_\lambda = r_\lambda s_\lambda$, proving (ii). In order to see (i), we shall distinguish between two cases. (1) $r+s\notin P_\lambda$: If $r_\lambda + s_\lambda = 0$ then $r_\lambda r = r_\lambda^2 = s_\lambda^2 = s_\lambda s = -r_\lambda s$, i. e., $r_\lambda(r+s) = 0$, wence it follows $r_\lambda\in P_\lambda$ or $r+s\in P_\lambda$ (Lemma 1 (a)). This contradiction means that $r_\lambda + s_\lambda\in S_\lambda$ (Lemma 4). Since $(r+s)(r_\lambda + s_\lambda) = rr_\lambda + sr_\lambda + rs_\lambda + ss_\lambda = r_\lambda^2 + s_\lambda r_\lambda + r_\lambda s_\lambda + s_\lambda^2 = (r_\lambda + s_\lambda)^2$, we have $(r+s)_\lambda = r_\lambda + s_\lambda$, proving (i). (2) $r+s\in P_\lambda$: Since $0 = (r+s)r_\lambda = r_\lambda^2 + sr_\lambda = r_\lambda^2 + s_\lambda r_\lambda$ and $0 = (r+s)s_\lambda = rs_\lambda + s_\lambda^2 = r_\lambda s_\lambda + s_\lambda^2$, we obtain $(r_\lambda + s_\lambda)^2 = 0$, and so $r_\lambda + s_\lambda = 0$, proving (i). Finally, in case $r\in P_\lambda$ and $s\in P_\lambda$, there is nothing to prove.

2. Proofs of theorems. The notations employed in the preceding section will be used here.

Proof of Theorem 1. (1)$\Longrightarrow$(2): Let $f: R \longrightarrow \prod_{\gamma\in\Gamma} A_\gamma$ be the map defined by $f(r) = (re_\gamma)$. Then, f is a ring homomorphism, and by Lemma 3 $\mathrm{Ker}\, f = \{r\in R \mid rE=0\} = 0$. If f is shown to be an isomorphism, $A_\gamma(\gamma\in\Gamma)$ are adapted for the division rings in (2). Now, let (r^γ) be an arbitrary element of $\prod_{\gamma\in\Gamma} A_\gamma$. By $e_\gamma e_\delta \leqq e_\gamma$ and e_δ, we can easily see that $e_\gamma e_\delta = 0$ for every $\gamma\neq\delta$. Hence, $T=\{r^\gamma \mid \gamma\in\Gamma\}$ is an orthogonal subset of R and there exists $r=\sup\, T$. We shall prove now $re_\delta = r^\delta$ for every $\delta\in\Gamma$. By $r^\delta \leqq r$, we obtain $r^\delta re_\delta = (r^\delta)^2$, i. e., $r^\delta \leqq re_\delta$. On the other hand, to be easily seen, $r^\gamma(r^\delta - re_\delta + r) = (r^\gamma)^2$, i. e, $r^\gamma \leqq r^\delta - re_\delta + r$ for every $\gamma\in\Gamma$. Hence, $r\leqq r^\delta - re_\delta + r$, and so $(r^\delta - re_\delta + r)\, r = r^2$, whence it follows $r^\delta(re_\delta) = r^\delta r = r^2 e_\delta = (re_\delta)^2$. Combining this with $r^\delta \leqq re_\delta$, we obtain $re_\delta = r^\delta$.

(2)$\Longrightarrow$(1): Let R be the direct product of division rings $R_\kappa\,(\kappa\in K)$. Then, it is clear that R is orthogonally complete. If $x=(x^\kappa)$ is an arbitrary non-zero element of R, then there exists $\alpha\in K$ with $x^\alpha\neq 0$. Then, we can easily see that x^α is a hyperatom and $x^\alpha \leqq x$.

Proof of Theorem 2. (1)$\Longrightarrow$(2): Let $g: R \longrightarrow \prod_{\lambda\in\Lambda} B_\lambda$ be the map defined by $g(r) = (g_\lambda(r))$. Then, by Lemma 5, g is a ring homomorphism with $\mathrm{Ker}\, g = \bigcap_{\lambda\in\Lambda} \mathrm{Ker}\, g_\lambda = \bigcap_{\lambda\in\Lambda} P_\lambda = 0$, and P_λ coincides with the annihilator of the integral domain B_λ. If g is shown to be an isomorphism, $B_\lambda(\lambda\in\Lambda)$ are adapted for the integral domains in (2). Now, we shall show that g is a surjection. Let (r^λ) be an arbitrary non-zero element of $\prod_{\lambda\in\Lambda} B_\lambda$, $N=\{\lambda\in\Lambda \mid r^\lambda = 0\}$, and $M=\Lambda\backslash N$. Since the set $T=\{r^\lambda \mid \lambda\in M\}$ is an orthoghnal subset of R, by hypothesis there exists $r=\sup\, T$. To our end, it suffices to show that $r\in P_\lambda$ if and only if $\lambda\in N$. Assume first that $r\in P_\lambda$. If $\lambda\in M$, then $r^\lambda = r_\lambda \in S_\lambda$, but then $r_\lambda^2 = rr_\lambda = 0$, a contradiction. Conversely, assume that $\lambda\in N$. If $r\notin P_\lambda$, then $r^\mu r_\lambda = 0 = r_\lambda r^\mu$ for every $\mu\in M$. Hence, $r^\mu(r_\lambda + r) = r^\mu r = (r^\mu)^2$, namely, $r^\mu \leqq r_\lambda + r$ for every $\mu\in M$. This implies $r\leqq r_\lambda + r$, and so $r(r_\lambda + r) = r^2$. However, the last contradicts $r(r_\lambda + r) = r_\lambda^2 + r^2$.

(2)$\Longrightarrow$(1): Assume that R is the direct product of the integral domains $R_\kappa\,(\kappa\in K)$ and $\prod_{\kappa\neq\alpha} R_\kappa\,(\alpha\in K)$ exhaust the proper prime ideals of R. Then, R is orthogonally complete evidently. Moreover, if $x=(x^\kappa)$ is an arbitrary element of $R\backslash \prod_{\kappa\neq\alpha} R_\kappa$ then we can easily see that x^α is a superatom of R not contained in $\prod_{\kappa\neq\alpha} R_\kappa$ and $x^\alpha \leqq x$.

Corollary 2. *If R is a reduced ring with* 1 *which is orthogonally* *complete and superatomic, then R is a finite direct sum of integral domains.*

Proof. In any rate, by the proof of Theorem 2, $R = \prod_{\lambda \in \Lambda} B_\lambda$ and $\prod_{\lambda \neq \alpha} B_\lambda$ $(\alpha \in \Lambda)$ exhaust the proper prime ideals of R. If Λ is infinite, then the proper ideal $\bigoplus_{\lambda \in \Lambda} B_\lambda$ is contained in some maximal ideal, which is a contradiction.

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Added in proof. A recent result of O. Goldman [J. Algebra 34 (1975), 64-73] enables us to see that the following condition is equivalent to those in Theorem 1:

(3) *R is a reduced ring with* 1 *which is complete in its intrinsic topology.*