Mathematical Journal of Okayama University

Volume 19, *Issue* 1 1976 *Article* 7

DECEMBER 1976

Semiperfect rings with quasi-projective left ideals

S. C. Goel* S. K. Jain†

*Ohio University

†Ohio University

SEMIPERFECT RINGS WITH QUASI-PROJECTIVE LEFT IDEALS

S. C. GOEL and S. K. JAIN

§0. **Introduction.** In this paper we continue the study of rings with quasi-projective left ideals initiated by Jain-Singh [5]. A ring R is called *semiperfect* if idempotents modulo the Jacobson radical N can be lifted and if R/N is semisimple artinian. Equivalently, R has a complete orthogonal set $e_1, e_2, \cdots, e_n$ of primitive idempotents with $e_1+\cdots+e_n=1$. R is a *semi-primary ring* if N is nilpotent and R/N is semisimple artinian. A module M is said to be *quasi-projective* if for every submodule K of M the induced sequence $\mathrm{Hom}(M, M)\to\mathrm{Hom}(M, M/K)\to 0$ is exact. A ring R is called a *left qp-ring* if each of its left ideals is quasi-projective. We call a ring R to be a *left weakly qp-ring* if each of its left ideals generated by at most two elements is quasi-projective.

§1. In Lemmas 1–4, we assume that R is a semi-perfect left weakly qp-ring with $\{e_i\}$, $1\leq i\leq n$, as a complete set of primitive orthogonal idempoetents and the Jacobson radical N as nil. For convenience e or f shall denote an arbitrary element in the set $\{e_i\}$.

Lemma 1. *Let A and B be two indecomposable principal left ideals of R. Then either $A\cap B=(0)$ or A and B are comparable.*

The proof follows from Miyashita [7, Th. 3. 3] as shown in [5, Lemma 2]. An application of the above lemma provides

Lemma 2. *$Reae$, $Rebe$ with $a, b\in N$ and $Reae\neq Rebe$ are not comparable if and only if $Reae$ and $Rebe$ are isomorphic and minimal in the family $\mathscr{C}=\{Reae \mid eae\in eNe\}$.*

Proof. Suppose $Reae$ and $Rebe$ are not comparable. Then $Reae\cap Rebe=(0)$. Since $Reae\oplus Rebe$ is quasi-projective and their projective covers are same, it follows by [2, Lemma 3. 2] that $Reae\cong Rebe$. Let $Rece$ be a nonzero submodule of $Reae$. Then $Rece\cap Rebe=(0)$ and as before $Rece\cong Rebe\cong Reae$. We claim that $Rece=Reae$. If $Rece\subsetneqq Reae$ then $eRece\subsetneqq eReae$. So there exists $ene\in eNe$ such that $ece=eneae$. Let f be an isomorphism from $eReae$ to $eRece$ and let $f(eae)=exece=exeneae$. Then

Mathematical Journal of Okayama University, Vol. 19 [1976], Iss. 1, Art. 7

$f^k(eae)=(exene)^k eae=0$ for some positive integer k, which is a contradiction. Therefore, $Rece=Reae$, proving $Reae$ is minimal in $\mathscr{C}$. Conversely, if $Reae$ and $Rebe$ are minimal then they cannot be comparable. This proves the lemma.

Lemma 3. *If $eae\ (\neq 0)\in eNe$ then $Reae$ is not projective and $eRf=(0)$.*

Proof. On the contrary, suppose $Reae$ is projective then $Re\cong Reae$, since e is primitive. As in Lemma 2 this implies that $Re=Reae$, which is impossible. Therefore $Reae$ is not projective. If $eRf\neq 0$, then $ebf\neq 0$ for some $b\in R$. Since $Re\oplus Rebf$ is quasi-projective the sequence $Re\to Rebf\to 0$ must split, and thus $Re\cong Rebf$. Again $Reae\oplus Rebf$ being quasi-projective implies that the sequence $Rebf\cong Re\to Reae\to 0$ splits. Therefore $Re\cong Reae$ which is a contradiction to the fact that $Reae$ is not projective. Hence $eRf=(0)$.

An immediate consequence of the above result is

Corollary. *A prime semiperfect left weakly qp-ring with Jacobson radical nil is a simple artinian ring.*

Lemma 4. *For a given idempotent e and the class $\mathscr{C}=\{Reae\mid eac\in eNe\}$, one and only one of the following holds.*

(a) *There exists an infinite properly ascending chain of principal left ideals.*

(b) *There exists a unique maximal left ideal in $\mathscr{C}$. In this case $ReNe=Reae$ for some $eae\in eNe$, $\mathscr{C}$ is finite and totally ordered.*

(c) *$\mathscr{C}$ has more than one element and all its members are maximal and minimal in $\mathscr{C}$. In this case all the members of $\mathscr{C}$ are isomorphic.*

Proof. We shall only prove that if (a) does not hold then (b) or (c) must hold. Other implications are clear. Suppose (a) does not hold, then we have a. c. c. in $\mathscr{C}$. Let $Reae$ be a maximal element in $\mathscr{C}$. Case (i) : $Reae$ is unique. Let $ebe\in eNe$, then $Reae\cap Rebe=(0)$ implies $Reae\cong Rebe$ by Lemma 2, and uniqueness of $Reae$ yields $Reae=Rebe$. Thus by Lemma 1, $ReNe=Reae$, which gives $eNe=eReNe=eReae$. It is easy to check that any left ideal $eRexe$, $exe\in eNe$, is of the form $eRe(eae)^m$ for some integer m. Hence $\mathscr{C}$ is totally ordered with all elements of the type $R(eae)^m$. Clearly $\mathscr{C}$ is finite. Case (ii) : $Reae$ is not unique. Let $Reae$ and $Rebe$ be distinct maximal elements in $\mathscr{C}$. Since $Reae$ and $Rebe$ are not comparable, $Reae\cap Rebe=(0)$. Then $Reae\cong Rebe$ and minimal. This implies that for any $Rece \in \mathscr{C}$, $Rece$ cannot be comparable with $Reae$ and hence $Rece \cong Reae$ proving the lemma.

Lemma 5. *A direct sum of left weakly qp-rings is a left weakly qp-ring.*

Proof. The proof is straight forward.

Theorem 1. *Let R be a local ring with Jacobson radical nil. Then the following are equivalent.*
(a) *R is a left weakly qp-ring with a. c. c. on principal left ideals.*
(b) *Either $N^2=(0)$ or R is a principal left ideal ring with d. c. c.*
(c) *R is a left qp-ring with a. c. c. on principal left ideals.*

Proof. (a $\Longrightarrow$ b) We know by Lemma 4 either $N=Ra$, $a \in N$, or every principal left ideal is maximal and minimal. If $N=Ra$ then every left ideal in R is of the form Ra^i and nilpotency of a yields R is left artinian. In case every principal left ideal is minimal then $N=\mathrm{Soc}(R)$ and hence $N^2=(0)$.

(b $\Longrightarrow$ c) If $N^2=(0)$ then R is clearly a left qp-ring with a. c. c. on principal left ideals. On the other hand if R is a principal left ideal ring then $N=Ra$, $a \in N$, showing that R is a duo ring. Thus R is a left qp-ring.

(b $\Longrightarrow$ a) Obvious.

Lemma 6. *Let R be a semi-perfect left weakly qp-ring with Jacobson radical nil. Then eRe is a left weakly qp-ring for each primitive idempotent e in R.*

Proof. Let $eReae$ be a principal left ideal in eRe. Since $eReae \cong eRe/\mathrm{ann}_{eRe}(eae)$, it is enough to show that $\mathrm{ann}_{eRe}(eae)$ is a two sided ideal in eRe. Since $Reae$ is a quasi-projective left R-module and Re is its projective cover, we have by [9, Prop. 2. 2], $\mathrm{ann}_{Re}(eae)\, eRe \subseteq \mathrm{ann}_{Re}(eae)$ and hence $\mathrm{ann}_{eRe}(eae) \subseteq \mathrm{ann}_{eRe}(eae)$. So by [9, Prop. 2. 1] or by [7, Th. 2. 4 (2)], $eReae$ is quasi-projective as eRe-module. Consider now $eReae+eRebe$ with eae, $ebe \in eNe$. By Lemma 2 either $Reae$ and $Rebe$ are comparable or isomorphic with zero intersection. In case $Reae \cap Rebe=(0)$, we get $eReae \oplus eRebe$ is quasi-projective as eRe-module since $eReae \cong eRebe$. In the other case $eReae+eRebe$ is obviously quasi-projective since $eReae$ and $eRebe$ are comparable.

It is well known that perfect hereditary rings are semiprimary. We prove the following more general result.

Mathematical Journal of Okayama University, Vol. 19 [1976], Iss. 1, Art. 7

Theorem 2. *Let R be a semiperfect ring with Jacobson radical nil and a. c. c. on principal left ideals. If R is a left weakly qp-ring then R is semiprimary.*

Proof. By Lemma 6 eRe is a left weakly qp-ring for each primitive idempotent e. It is immediate that eRe has a. c. c. on principal left ideals and the Jacobson radical eNe is nil. Hence by Theorem 1, eRe is a left qp-ring for each primitive idempotent. Again using Theorem 1 it follows that eNe is nilpotent. This yields N is nilpotent, proving R is semiprimary.

Remark. The Theorem 2, in particular, implies that all the results proved for perfect left qp-rings in [5] hold for left perfect left qp-rings.

Theorem 3. *Let R be a semiprime left noetherian left qp-ring then R is left hereditary.*

Proof. Let I be an essential left ideal in R. Then I contains a regular element and so contains a copy of R. Since I is quasi-projective by de Robert [8], I is projective relative to R. Again by de Robert I is projective relative to any finitely generated module. Since I is finitely generated, it follows that I is projective proving that R is left hereditary.

The above result was also noticed by Surjeet Singh independently. Next we note that the class of left qp-rings is not closed under Morita equivalence as follows from the following lemma and the example.

Lemma 7. *Let S be a primary nonlocal left qp-ring with Jacobson radical nil. Then S is simple artinian.*

Proof. Since S is a nonlocal primary ring, $S=R_n$, $n>1$, for some local ring R [4, Th. 1, p. 56]. We claim R is a division ring. Let a be a non-zero element of R. Let K be the principal left ideal of R_n generated by $x=ae_{11}+e_{22}+\cdots+e_{nn}$. Now K is quasi-projective as R_n-module. This implies by Miyashita [7, Th. 2. 8 (1)] or by Golan [3, Cor. 1. 2 (2)] that $e_{11}K$ is quasi-projective as R-module. Since $e_{11}K\cong Ra\oplus R\oplus\cdots\oplus R$, we obtain $Ra\oplus R$ is quasi-projective. Hence Ra is projective. Thus $\mathrm{ann}(a)=(0)$ since R is local. Therefore, R is a domain and so a division ring, proving that R_n is simple artinian.

Example. Let $R=Z/(p^2)$ where Z is the ring of integers and p is a prime number. By Theorem 1, R is a qp-ring. However, by Lemma 7, $S=R_n$, $n>1$, cannot be a qp-ring.

REFERENCES

[1] H. BASS: Finitistic dimension and homological generalizations of semi-primary rings, Trans. Amer. Math. Soc. **95** (1960), 466—488.

[2] K. FULLER: On direct representations of quasi-injectives and quasi-projectives, Arch. Math. **20** (1969), 495—502.

[3] J. GOLAN: Characterization of rings using guasi-projective modules II, Proc.. Amer. Math. Soc. **28** (1971), 337—343.

[4] N. JACOBSON: Structure of Rings, Amer. Math. Soc. Colloq. Publ. **37**, Providence, 2nd ed., 1964.

[5] S. K. JAIN and S. SINGH: Rings with quasi-projective left ideals, Pacific J. Math. **60** (1975), 169—181.

[6] A. KOEHLER: Quasi-projective and puasi-injective modules, Pacific J. Math. **36** (1971), 713—720.

[7] Y. MIYASHITA: Quasi-projective modules, perfect modules and a theorem for modular lattices, J. Fac. Sci. Hokkaido Univ., Ser. I, **19** (1966), 86—110.

[8] E. DE ROBERT: Projectifs et injectifs relatifs, C. R. Acad. Sci. Paris Ser. A, **268** (1969), 361—364.

[9] L. WU and J. JANS: On quasi-projectives, Illinois J. Math. **11** (1967), 439—448.

OHIO UNIVERSITY
ATHENS, OHIO 45701
U. S. A.

(*Received July 7, 1976*)