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Abstract

Let A be an ample line bundle on a projective toric variety X of dimension n ($\geq$ 2). It is known that the d-th tensor power A$\otimes$d embedds X as a projectively normal variety in Pr := P(H0(X,L$\otimes$d)) if d $\geq$ n − 1. In this paper first we show that when dimX = 2 the line bundle A$\otimes$d satisfies the property Np for p $\leq$ 3d − 3. Second we show that when dimX = n $\geq$ 3 the bundle A$\otimes$d satisfies the property Np for p $\leq$ d − n + 2 and d $\geq$ n − 1.

KEYWORDS: toric variety, syzygy

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ON HIGHER SYZYGIES OF PROJECTIVE TORIC VARIETIES

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Abstract. Let A be an ample line bundle on a projective toric variety X of dimension n (≥ 2). It is known that the d-th tensor power $A^{\otimes d}$ embedds X as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X, L^{\otimes d}))$ if $d \geq n-1$. In this paper first we show that when $\dim X = 2$ the line bundle $A^{\otimes d}$ satisfies the property N_p for $p \leq 3d-3$. Second we show that when $\dim X = n \geq 3$ the bundle $A^{\otimes d}$ satisfies the property N_p for $p \leq d-n+2$ and $d \geq n-1$.

Introduction

The purpose of this article is to study the minimal free resolution of homogeneous coordinate rings of toric varieties.

Let X be a projective toric variety of dimension n and L a very ample line bundle on X. Since projective toric variety of dimension one is isomorphic to the projective line, we may assume that $n \geq 2$.

Koelman showed that an ample line bundle on a projective toric surface X is very ample and embedds X as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X, L))$ [10], and obtained a criterion when the surface is defined by only quadrics [11]. When $n \geq 3$ an ample line bundle is not very ample in general. Ewald and Wessels [3] showed that for an ample line bundle A on X the d-th tensor power $L = A^{\otimes d}$ is very ample for $d \geq \dim X - 1$. Ogata and Nakagawa [13] showed that $L = A^{\otimes d}$ embedds X as a projectively normal variety if $d \geq \dim X - 1$ and that the homogeneous ideal I of X in $\mathbb{P}^r := \mathbb{P}(H^0(X, L))$ is generated by quadrics if $d \geq \dim X$. In this paper, we study higher syzygies of the homogeneous ideal of X in $\mathbb{P}^r$, especially the property N_p introduced by Green and Lazarsfeld [7].

Definition 1. Let X be a projective variety and L a very ample line bundle on X defining an embedding $X \hookrightarrow \mathbb{P}^r := \mathbb{P}(H^0(X, L))$. Denote by $S = \mathrm{Sym}\, H^0(X, L)$ the homogeneous coordinate ring of the projective space $\mathbb{P}^r$. Consider the graded S-module $R = R(L) = \oplus_{i \geq 0} H^0(X, L^{\otimes i})$, the homogeneous coordinate ring of X. Let $E_\bullet$ be a minimal graded free resolution of R:

$$\cdots \to E_2 \to E_1 \to E_0 \to R \to 0,$$

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where $E_i = \bigoplus_j S(-a_{ij})$. Then the line bundle L satisfies *Property* (N_0) if $E_0 = S$. For an integer $p \geq 1$ the line bundle L satisfies *Property* (N_p) if $E_0 = S$ and if $a_{ij} = i+1$ for $1 \leq i \leq p$.

Schenck and Smith [17] proved that for an ample line bundle A on a projective toric variety of dimension n, the bundle $A^{\otimes d}$ satisfies Property N_{d-n+1} for $d \geq n-1$.

This paper improves their results by separately considering the case $n = 2$ and $n \geq 3$.

Theorem 0.1. *Let X be a projective toric surface and A an ample line bundle on X. Then $A^{\otimes d}$ satisfies Property N_p for $p \leq 3d-3$.*

This is given by Proposition 2.3.

Theorem 0.2. *Let X be a projective toric variety of dimension n (≥ 3) and A an ample line bundle on X. Then $A^{\otimes d}$ satisfies Property N_p for $p \leq d-n+2$ and $d \geq n-1$.*

This is given by Proposition 3.3.

1. Polarized Toric Varieties

First we mention the facts about toric varieties needed in this paper following Oda's book [14], or Fulton's book [5].

Let N be a free $\mathbb{Z}$-module of rank n, M its dual and $\langle , \rangle : M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic n-torus over the field $\mathbb{C}$ of complex numbers, where $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$. Then $M = \mathrm{Hom}_{\mathrm{gr}}(T_N, \mathbb{C}^*)$ is the character group of T_N. For $m \in M$ we denote $\mathbf{e}(m)$ the corresponding character of T_N. Let Δ be a complete finite fan of N consisting of strongly convex rational polyhedral cones σ, that is, there exist a finite number of elements $v_1, v_2, \ldots, v_s$ in N such that

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_s,$$

and $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N\mathrm{emb}(\Delta) := \cup_{\sigma \in \Delta} U_\sigma$ of dimension n (see Section 1.2 [14], or Section 1.4 [5]). Here $U_\sigma = \mathrm{Spec}\ \mathbb{C}[\sigma^\vee \cap M]$ and $\sigma^\vee$ is the dual cone of σ with respect to the pairing $\langle , \rangle$. For the origin $\{0\}$, the affine open set $U_{\{0\}} = \mathrm{Spec}\ \mathbb{C}[M]$ is the unique dense T_N-orbit. We note that a toric variety is always normal.

Let L be an ample T_N-equivariant invertible sheaf on X. Then the polarized variety (X, L) corresponds to an integral convex polytope P in $M_{\mathbb{R}}$ of dimension n. We call the convex hull $\mathrm{Conv}\{u_0, u_1, \ldots, u_r\}$ in $M_{\mathbb{R}}$ of a finite subset $\{u_0, u_1, \dots, u_r\} \subset M$ an *integral convex polytope* in $M_{\mathbb{R}}$. The correspondence is given by the isomorphism

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\mathbf{e}(m), \tag{1.1}$$

where $\mathbf{e}(m)$ are considered as rational functions on X because they are functions on the open dense subset T_N of X (see Section 2.2 [14], or Section 3.5 [5]).

Let P_1 and P_2 be integral convex polytopes in $M_{\mathbb{R}}$. Then we can consider the Minkowski sum $P_1 + P_2 := \{x_1 + x_2 \in M_{\mathbb{R}}; x_i \in P_i \quad (i = 1, 2)\}$ and the multiplication by scalars $rP_1 := \{rx \in M_{\mathbb{R}}; x \in P_1\}$ for a positive real number r. If l is a natural number, then lP_1 coincides with the l times sum of P_1, i.e., $lP_1 = \{x_1 + \cdots + x_l \in M_{\mathbb{R}}; x_1, \dots, x_l \in P_1\}$. The l-th tensor power $L^{\otimes l}$ corresponds to the convex polytope $lP := \{lx \in M_{\mathbb{R}}; x \in P\}$. Moreover the multiplication map

$$H^0(X, L^{\otimes l}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (l+1)}) \tag{1.2}$$

transforms $\mathbf{e}(u_1) \otimes \mathbf{e}(u_2)$ for $u_1 \in lP \cap M$ and $u_2 \in P \cap M$ to $\mathbf{e}(u_1 + u_2)$ through the isomorphism (1.1). Therefore the equality $(lP \cap M) + (P \cap M) = (l+1)P \cap M$ is equivalent to the surjectivity of the multiplication map (1.2).

Proposition 1.1 (Nakagawa-Ogata [13])**.** *Let X be a projective toric variety of dimension n and L an ample line bundle on X. Then the multiplication map*

$$H^0(X, L^{\otimes i}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (i+1)})$$

is surjective for all $i \geq n - 1$.

This implies that $L^{\otimes d}$ satisfies Property N_0 for $d \geq n-1$. By employing an analogous method of Mumford [12] we obtained that $L^{\otimes d}$ satisfies Property N_1 for $d \geq n$ (see Corollary 2.2 in [13]). Schenck and Smith [17] generalizes for $L^{\otimes d}$ to satisfy Property N_p for $d \geq n - 1 + p$.

2. Toric Surfaces

Ogata[16] generalize the result of [11] to higher dimension by using the method of Fujita's regular ladder [4]. In this section we use the same method in the case of dimension two.

Let X be a projective toric surface and L an ample line bundle on X. We consider a general hyperplane section C of the linear system $|L|$. Since X is normal, we may assume that C is nonsingular. Set $L_C = L|C$, the restriction of L to the curve C. From easy calculation, we see that

$$h^0(C, L_C) = h^0(X, L) - 1 = {}^\sharp P \cap M - 1, \tag{2.1}$$

$$\begin{array}{lrcl}(2.2) & h^1(C,\mathcal{O}_C) & = & h^2(X,L^{-1}) = h^0(X,\omega_X\otimes L) = {}^\sharp\mathrm{Int}\ P\cap M,\\ (2.3) & h^1(C,L_C) & = & 0.\end{array}$$

Denote by $g = h^1(C,\mathcal{O}_C)$ the genus of C. Then from Riemann-Roch Theorem we have $\deg L_C = 2g-2+{}^\sharp\partial P\cap M$.

Lemma 2.1 (Green [6], (4.a.1)). *Let L be a line bundle of degree $2g+1+p$ ($p \geq 0$) on a smooth irreducible projective curve of genus g. Then L satisfies Property N_p.*

This is a generalization of Mumford [12] (the case $p = 0$) and of Fujita [4] (the case $p = 1$).

Lemma 2.2 (Green [6], (3.b.7)). *Let X be a compact complex manifold, L a line bundle on X, $Y \subset X$ a connected hypersurface in the linear system $|L|$ and L_Y denote the restriction of L to Y. Assume that*

$$H^1(X,L^{\otimes q}) = 0 \quad \text{for all } q \geq 0.$$

Then $\mathcal{K}_{p,q}(X,L) \cong \mathcal{K}_{p,q}(Y,L_Y)$ for all p,q.

Since $H^1(X,L^{\otimes q}) = 0$ for $q \geq 0$ and for any ample ample line bundle L on a toric variety X, this lemma implies that if L_Y satisfies Property N_p for a general smooth hypersurface Y in $|L|$, then L also satisfies Property N_p.

Proposition 2.3. *Let A be an ample line bundle on a projective toric surface X corresponding to an integral convex polygon Q in $M_\mathbb{R}$ given by the isomorphism*

$$H^0(X,A) \cong \bigoplus_{m\in Q\cap M} \mathbb{C}\mathbf{e}(m).$$

Then the d-th tensor power $A^{\otimes d}$ satisfies Property N_p for $p \leq d\ {}^\sharp\mathrm{Int}\ Q\cap M-3$. In particular, $A^{\otimes d}$ satisfies Property N_p for $p \leq 3d-3$.

Proof. Set $L = A^{\otimes d}$. Let C be a general hyperplane section of $|L|$. Set $L_C = L|C$. Denote by g the genus of C. Then we have

$$\begin{aligned}\deg L_C &= 2g-2+d\ {}^\sharp\mathrm{Int}\ Q\cap M\\ &= 2g+1+(d\ {}^\sharp\mathrm{Int}\ Q\cap M-3).\end{aligned}$$

From Lemma 2.1, the bundle L_C satisfies Property N_p for $p \leq d\ {}^\sharp\mathrm{Int}\ Q\cap M-3$. From Lemma 2.2 we obtain a proof of Proposition. □

This is a generalization of the case $(X,A) = (\mathbb{P}^2,\mathcal{O}(1))$ treated by Birkenhake in [1].

3. Higher Dimension

Lemma 3.1 (Ogata [15]). *Let A be an ample line bundle on a projective toric variety of dimension n ($n \geq 3$). Then $A^{\otimes d}$ satisfies Property N_1 for $d \geq n-1$.*

A very ample invertible sheaf L on a projective variety X defines an embedding $\Phi_L : X \to \mathbb{P}(H^0(X,L)) = \mathbb{P}^r$. Set $M_L := \Phi_L^* \Omega^1_{\mathbb{P}^r}(1)$ so that there exists the following exact sequence of vector bundles

$$0 \to M_L \to H^0(X,L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0. \tag{3.1}$$

Lemma 3.2 (Ein-Lazarsfeld [2]). *Assume that L is very ample and that $H^1(X, L^{\otimes k}) = 0$ for all $k \geq 1$. Then L satisfies Property N_p if and only if*

$$H^1(X, \wedge^a M_L \otimes L^{\otimes b}) = 0 \quad \textit{for } 1 \leq a \leq p+1 \textit{ and } b \geq 1.$$

Since in characteristic zero $\wedge^a M_L$ is a direct summand of $M_L^{\otimes a}$, we have only to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ as in [2] and [8].

Proposition 3.3. *Let $p \geq 2$ be an integer. Let A be an ample line bundle on a projective toric variety X of dimension n ($n \geq 3$). Then $A^{\otimes d}$ satisfies Property N_p for $p \leq n-2-d$.*

For a proof we have to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ for $1 \leq a \leq p+1$ and $b \geq 1$ with $L = A^{\otimes d}$. We need a lemma.

Lemma 3.4 (Mumford [12]). *Let F be a coherent sheaf on a projective algebraic variety X. Let A be a line bundle on X generated by global sections. If $H^i(X, F \otimes A^{\otimes(-i)}) = 0$ for all $i \geq 1$, then the multiplication map*

$$H^0(X, F \otimes A^{\otimes j}) \otimes H^0(X,A) \to H^0(X, F \otimes A^{\otimes(j+1)})$$

is surjective for all $j \geq 0$.

For a proof see Theorem 2 in [12].

Proof of Proposition 3.3. Let $q \geq 2$ be an integer and $L = A^{\otimes d}$ with $d \geq n+q-2$. We want to show that

$$H^1(X, M_L^{\otimes q} \otimes A^{\otimes j}) = 0 \quad \text{for } j \geq n+q-3, \tag{3.2}$$

$$H^i(X, M_L^{\otimes q} \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 2 \text{ and } j \geq 0. \tag{3.3}$$

From Proposition 1.1 and the exact sequence (3.1) we see that

$$H^i(X, M_L \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 1,\ j \geq 0 \text{ and } d \geq n-1. \tag{3.4}$$

First we shall show the vanishing of (3.2) for $q = 2$. Taking tensor product of (3.1) with $M_L \otimes A^{\otimes j}$ we obtain an exact sequence

$$0 \to M_L^{\otimes 2} \otimes A^{\otimes j} \to H^0(X,L) \otimes_{\mathbb{C}} M_L \otimes A^{\otimes j} \to M_L \otimes L \otimes A^{\otimes j} \to 0. \tag{3.5}$$

From (3.4) and (3.5) we have

$$H^i(X, M_L^{\otimes 2} \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 2,\ j \geq 0 \text{ and } d \geq n-1. \tag{3.6}$$

Second we shall show the vanishing of (3.2) for $q = 2$. From Lemmas 3.1 and 3.2, we see that $H^1(X, \wedge^2 M_L \otimes L^{\otimes b}) = 0$ for $b \geq 1$ and $d \geq n-1$. Taking wedge product in (3.1) and twisting by $L^{\otimes b}$, we obtain an exact sequence

$$0 \to \wedge^2 M_L \otimes L^{\otimes b} \to \wedge^2 H^0(X, L) \otimes_{\mathbb{C}} L^{\otimes b} \to M_L \otimes L^{\otimes (b+1)} \to 0. \tag{3.7}$$

The vanishing of the first cohomology group implies the surjectivity of the map

$$\wedge^2 H^0(X, L) \otimes H^0(X, L^{\otimes b}) \to H^0(X, M_L \otimes L^{\otimes (b+1)}). \tag{3.8}$$

Taking global sections of (3.7) with $b = 0$ we see that the surjective map (3.8) factors through $\wedge^2 H^0(X, L) \to H^0(X, M_L \otimes L)$. Thus we have that

$$H^1(X, M_L^{\otimes 2} \otimes L^{\otimes b}) = 0 \quad \text{for } b \geq 1 \text{ and } d \geq n-1 \tag{3.9}$$

from the exact sequence (3.5) replacing A by L. For line bundles L_1 and L_2, denote by $R(L_1, L_2)$ the kernel of the multiplication map $H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2)$. By taking a global section of the exact sequence

$$0 \to M_L \otimes A \to \Gamma(X, L) \otimes_{\mathbb{C}} A \to L \otimes A \to 0,$$

we have $H^0(X, M_L \otimes A) = R(L, A)$. Then we can rewrite the sequence (3.5) after taking its global sections as

$$\begin{array}{ccc} H^0(X, L) \otimes H^0(X, M_L \otimes A^{\otimes j}) & \longrightarrow & H^0(X, M_L \otimes L \otimes A^{\otimes j}) \\ \| & & \| \\ H^0(X, L) \otimes R(A^{\otimes j}, L) & \longrightarrow & R(L \otimes A^{\otimes j}, L). \end{array}$$

For simplicity we denote $A^{\otimes i}$ as A^i and $H^0(X, L)$ as $\Gamma(L)$. From Corollary 2.2 in [13], we have that

$$\Gamma(A^i) \otimes R(A^d, A^j) \to R(A^{d+i}, A^j)$$

is surjective for $d \geq n$, $i \geq 1$ and $j \geq 1$. Hence we showed the vanishing of (3.2) for $j \geq n$. We remain to show the vanishing of $H^1(X, M_L^{\otimes 2} \otimes A^{\otimes (n-1)})$

for $d \geq n$. Consider the diagram

$$
\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & R(A^{n-1}, A^{n-1}) \otimes \Gamma(A^d) & \xrightarrow{\alpha} & R(A^{n-1}, A^{d+n-1}) & \\
 & & & \downarrow & & \downarrow & \\
0 \to & \Gamma(A^{n-1}) \otimes R(A^{n-1}, A^d) & \longrightarrow & \Gamma(A^{n-1})^{\otimes 2} \otimes \Gamma(A^d) & \longrightarrow & \Gamma(A^{n-1}) \otimes \Gamma(A^{d+n-1}) & \to 0 \\
 & \downarrow \beta & & \downarrow & & \downarrow & \\
0 \to & R(A^{2n-2}, A^d) & \longrightarrow & \Gamma(A^{2n-2}) \otimes \Gamma(A^d) & \longrightarrow & \Gamma(A^{d+2n-2}) & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}
$$

If α is surjective, then β is also surjective. The vanishing (3.9) implies the surjectivity of

$$R(A^{n-1}, A^{n-1}) \otimes \Gamma(A^{n-1}) \to R(A^{n-1}, A^{2n-2}).$$

Since $\Gamma(A^i) \otimes \Gamma(A^{n-1}) \to \Gamma(A^{n+i-1})$ is surjective for $i \geq 1$, the map α is surjective, hence,

$$\Gamma(A^d) \otimes R(A^{n-1}, A^d) \to R(A^{d+n-1}, A^d)$$

is surjective. This map is the same as

$$\Gamma(L) \otimes \Gamma(M_L \otimes A^{n-1}) \to \Gamma(M_L \otimes L \otimes A^{n-1})$$

with $L = A^d$. Thus we obtain that $H^1(X, M_L^{\otimes 2} \otimes A^{n-1}) = 0$.

For $q \geq 2$ and $L = A^d$ with $d \geq n + q - 2$, if we have the equalities (3.2) and (3.3), then from the exact sequence

$$0 \to M_L^{\otimes(q+1)} \otimes A^j \to \Gamma(L) \otimes M_L^{\otimes q} \otimes A^j \to M_L^{\otimes q} \otimes L \otimes A^j \to 0 \tag{3.10}$$

we have

$$H^i(X, M_L^{\otimes(q+1)} \otimes A^j) = 0 \quad \text{for } i \geq 2 \text{ and } j \geq 0,$$

and from Lemma 3.4 we see the surjectivity of

$$\Gamma(M_L^{\otimes q} \otimes A^j) \otimes \Gamma(A) \to \Gamma(M_L^{\otimes q} \otimes A^{j+1})$$

for $j \geq n+q-2$. From this and the exact sequence (3.10) we have

$$H^1(X, M_L^{\otimes(q+1)} \otimes A^j) = 0 \quad \text{for } j \geq n+q-2.$$

In particular, we have

$$H^1(X, M_L^{\otimes(q+1)} \otimes L^{\otimes b}) = 0$$

for $b \geq 1$ and $L = A^{\otimes d}$ with $d \geq n+q-2$. □

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