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Paul F. Dubois* Sudarshan K. Sehgal†

*University of Alberta

†University of Alberta

ANOTHER PROOF OF THE INVARIANCE OF ULM'S FUNCTIONS IN COMMUTATIVE MODULAR GROUP RINGS

PAUL F. DUBOIS and SUDARSHAN K. SEHGAL

In this note we give a short and natural proof of the following theorem due to Berman and Mollov [1] and May [2].

Theorem. *Let* Z_pG *be the group ring of a p-primary group* G *over* Z_p, *the field of* p *elements. Suppose* $\theta : Z_pG \simeq Z_pH$. *Then* G *and* H *have the same Ulm's functions.*

The proof is a direct consequence of a lemma of Jennings [3] which we give below. First, we need some notation. We write all groups multiplicatively and define $G^p = \{g \in G \mid g = x^p \text{ for some } x \in G\}$. Inductively, for ordinals β we have

$$G^{p^{\beta+1}} = \left(G^{p^\beta}\right)^p \quad \text{and} \quad G^{p^\beta} = \bigcap_{\alpha<\beta} G^{p^\alpha}$$

for β a limit ordinal.

If K is a subgroup of G, by $\triangle(G;K)$ we mean the ideal of Z_pG generated by elements of the form $1-k$, $k \in K$. Sometimes we write $\triangle(K)$ if the context is clear. We denote $\{x \in K \mid x^p = 1\}$ by $K[p]$.

Lemma. *Let* G *be a p-primary abelian group and* N *a subgroup. Then*

(1) $G/G^p \simeq \triangle(G)/\triangle^2(G)$, *and*

(2) $N/N^p \simeq \triangle(G;N)/\triangle(G)\cdot\triangle(G;N)$.

Proof. Define $\lambda : G \to \triangle(G)/\triangle^2(G)$ by $\lambda(g) = g-1+\triangle^2(G)$. Since

(*) $g_1g_2-1 = (g_1-1)+(g_2-1)+(g_1-1)(g_2-1)$,

λ is an epimorphism with $\text{kernel} = \{g \in G \mid g-1 \in \triangle^2(G)\} = G^p$ by Jennings [3], proving (1). Actually, Jennings proved this equality for finite groups but since in an equation $g-1 = \delta \in \triangle^2(G)$, only a finite number of elements of G occur, his result is applicable to our case.

For the second part, define

$$\mu : N \to \triangle(G;N)/\triangle(G)\cdot\triangle(G;N) \text{ by } \mu(n) = \overline{n-1}.$$

It follows from (*) and

$$g(n-1)=n-1+(g-1)(n-1)$$

that μ is an epimorphism with kernel $=\{n \mid n-1 \in \triangle(G)\cdot\triangle(G;N)\}$. It remains to prove:

(**) $$n-1\in\triangle(G)\cdot\triangle(G;N) \Rightarrow n\in N^p.$$

Choose a transversal $\{g_i\}$ of N in G with $g_1=1$. Define for $g_i n\in G$, $\sigma(g_i n)=n$ and extend this linearly to $\sigma: Z_pG\rightarrow Z_pN$. Now,

$$n-1=\sum_i \gamma_i(n_i-1),\ \gamma_i\in\triangle(G),\ n_i\in N.$$

Therefore

$$n-1=(n-1)^\sigma=\sum_i \gamma_i^\sigma(n_i-1) \text{ and } n-1\in\triangle^2(N;N).$$

Hence, $n\in N^p$. This proves (**) and therefore (2).

Remark. The above lemma is a special case of a similar result that holds for arbitrary (not necessarily abelian or finite) groups. Also, there is a corresponding result for integral group rings (see, Sehgal [4]). For the purpose of this paper the above will suffice.

Proof of Theorem. Now, suppose $\theta: Z_pG\simeq Z_pH$. We may assume here that θ is normalized; if $\theta(g)=\sum_{h\in H}\alpha_h h$ then $\sum_{h\in H}\alpha_h=1$. By noting that $\theta(g^p)=(\sum_{h\in H}\alpha_h h)^p=\sum_{h\in H}\alpha_h^p h^p$, we have that θ maps Z_pG^p isomorphically onto Z_pH^p. By a simple induction

$$\theta: Z_pG^{p^\beta}\simeq Z_pH^{p^\beta} \quad \text{for all ordinals } \beta.$$

We show first that the finite Ulm invariants are equal. The ith Ulm invariant, $i<\omega$ (the first limit ordinal), is the dimension of $(G^{p^i})[p]/(G^{p^{i+1}})[p]$. For convenience let us denote $(G^{p^i})[p]$ by L_i.

By the lemma we have an isomorphism

$$L_i\simeq\triangle(G;L_i)/\triangle(G)\cdot\triangle(G;L_i).$$

Under θ, $\triangle(L_i)$ is isomorphic to $\triangle(M_i)$ where $M_i=(H^{p^i})[p]$. Thus we obtain for each i the commutative diagram below:

$$\begin{array}{ccccccc}
L_i & \simeq & \triangle(L_i)/\triangle(G)\triangle(L_i) & \simeq & \triangle(M_i)/\triangle(H)\triangle(M_i) & \simeq & M_i \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
L_{i+1} & \simeq & \triangle(L_{i+1})/\triangle(G)\triangle(L_{i+1}) & \simeq & \triangle(M_{i+1})/\triangle(H)\triangle(M_{i+1}) & \simeq & M_{i+1}
\end{array}$$

and thus $L_i/L_{i+1}\simeq M_i/M_{i+1}$.

The observation that $Z_pG^{p^\beta}\simeq Z_pH^{p^\beta}$ allows us to conclude that even the transfinite Ulm invariants are equal.

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UNIVERSITY OF ALBERTA

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