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# ON THE ATTACHING MAP IN THE STIEFEL MANIFOLD OF 2-FRAMES

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**0. Introduction.** Let $\mathbf{F}=\mathbf{R}$(real), $\mathbf{C}$(complex) or $\mathbf{H}$(quaternionic) and $d=\dim_{\mathbf{R}}\mathbf{F}$. Let $\iota_n\in\pi_n(S^n)$ be the identity map, $\eta_n\in\pi_{n+1}(S^n)$ for $n\geq 2$ and $\nu_n\in\pi_{n+3}(S^n)$ for $n\geq 4$ the Hopf maps. Throughout the paper $O_{n,k}(\mathbf{F})$ stands for the Stiefel manifold consisting of orthonormal $k$-frames in $\mathbf{F}^n$, $Q_{n,k}(\mathbf{F})\subset O_{n,k}(\mathbf{F})$ does for the stunted quasiprojective space and $Q_{2n+1,2}(\mathbf{F})=S^{2dn-1}\cup_{\omega_n(\mathbf{F})}e^{d(2n+1)-1}$, where $\omega_n(\mathbf{R})=2\iota_{2n-1}$, $\omega_n(\mathbf{C})=\eta_{4n-1}$ and $\omega_n(\mathbf{H})=(2n+1)\nu_{8n-1}$. We have a cellular decomposition :

$$O_{2n+1,2}(\mathbf{F})=Q_{2n+1,2}(\mathbf{F})\cup_{\gamma_n(\mathbf{F})}e^{(4n+1)d-2}.$$

The purpose of the present note is to determine the $(d-k)$-fold suspension $\Sigma^{d-k}\gamma_n(\mathbf{F})\in\pi_{(4n+2)d-k-3}(\Sigma^{d-k}Q_{2n+1,2}(\mathbf{F}))$ for $0\leq k\leq d$. We shall freely use the notation and results of [16], [10] and [11]. We shall also use the EHP-sequences and the information about the (relative) Whitehead products $[\ ,\ ]$. We denote by $\#\alpha$ the order of $\alpha$. Our result is stated as follows.

**Theorem 1.** i) $\#\Sigma^d\gamma_n(\mathbf{F})=2$ *and* $\#\Sigma\gamma_n(\mathbf{C})=2$. ii) $\#\Sigma^k\gamma_1(\mathbf{H})=2$ *for* $1\leq k\leq 3$ ; $\#\Sigma^k\gamma_n(\mathbf{H})=8$ *for* $n\geq 2$ *and* $k=1$ *or* $2$ ; $\#\Sigma^3\gamma_n(\mathbf{H})=4$ *for* $n\geq 2$.

**Theorem 2.** $\pi_{(4n+1)d-3}(X)\cong K\{\gamma_n(\mathbf{F})\}\oplus\pi_{(4n+1)d-3}(W)$, *where* $X=Q_{2n+1,2}(\mathbf{F})$, $W=O_{2n+1,2}(\mathbf{F})$ *and* $K=\mathbf{Z}$ *if* $d\neq 1$ *or* $d=n=1$ ; $K=\mathbf{Z}_8$ *if* $d=1$ *and* $n=3$ *or* $n\geq 5$ ; $K=\mathbf{Z}_4$ *if* $d=1$ *and* $n=2$ *or* $4$.

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The paper is organized as follows. § 1 is devoted to prepare some lemmas due to James and Toda. § 2 is to summarize the behavior of the $J$-image of the characteristic element for $O_{2n+1,2}(\mathbf{F})$. §§ 3—5 are devoted to prove the theorems and to determine the generalized Hopf invariant of $\gamma_n(\mathbf{R})$.

**1. Some results of James and Toda.** Let $X=S^q\cup_\alpha e^n$ for $q\leqq n-1$ and $B=X\cup_\gamma e^{n+q}$, where $B$ is regarded as the $q$-sphere bundle over $S^n$ [5].

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Let $i: S^q \to X$, $j: (X, *) \to (X, S^q)$ be the inclusions and $p: (X, S^q) \to (S^n, *)$ a map collapsing $S^q$ to the base point. Let $\varkappa = \varkappa_n: (CS^{n-1}, S^{n-1}) \to (X, S^q)$ be a characteristic map, where $CS^m$ is a cone on $S^m$. By (5.1) of [6] and (3.3) of [2], we have

$$(1) \qquad j_*\gamma = (-1)^{nq}[\iota_q, \varkappa].$$

By Lemma 4.4.3 of [1] and by Lemma 2.32 and Corollary 3.6 of [15], we have the following

**Lemma 1.** *Let $\beta \in \pi_{n-1}(SO_{q+1})$ be the characteristic element for $B$ and $\theta \in \pi_{n+q}(S^{q+1})$ an element obtained from $\beta$ by the Hopf construction. Then $\Sigma\gamma = \pm(\Sigma i)_*\theta$ and $H(\theta) = \pm\Sigma^{q+1}\alpha$.*

We denote by $\hat{\alpha} \in \pi_{k+1}(CS^n, S^n)$ for $\alpha \in \pi_k(S^n)$ an element satisfying $\partial'\hat{\alpha} = \alpha$, where $\partial': \pi_{k+1}(CS^n, S^n) \to \pi_k(S^n)$ is the boundary isomorphism. We denote by $\Sigma': \pi_r(X, S^q) \to \pi_{r+1}(\Sigma X, S^{q+1})$ the relative suspension homomorphism [15]. By Theorem 2.1 of [3], we have an exact sequence for $t = n+2q+3k-2$ $(k \geq 0)$:

$$(2) \qquad \begin{aligned} &\pi_t(\Sigma^k X, S^{q+k}) \xrightarrow{(\Sigma^k p)_*} \cdots \to \pi_r(\Sigma^k X, S^{q+k}) \xrightarrow{(\Sigma^k p)_*} \\ &\pi_r(S^{n+k}) \xrightarrow{H'} \pi_{r-n-k}(S^{q+k}) \xrightarrow{Q} \pi_{r-1}(\Sigma^k X, S^{q+k}) \to \cdots, \end{aligned}$$

where $H' = (\Sigma^k\alpha)_*\Sigma^{-n-k}H$ and $Q(\ \ ) = [\ , (\Sigma')^k\varkappa]$.

**Lemma 2.** i) $\operatorname{Ker}\{(\Sigma^k i)_*: \pi_r(S^{q+k}) \to \pi_r(\Sigma^k X)\} = (\Sigma^k\alpha)_*\pi_r(S^{n+k-1})$ *for $r = n+q+k-1$ if $k = 0$ or $k \geq 2$.*

ii) $\operatorname{Ker}(\Sigma i)_* = \{[\iota_{q+1}, \Sigma\alpha]\} + (\Sigma\alpha)_*\pi_{n+q}(S^n)$.

*Proof.* i) for $k = 0$ is just (3.2) of [6]. Recall $\operatorname{Ker}(\Sigma^k i)_* = \operatorname{Im}\partial$, where $\partial: \pi_{r+1}(\Sigma^k X, S^{q+k}) \to \pi_r(S^{q+k})$ is the connecting map. Since $\pi_{r+1}(\Sigma^k X, S^{q+k}) \cong \pi_{r-1}(S^{n+k})$ for $k \geq 2$ and $\partial((\Sigma')^k\varkappa \circ \hat{\beta}) = \Sigma^k\alpha \circ \beta$ for $\beta \in \pi_t(S^{n+k-1})$, we have the assertion for $k \geq 2$.

By (2), we have $\pi_{n+q+1}(\Sigma X, S^{q+1}) = \mathbf{Z}\{[\iota_{q+1}, \Sigma'\varkappa]\} \oplus \pi_{n+q+1}(S^{n+1})$. This leads us to ii) and completes the proof.

As is well known, we have the following

**Remark 1.** i) Let $G$ be a group generated by $\Delta(\iota_{4n-1}) = [\iota_{2n-1}, \iota_{2n-1}]$. Then $G = 0$ if $n = 1, 2$ or $4$ and $G = \mathbf{Z}_2$ if otherwise. We have a short exact sequence

(3) $$0 \to G\{\Delta(\iota_{4n-1})\} \hookrightarrow \pi_{4n-3}(S^{2n-1}) \xrightarrow{\Sigma} \pi_{4n-2}(S^{2n}) \to 0$$

which is split if $n = 1, 2, 4$ or $n$ is not a power of 2.

ii) We have

(4) $$\pi_{4n-1}(S^{2n}) \cong \mathbf{Z}\{\Delta(\iota_{4n-1})\} \oplus \Sigma\pi_{4n-2}(S^{2n-1}) \text{ for } n = 3 \text{ or } n \geq 5.$$

By Propositions 2.7 and 2.2 of [16], $H(\Delta(\iota_{4n+1})) = \pm 2\iota_{4n-1}$ and $H(2\iota_{2n} \circ \Delta(\iota_{4n+1})) = \Sigma(2\iota_{2n-1} \wedge 2\iota_{2n-1}) \circ H(\Delta(\iota_{4n+1})) = \pm 8\iota_{4n-1}$. So, by (4), we have $[\iota_{2n}, 2\iota_{2n}] \notin (2\iota_{2n})_*\pi_{4n-1}(S^{2n})$. By this and [12], we have the following

**Remark 2.** $[\iota_{q+1}, \Sigma\alpha] \notin (\Sigma\alpha)_*\pi_{n+q}(S^n)$ for some $n$, where $X = Q_{2n+1,2}(\mathbf{F})$, $\alpha = \omega_n(\mathbf{F})$ and $q = 2dn-1$.

Let $\mathbf{R}P^n$ be the real $n$-dimensional projective space and $\mathbf{R}P^n_k = \mathbf{R}P^n/\mathbf{R}P^{k-1}$ the stunted space.

**Lemma 3.** *$\pi_{(4n+1)d-3}(X, S^{2dn-1}) \cong \pi_{(4n+1)d-3}(S^{(2n+1)d-1}) \oplus L\{[\iota_{2dn-1}, \varkappa]\}$, where $X = Q_{2n+1,2}(\mathbf{F})$, $\varkappa = \varkappa_{(2n+1)d-1}$ and $L = \mathbf{Z}$ if $d \neq 1$ or $d = n = 1$; $L = \mathbf{Z}_4$ if $d = 1$ and $n = 3$ or $n \geq 5$; $L = \mathbf{Z}_2$ if $d = 1$ and $n = 2$ or $4$.*

*Proof.* First we shall give a proof in the real case. By (2), we have an exact sequence for $n \geq 2$:

$$\pi_{4n-1}(S^{2n}) \xrightarrow{H'} \pi_{2n-1}(S^{2n-1}) \xrightarrow{Q} \pi_{4n-2}(X, S^{2n-1}) \xrightarrow{p_*} \pi_{4n-2}(S^{2n}) \to 0.$$

By (4), $\operatorname{Im} H' \cong 4\mathbf{Z}$ for $n = 3$ or $n \geq 5$. $\operatorname{Im} H' \cong 2\mathbf{Z}$ for $n = 2$ or $4$. So we have a short exact sequence

(5) $$0 \to L \hookrightarrow \pi_{4n-2}(X, S^{2n-1}) \xrightarrow{p_*} \pi_{4n-2}(S^{2n}) \to 0.$$

We set $m = 2n-1$ and $\alpha = \Delta(\iota_{2m+1})$. By (2.18) of [15], we have $\hat{\alpha} = [\iota_m, \hat{\iota}_m]$, where $\hat{\iota}_m$ coincides with the identity map of $(CS^m, S^m)$. So, by (2.16–18) of [15] or by (3.4–6) of [2], we have $\varkappa\hat{\alpha} = \varkappa_*[\iota_m, \hat{\iota}_m] = [2\iota_m, \varkappa] = 2[\iota_m, \varkappa]$. Let $\beta \in \pi_{2m-1}(S^m)$ be an element such that $\#\beta = \#\Sigma\beta$. Then $p_*(\varkappa \circ \hat{\beta}) = \Sigma\beta$ and $\#(\varkappa \circ \hat{\beta}) = \#\Sigma\beta$. Therefore, if (3) is split, so is (5).

Suppose that (3) is not split. Then there exists an element $\beta \in \pi_{2m-1}(S^m)$ such that $2\beta = \Delta(\iota_{2m+1})$ and $\#\Sigma\beta = 2$. Since $2(\varkappa\hat{\beta}) = \varkappa\hat{\alpha} = 2[\iota_m, \varkappa]$, we have $\#\delta = 2$ and $p_*\delta = \Sigma\beta$ for $\delta = \varkappa\hat{\beta} - [\iota_m, \varkappa]$. So (5) is also split in this case. This leads us to the assertion of the real case except for $n = 1$.

We have $X = \mathbf{R}P^2$ and $O_{3,2}(\mathbf{R}) = \mathbf{R}P^3$ if $d = n = 1$. So, by use of the homotopy exact sequence of a pair $(X, S^1)$, we have $\pi_2(X, S^1) \cong \mathbf{Z}\{\gamma_1(\mathbf{R})\} \oplus \mathbf{Z}\{\varkappa\}$. Since $p_*\varkappa = \iota_2$ and $j_*\gamma_1(\mathbf{R}) = [\iota_1, \varkappa]$ by (1), we have the splitting of (5).

For $d = 2$ or 4, we have, by (2), a short exact sequence for $r = (4n+1)d-3$ :

$$0 \to \pi_{2dn-1}(S^{2dn-1}) \xrightarrow{Q} \pi_r(X, S^{2dn-1}) \xrightarrow{p_*} \pi_r(S^{(2n+1)d-1}) \to 0.$$

Since $\Sigma: \pi_{r-1}(S^{(2n+1)d-2}) \to \pi_r(S^{(2n+1)d-1})$ is isomorphic onto, the sequence is split. This completes the proof.

By (11.8) and Theorem 11.7 of [16], we have the following

**Lemma 4.** *There exists a mapping* $\delta: \Sigma^{n-1}\mathbf{R}P_n^{n+k-1} \to S^n$ *such that* Ker $\{\Sigma^k: \pi_i(S^n) \to \pi_{i+k}(S^{n+k})\} = \delta_*\pi_i(\Sigma^{n-1}\mathbf{R}P_n^{n+k-1})$ *for* $i \le 3n-3$. *In the* 2-*components, the assertion holds for* $i \le 4n-4$.

By Proposition 7.10 of [4], $Q_{n,k}(\mathbf{F})$ is a stable retract of $O_{n,k}(\mathbf{F})$. Especially we have $\Sigma^{d+1}\gamma_n(\mathbf{F}) = 0$.

Hereafter, by abuse of notation, we often use the inclusion $i$ and the projection $p$ to denote $\Sigma^r i$ and $\Sigma^s p$ for integers $r$ and $s$, respectively.

Let $\sigma_n \in \pi_{n+7}(S^n)$ for $n \ge 8$ be the Hopf map and $\iota_X$ the identity class of $X = Q_{2n+1,2}(\mathbf{F})$. Then $X \wedge X$ is homotopy equivalent to a mapping cone

$$\Sigma^{2dn-1}X \cup_{\lambda_n(\mathbf{F})} C(\Sigma^{(2n+1)d-2}X),$$

where $\lambda_n(\mathbf{F}) = \iota_X \wedge \omega_n(\mathbf{F})$.

In the 2-components, stable Toda brackets $\langle 2\iota, \eta, 2\iota\rangle$, $\langle \eta, \nu, \eta\rangle$ and $\langle \nu, 8\iota, \nu\rangle$ consist of single elements $\eta^2$, $\nu^2$ and $8\sigma$, respectively. By this and by Lemma 3.5 and Theorem 3.6 of [16] and by their proofs, we have the following

**Lemma 5.** $\lambda_n(\mathbf{R}) = i\eta_{4n-2}p$, $\lambda_n(\mathbf{C}) = 3ai\nu_{8n-2}p$ *and* $\lambda_n(\mathbf{H}) = 15bi\sigma_{16n-2}p - (\Sigma^{8n-1}\tilde{\theta})p$ *for* $n \ge 1$ *and odd integers* $a$ *and* $b$, *where* $\tilde{\theta}$ *is a coextension of* $\theta = 2\Sigma^3\omega_n(\mathbf{H})$ *with respect to* $\omega_n(\mathbf{H})$.

**2. The *J*-image of the characteristic element.** Let $\gamma'_n(\mathbf{F}) \in \pi_{d(n+1)-2}(O_n(\mathbf{F}))$ be the characteristic map [11], where $O_n(\mathbf{F}) = O_n$, $U_n$ or $Sp_n$ according as $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. Let $J: \pi_k(O_n(\mathbf{F})) \to \pi_{k+dn}(S^{dn})$ be the $J$-homomorphism and $j_n(\mathbf{F}) = J(\gamma'_n(\mathbf{F})) \in \pi_{(2n+1)d-2}(S^{dn})$. Then $j_n(\mathbf{F})$ is an element obtained from the characteristic element $\gamma'_n(\mathbf{R})$, $c\gamma'_n(\mathbf{C})$ or $rc\gamma'_n(\mathbf{H})$ by the Hopt construction, where $r: U_n \hookrightarrow SO_{2n}$ and $c: Sp_n \hookrightarrow SU_{2n}$ are the canonical maps. We recall the following relations: $j_n(\mathbf{R}) = \Delta(\iota_{2n+1}) = \pm[\iota_n, \iota_n]$, $\Sigma j_n(\mathbf{C}) = j_{2n+1}(\mathbf{R})$, $\Sigma^2 j_n(\mathbf{H}) = j_{2n+1}(\mathbf{C})$, $H(j_n(\mathbf{C})) = (n-1)\eta_{4n-1}$ and $H(j_n(\mathbf{H})) = \pm(n+1)\nu_{8n-1}$. By Lemma 1, we have

$$\Sigma\gamma_n(\mathbf{F}) = \pm i_* j_{2n}(\mathbf{F}),\ \Sigma^d\gamma_n(\mathbf{F}) = i_*\Delta(\iota_{2d(2n+1)-1}) \text{ and } H(j_{2n}(\mathbf{F})) = \pm\Sigma^{2dn}\omega_n(\mathbf{F}). \tag{6}$$

By [8], [14] and [16], we have

$$\Delta(\eta_{2n+1}) \neq 0 \text{ if and only if } n = 4, 5 \text{ or } n \not\equiv 3 \bmod 4 \text{ and } n \geq 8\ ;\ \Delta(\eta^2_{2n+1}) \neq 0 \text{ if and only if } n = 4 \text{ or } n \equiv 0,\ 1 \bmod 4 \text{ and } n \geq 6. \tag{7}$$

We denote by $(a, b)$ the greatest common divisor of integers $a$ and $b$.

**Lemma 6.** i) *In the 2-component, there exists an element* $\lambda \in \pi_{16n-1}(S^{8n-3})$ *such that* $\pm(2n+1)\Delta(\nu_{16n+1}) = 2j_{2n}(\mathbf{H}) - \Sigma^3\lambda$ *and* $H(\lambda) = \nu^2_{16n-7}$. *There exists* $\lambda' \in \pi_{16n-3}(S^{8n-5})$ *such that* $2\lambda = \Sigma^2\lambda'$ *and* $H(\lambda') \equiv \varepsilon_{16n-11} \bmod \eta_{16n-11}\sigma_{16n-10}$. *We set* $\lambda = \nu_5\sigma_8$ *and* $\lambda' = \pm\varepsilon'$ *for* $n = 1$.

ii) $\# j_{2n}(\mathbf{C}) = 2$ ; $\# j_{2n+1}(\mathbf{C}) = 4$ *and* $2j_{2n+1}(\mathbf{C}) = \Delta(\eta_{8n+5})$ *for* $n \geq 2$ ; $\#\Sigma j_{2n}(\mathbf{H}) = 8$ *and* $4\Sigma j_{2n}(\mathbf{H}) = \Delta(\eta^2_{16n+3})$ ; $\# j_{2n}(\mathbf{H}) = 24/(3, 2n+1)$ *and* $\{12/(3, 2n+1)\} j_{2n}(\mathbf{H}) = j_{4n}(\mathbf{C}) \circ \eta^2_{16n}$.

*Proof.* i) for $n \geq 2$ is obtained from Lemma 11.17 and Proposition 11.15 of [16]. For $n = 1$, the assertion holds [16].

We recall that $\pi_{4n}(SO_{4n}) \cong (\mathbf{Z}_2)^2$ or $(\mathbf{Z}_2)^3$ according as $n$ is odd or even [7]. Since $j_n(\mathbf{C}) = J(r\gamma'_n(\mathbf{C}))$ and $H(j_{2n}(\mathbf{C})) = \eta_{8n-1}$, we have the first of ii).

Since $\pi_6(SO_6) = 0$, we have $j_3(\mathbf{C}) = 0$. We consider an anti-commutative diagram between exact sequences for $n \geq 2$ :

$$\begin{array}{ccccc}
\pi_{4n+3}(S^{4n+2}) & \xrightarrow{\partial} & \pi_{4n+2}(SO_{4n+2}) & \xrightarrow{i_*} & \pi_{4n+2}(SO_{4n+3}) \\
\downarrow\Sigma^{4n+3} & & \downarrow J & & \downarrow J \\
\pi_{8n+6}(S^{8n+5}) & \xrightarrow{\Delta} & \pi_{8n+4}(S^{4n+2}) & \xrightarrow{\Sigma} & \pi_{8n+5}(S^{4n+3}).
\end{array}$$

By [7], $\pi_{4n+2}(SO_{4n+2+k}) \cong \mathbf{Z}_{2(2-k)}$ for $k = 0$ or $1$ and $2r\gamma'_{2n+1}(\mathbf{C}) = \partial\eta_{4n+2}$. So we have $2j_{2n+1}(\mathbf{C}) = \Delta(\eta_{8n+5})$. By (7), we have the second of ii).

We recall that $\pi_{8n+2}(SO_{8n}) = \{\gamma'_{8n}(\mathbf{R}) \circ \nu_{8n-1}, rc\gamma'_{2n}(\mathbf{H})\} \cong \mathbf{Z}_{24} \oplus \mathbf{Z}_8$ and $\pi_{8n+2}(SO_{8n+1}) = \mathbf{Z}_8\{r'c\gamma'_{2n}(\mathbf{H})\}$ [11]. $J(\gamma'_{8n}(\mathbf{R}) \circ \nu_{8n-1}) = j_{8n}(\mathbf{R}) \circ \nu_{16n-1} = \pm\Delta(\nu_{16n+1})$, $J(rc\gamma'_{2n}(\mathbf{H})) = j_{2n}(\mathbf{H})$ and $J(r'c\gamma'_{2n}(\mathbf{H})) = \Sigma j_{2n}(\mathbf{H})$. By Theorem 4 of [11], $\{12/(3,2n+1)\}j_{2n}(\mathbf{H}) = j_{4n}(\mathbf{C}) \circ \eta^2_{16n}$ and $\{12/(3,2n+1)\}\Sigma j_{2n}(\mathbf{H}) = \Sigma j_{4n}(\mathbf{C}) \circ \eta^2_{16n+1} = \Delta(\eta^2_{16n+3})$. By (7) and an EHP-sequence

$$\pi_{16n+5}(S^{16n+3}) \xrightarrow{\Delta} \pi_{16n+3}(S^{8n+1}) \xrightarrow{\Sigma} \pi_{16n+4}(S^{8n+2}),$$

we have the rest of ii). This completes the proof.

Let $G_k$ be the $k$-th stable homotopy group of spheres. By [14] and Lemma 6, we have the following

**Remark 3.** i) $\pi_{8n}(S^{4n}) \cong G_{4n} \oplus \mathbf{Z}_2\{\Delta(\eta_{8n+1})\} \oplus \mathbf{Z}_2\{j_{2n}(\mathbf{C})\}$; $\pi_{8n-1}(S^{4n-1}) \cong G_{4n} \oplus \mathbf{Z}_2\{\delta\}$, where $H(\delta) = \eta^2_{8n-3}$ and $\Sigma\delta = \Delta(\eta_{8n+1})$.

ii) $\pi_{16n+4}(S^{8n+2}) \cong G_{8n+2} \oplus \mathbf{Z}_4\{j_{4n+1}(\mathbf{C})\}$; $\pi_{16n+3}(S^{8n+1}) \cong G_{8n+2} \oplus \mathbf{Z}_8\{\Sigma j_{2n}(\mathbf{H})\}$; $\pi_{16n+2}(S^{8n}) \cong G_{8n+2} \oplus (\mathbf{Z}_8 \oplus \mathbf{Z}_{24})\{\Delta(\nu_{16n+1}), j_{2n}(\mathbf{H})\}$; $\pi_{16n+2-k}(S^{8n-k}) \cong G_{8n+2} \oplus \mathbf{Z}_8\{\Sigma^{3-k}\lambda\}$ for $k = 1$ or $2$.

**3. The complex or quaternionic case.** Hereafter we set $X = Q_{2n+1,2}(\mathbf{F})$ and $\gamma = \gamma_n(\mathbf{F})$.

**Proposition 7.** $\#\Sigma\gamma = \#\Sigma^2\gamma = 2$ *for* $\mathbf{F} = \mathbf{C}$.

*Proof.* By (6), $\Sigma^2\gamma = i_*\Delta(\iota_{8n-3})$. Assume that $\Sigma^2\gamma = 0$. Then, by Lemma 2, there exists an element $\beta \in \pi_{8n-1}(S^{4n})$ satisfying $\Delta(\iota_{8n+3}) = \eta_{4n+1} \circ \Sigma^2\beta$. So we have $j_{2n}(\mathbf{C}) = \eta_{4n} \circ \Sigma\beta + a\Delta(\eta_{8n+1})$ for $a = 0$ or $1$. Apply $H$ to this relation. Then $\eta_{8n-1} = 0$ and this is a contradiction. Therefore $\Sigma^2\gamma \neq 0$ and $\#\Sigma^2\gamma = 2$.

By (6) and Lemma 6, $2\Sigma\gamma = 2i_*j_{2n}(\mathbf{C}) = 0$. So we have $\#\Sigma\gamma = 2$. This completes the proof.

Hereafter in this section, we shall deal with the quaternionic case.

**Lemma 8.** $\#\Sigma\gamma = \#\Sigma^2\gamma$ *and* $8\Sigma\gamma = 0$.

*Proof.* In an EHP-sequence

$$\pi_{16n+4}(\Sigma(\Sigma X \wedge \Sigma X)) \xrightarrow{\Delta} \pi_{16n+2}(\Sigma X) \xrightarrow{\Sigma} \pi_{16n+3}(\Sigma^2 X),$$

the left group is isomorphic to $\pi_{16n+4}((S^{16n+1} \cup_{(2n+1)\nu_{16n+1}} e^{16n+5}) \vee S^{16n+5}) \cong \mathbf{Z}_{(24,2n+1)}\{i\nu_{16n+1}\}$ by Lemma 5. Hence $\Sigma$ is monomorphic if $2n+1 \equiv 1$ or $2$ mod 3 and so is on the 2-component if $2n+1 \equiv 0$ mod 3. By Lemma 6, $\#\Sigma j_{2n}(\mathbf{H}) = 8$ and $8j_{2n}(\mathbf{H}) = 0$ if $2n+1 \equiv 0$ mod 3. Therefore we have $8\Sigma\gamma = 0$.

This completes the proof.

**Proposition 9.** i) $\#\Sigma^4\gamma = 2$.
ii) $\#\Sigma^k\gamma = 2$ *for* $n = 1$ *and* $1 \leq k \leq 3$.
iii) $\#\Sigma^3\gamma = 4$ *for* $n \geq 2$.

*Proof.* By use of the homotopy exact sequence of a pair $(\Sigma^5X, S^{8n+4})$, we have $\pi_{16n+7}(\Sigma^5X) \cong \mathbf{Z}\{i\Delta(\iota_{16n+9})\} \oplus K$, where $K$ is a finite abelian group. In an EHP-sequence

$$\begin{array}{ccccc} \pi_{16n+7}(\Sigma^5X) & \xrightarrow{H} & \pi_{16n+7}(\Sigma(\Sigma^4X\wedge\Sigma^4X)) & \xrightarrow{\Delta} & \pi_{16n+5}(\Sigma^4X), \\ \cup\!\!\!\!\oplus & & \wr\| & & \\ \mathbf{Z}\{i\Delta(\iota_{16n+9})\} & & \pi_{16n+7}(S^{16n+7}) & & \end{array}$$

$H(i\Delta(\iota_{16n+9})) = \pm 2i\iota_{16n+7}$. So we have $\#(i\Delta(\iota_{16n+7})) = 2$. So, by (6), we have i).

By i) of Lemma 6, $2j_2(\mathrm{H}) \equiv 3\nu_8\circ\sigma_{11} \pm 3\Delta(\nu_{17})$. So, by Lemma 8 and its proof, $2i_*j_2(\mathrm{H}) = 0$ and $2\Sigma^k\gamma = 0$ for $n = 1$ and $1 \leq k \leq 3$. So, by i), we have ii).

By Lemmas 6, 8 and i), $\#\Sigma^3\gamma = 2$ or $4$. Assume that $2\Sigma^3\gamma = 2i_*\Sigma^2j_{2n}(\mathrm{H}) = 0$. Then, by (4) and Lemma 2, there exists an element $\alpha \in \pi_{16n-2}(S^{8n-1})$ satisfying $2\Sigma^2j_{2n}(\mathrm{H}) = (2n+1)\nu_{8n+2}\circ\Sigma^6\alpha$. So $2\Sigma j_{2n}(\mathrm{H}) \equiv (2n+1)\nu_{8n+1}\circ\Sigma^5\alpha \mod \Delta(\eta^2_{16n+3}) = 4\Sigma j_{2n}(\mathrm{H})$. Therefore $\pm 2j_{2n}(\mathrm{H}) = (2n+1)\nu_{8n}\circ\Sigma^4\alpha + x\Delta(\nu_{16n+1})$ for an integer $x$. Since $2(2n+1)\nu_{16n-1} = 2H(j_{2n}(\mathrm{H})) = \pm 2x\nu_{16n-1}$, we have $x \equiv \pm(2n+1) \mod 12$. By Lemma 6, we have $\Sigma^3\lambda = \pm(2n+1)\nu_{8n}\circ\Sigma^4\alpha$ since $4H(j_{2n}(\mathrm{H})) \neq 0$. By use of the EHP-sequences, we have $\pm\lambda \equiv (2n+1)\nu_{8n-3}\circ\Sigma\alpha \mod \Delta(\nu^2_{16n-5})$. Applying $H$ to this relation, we have $\nu^2_{16n-7} \equiv 0 \mod H(\Delta(\nu^2_{16n-5})) = 2\iota_{16n-7}\circ\nu^2_{16n-7} = 0$. This is a contradiction and hence we have iii). This completes the proof.

**Proposition 10.** $\#\Sigma^k\gamma = 8$ *if* $n \geq 2$ *and* $k = 1$ *or* $2$.

*Proof.* By Lemma 8, it suffices to work in the 2-components and to prove the assertion for $k = 2$. By Lemma 8 and Proposition 9, $\#\Sigma^2\gamma = 4$ or $8$. Assume that $4\Sigma^2\gamma = 4i_*\Sigma j_{2n}(\mathrm{H}) = 0$. Then, by Lemmas 2 and 6, there exists an element $\alpha \in \pi_{16n}(S^{8n+1})$ satisfying $\Sigma^6\lambda' = \nu_{8n+1}\circ\Sigma^3\alpha$. By (7), $\Delta(\eta_{16n-3}) \neq 0$ and $\Delta(\eta^2_{16n-5}) \neq 0$. So, by use of the EHP-sequences, there exists an element $\beta \in \pi_{16n-4}(S^{8n-3})$ satisfying $\alpha = \Sigma^4\beta$. By an EHP-sequence

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$$\pi_{16n+4}(S^{16n+1}) \xrightarrow{\Delta} \pi_{16n+2}(S^{8n}) \xrightarrow{\Sigma} \pi_{16n+3}(S^{8n+1}),$$

$\Sigma^5\lambda' - \nu_{8n}\circ\Sigma^6\beta = a\Delta(\nu_{16n+1})$ for an integer $a$. Applying $H$ to this relation, we have $\pm 2a\nu_{16n-1} = 0$ and $a = 4b$ for an integer $b$. By Lemma 6, $4b\Delta(\nu_{16n+1}) = -2b\Sigma^5\lambda'$. So we have $(1+2b)\Sigma^5\lambda' = \nu_{8n}\circ\Sigma^6\beta$. Since $\Sigma: \pi_{16n+k}(S^{8n-2+k}) \to \pi_{16n+k+1}(S^{8n-1+k})$ is monomorphic for $k = 0$ or $1$, $(1+2b)\Sigma^3\lambda' = \nu_{8n-2}\circ\Sigma^4\beta$. We set $m = 8n-5$. By Lemma 4, there exists a mapping $\delta: \Sigma^{m-1}\mathbf{R}P^{m+2}_m \to S^m$ such that Ker $\{\Sigma^3: \pi_{2m+7}(S^m) \to \pi_{2m+10}(S^{m+3})\} = \delta_*\pi_{2m+7}(\Sigma^{m-1}\mathbf{R}P^{m+2}_m)$. $\mathbf{R}P^{m+2}_m = \Sigma^{m-3}\mathbf{R}P^5_3$ and $\pi_{2m+7}(\Sigma^{m-1}\mathbf{R}P^{m+2}_m) = \pi^S_{11}(\mathbf{R}P^5_3)$ (the stable group). Therefore we have

$$(8)\qquad (1+2b)\lambda' - \nu_m\circ\Sigma\beta \in \delta_*\pi^S_{11}(\mathbf{R}P^5_3).$$

Recall $\mathbf{R}P^5_3 = (S^3 \cup_{2\iota_3} e^4) \cup_{i\eta_3} e^5$. By [9], $\pi^S_9(\mathbf{R}P^2) = \mathbf{Z}_2\{\widetilde{8\sigma}\} \oplus \mathbf{Z}_2\{i\eta\sigma\} \oplus \mathbf{Z}_2\{i\varepsilon\}$. By use of a cofibre sequence starting with $i\eta_3$, we have an exact sequence

$$\mathbf{Z}_{16}\{\sigma\} \xrightarrow{(i\eta)_*} \pi^S_9(\mathbf{R}P^2) \xrightarrow{i'_*} \pi^S_{11}(\mathbf{R}P^5_3) \xrightarrow{p'_*} \mathbf{Z}_2\{\nu^2\} \to 0,$$

where $i': \Sigma^2\mathbf{R}P^2 \hookrightarrow \mathbf{R}P^5_3$ and $p': \mathbf{R}P^5_3 \to S^5$ are the canonical maps. Let $\tilde{\nu}'$ be an element of the Toda bracket $\langle i', i\eta, \nu\rangle \subset \pi^S_{11}(\mathbf{R}P^5_3)$. Then $2\tilde{\nu}'\nu \in \langle i', i\eta, \nu^2\rangle\circ 2\iota = -i'\langle i\eta, \nu^2, 2\iota\rangle \supset i''\langle\eta, \nu^2, 2\iota\rangle \ni i''\varepsilon \bmod i''\eta\sigma = 0$, where $i'' = i'\circ i: S^3 \hookrightarrow \mathbf{R}P^5_3$. So we have $2\tilde{\nu}'\nu = i\varepsilon$ and $\pi^S_{11}(\mathbf{R}P^5_3) = \mathbf{Z}_2\{i'\widetilde{8\sigma}\} \oplus \mathbf{Z}_4\{\tilde{\nu}'\nu\}$.

On the other hand, $H(\delta) \in [\Sigma^{2m-4}\mathbf{R}P^5_3, S^{2m-1}] \cong \{\mathbf{R}P^5_3, S^3\}$. We recall that $\{\mathbf{R}P^2, S^1\} = \mathbf{Z}_2\{\eta p\}$ and $\{\mathbf{R}P^2, S^0\} = \mathbf{Z}_4\{\bar{\eta}\}$, where $\bar{\eta}$ is an extension of $\eta$. By use of the above cofibre sequence, we have an exact sequence

$$0 \leftarrow \mathbf{Z}_2\{\eta p\} \xleftarrow{i'^*} \{\mathbf{R}P^5_3, S^3\} \xleftarrow{p'^*} \mathbf{Z}_2\{\eta^2\} \xleftarrow{(i\eta)^*} \mathbf{Z}_4\{\bar{\eta}\}.$$

Let $\bar{p} \in \{\mathbf{R}P^5_3, S^4\}$ be an extension of $p$ with respect to $i\eta$. Then $\{\mathbf{R}P^5_3, S^3\} = \mathbf{Z}_2\{\eta\bar{p}\}$. $\eta\bar{p}\circ i'\widetilde{8\sigma} = \eta p\widetilde{8\sigma} = 8\eta\sigma = 0$ and $\eta\bar{p}\circ\tilde{\nu}'\nu \in \eta\circ\langle p, i\eta, \nu\rangle\circ\nu \subset \eta\circ G_4\circ\nu = 0$. So we have $(\eta\bar{p})_*\pi^S_{11}(\mathbf{R}P^5_3) = 0$. Applying $H$ to (8), we have $(1+2b)\ H(\lambda') \in H(\delta)_*\pi^S_{11}(\mathbf{R}P^5_3) \subset (\eta\bar{p})_*\pi^S_{11}(\mathbf{R}P^5_3) = 0$. By Lemma 6, $H(\lambda') \equiv \varepsilon_{2m-1} \bmod \eta_{2m-1}\sigma_{2m}$. This is a contradiction and completes the proof.

**4. The real case.** We set $Y = \Sigma^{2n-3}\mathbf{R}P^2$ for $n \geq 2$, $X = Q_{2n+1,2}(\mathbf{R}) = \Sigma^{2n-2}\mathbf{R}P^2$ and $\gamma = \gamma_n(\mathbf{R})$ for $n \geq 1$.

**Proposition 11.** $\#\Sigma\gamma = 2$ *for* $n \geq 1$.

*Proof.* By (6), $\Sigma\gamma = i_*\Delta(\iota_{4n+1})$. Assume that $\Sigma\gamma = 0$. Then, by Lemma 2, we have $\Delta(\iota_{4n+1}) \in \{2\Delta(\iota_{4n+1})\} + (2\iota_{2n})_*\pi_{4n-1}(S^{2n})$. Applying $H$ to this relation, we have $\pm 2\iota_{4n-1} \in \{4\iota_{4n-1}\} + \{8\iota_{4n-1}\}$. This is a contradiction and completes the proof.

By Propositions 7, 9, 10 and 11, we have completed the proof of Theorem 1.

We recall that $\pi_{4n-2}(\Sigma(Y \wedge Y)) = \mathbf{Z}_4\{\tilde{i}'\}$ and $2\tilde{i}' = i'\eta_{4n-3}$ for $n \geq 2$, where $i' : S^{4n-3} \hookrightarrow \Sigma(Y \wedge Y)$ is the inclusion [10].

**Lemma 12.** *Let $n \geq 2$. Then $2\gamma = \pm\Delta(\Sigma^2\tilde{i}')$, $\#\gamma = 4$ for even $n$ and $\#\gamma = 8$ for odd $n$.*

*Proof.* First we shall show $\#\gamma = 4$ for $n = 2$ or $4$. We consider a commutative diagram between exact sequences:

$$\begin{array}{ccccccc}
\pi_6(S^3) & \xrightarrow{i_*} & \pi_6(X) & \xrightarrow{j_*} & \pi_6(X, S^3) & \xrightarrow{\partial} & \pi_5(S^3) \\
\| & & \downarrow i''_* & & \downarrow p_* & & \| \\
\pi_6(S^3) & \xrightarrow{i'_*} & \pi_6(V_{5,2}) & \xrightarrow{p'_*} & \pi_6(S^4) & \xrightarrow{\partial'} & \pi_5(S^3).
\end{array}$$

By (2) and Lemma 3, we have $\pi_6(X, S^3) = \mathbf{Z}_2\{[\iota_3, x]\} \oplus \mathbf{Z}_2\{x\hat{\eta}_3^2\}$ and $\pi_7(X, S^3) = \mathbf{Z}_2\{[\eta_3, x]\} \oplus \mathbf{Z}_2\{x\hat{\nu'}\}$. $\partial[\iota_3, x] = 2[\iota_3, \iota_3] = 0$, $\partial(x\hat{\eta}_3^2) = 2\iota_3 \circ \eta_3^2 = 0$, $\partial[\eta_3, x] = [\eta_3, 2\iota_3] = 0$ and $\partial(x\hat{\nu'}) = 2\iota_3 \circ \nu' = 2\nu'$. So we have Im $i_* \cong \mathbf{Z}_2$ and $j_*$ is epimorphic. We recall that $\pi_6(V_{5,2}) \cong \mathbf{Z}_2[13]$ and $\partial'\eta_4^2 = 2\iota_3 \circ \eta_3^2 = 0$. So we have $\pi_6(V_{5,2}) = \mathbf{Z}_2\{i''_*\tilde{\eta}_3\eta_5\}$ and $i'_*\nu' = 0$. Therefore we have $i_*\nu' = a\gamma$ for $a = 1$ or $2$. By (1) and Lemma 3, we have $0 = aj_*\gamma = a[\iota_3, x]$ and hence we have $a = 2$, $2\gamma = i\nu'$ and $\pi_6(X) = \mathbf{Z}_4\{\gamma\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5\}$.

By (2) and Lemma 3, we have $\pi_{14}(X, S^7) = \mathbf{Z}_2\{[\iota_7, x]\} \oplus \mathbf{Z}_2\{x\hat{\nu}_7^2\}$ and $\pi_{15}(X, S^7) = \mathbf{Z}_2\{[\eta_7, x]\} \oplus \mathbf{Z}_8\{x\hat{\sigma'}\}$. The connecting map $\partial$ is trivial except for the following: $\partial(x\hat{\sigma'}) = 2\iota_7 \circ \sigma' = 2\sigma'$. So, by a parallel argument to the above, we have $2\gamma = i_*\sigma'$ and $\pi_{14}(X) = \mathbf{Z}_4\{\gamma\} \oplus \mathbf{Z}_2\{\tilde{\nu}_7^2\}$ for $n = 4$. We note that $\pi_{14}(V_{9,2}) \cong \mathbf{Z}_2$ [13].

By Proposition 11 and an EHP-sequence

$$\pi_{4n}(\Sigma(X \wedge X)) \xrightarrow{\Delta} \pi_{4n-2}(X) \xrightarrow{\Sigma} \pi_{4n-1}(\Sigma X),$$

we have $2\gamma = a\Delta(\Sigma^2\tilde{i}')$ for an integer $a$. If $a$ is even, $2\gamma = (a/2)i_*\Delta(\eta_{4n-1})$. So we have $2\gamma = 0$ for $n = 2$ or $4$ and $2[\iota_{2n-1}, x] = 2j_*\gamma = 0$ for $n = 3$ or $n \geq 5$ by (1). This contradicts the above and Lemma 3. Hence we have the first assertion.

By (7), $\Delta(\eta_{4n-1})$ is trivial for even $n$ and nontrivial for odd $n$. So $2\Delta(\Sigma^2\tilde{i}') = i_*\Delta(\eta_{4n-1}) = 0$ and $4\gamma = 0$ for even $n$. By (1) and Lemma 3, $2\gamma \neq 0$. This leads us to the second assertion.

For odd $n$, it suffices to show $i_*\Delta(\eta_{4n-1}) \neq 0$. By [10], we have $i_*\Delta(\eta_{11}) = i_*\nu_5\eta_8^2 \neq 0$. Assume that it is trivial for $n \geq 5$. Then, by Lemma 2, there exists an element $\beta \in \pi_{4n-2}(S^{2n-1})$ satisfying $\Delta(\eta_{4n-1}) = 2\iota_{2n-1}\circ\beta$. By (7), $\Delta(\eta_{4n-3}) \neq 0$ for odd $n \geq 5$. So we have $\beta = \Sigma\beta'$ for some $\beta' \in \pi_{4n-3}(S^{2n-2})$. Therefore $\Delta(\eta_{4n-1}) = 2\Sigma\beta'$ and $\Delta(\eta_{4n-1}^2) = 2\Sigma\beta'\circ\eta_{4n-2} = 0$. By (7), $\Delta(\eta_{4n-1}^2) \neq 0$ for odd $n \geq 5$. This is a contradiction and completes the proof.

We set $X = Q_{2n+1,2}(\mathbf{F})$, $W = O_{2n+1,2}(\mathbf{F})$, $r = 2dn-1$ and $s = 2r+d-1 = (4n+1)d-3$. We consider a commutative diagram among exact sequences for $n \geq 2$ :

$$
\begin{array}{ccccccc}
 & & \pi_s(W,X) & \xrightarrow[\cong]{\Sigma''} & \pi_r(S^r) & & \\
 & & \downarrow\partial'' & & \downarrow Q & & \\
\pi_{s-1}(S^r) & \xrightarrow{i_*} & \pi_{s-1}(X) & \xrightarrow{j_*} & \pi_{s-1}(X,S^r) & \xrightarrow{\partial} & \pi_{s-2}(S^r) \\
\| & & \downarrow i''_* & & \downarrow p_* & & \| \\
\pi_{s-1}(S^r) & \xrightarrow{i'_*} & \pi_{s-1}(W) & \xrightarrow{p'_*} & \pi_{s-1}(S^{r+d}) & \xrightarrow{\partial'} & \pi_{s-2}(S^r) \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where $\Sigma'' = \Sigma^{-(r+d-1)}\circ p''_*$ for the canonical map $p'' : (W,X) \to (S^s,*)$. By (1), Lemmas 3 and 12, we have Theorem 2.

**5. Determination of $H(\gamma_n(\mathbf{R}))$.** We shall show $H(\gamma) = \pm\tilde{i}'$, where $\gamma = \gamma_n(\mathbf{R})$ and $n \geq 2$. We set $Y = \Sigma^{2n-3}\mathbf{R}P^2$ and $X = \Sigma Y$.

**Lemma 13.** *Let $n \geq 2$. Then we have*

i) $i_*\Delta(\eta_{4n-3}) = 0$ ;

ii) $\mathrm{Im}\{\Sigma' : \pi_{4n-3}(Y, S^{2n-2}) \to \pi_{4n-2}(X, S^{2n-1})\} \cong \pi_{4n-2}(S^{2n})$.

*Proof.* By (7), $\Delta(\eta_{4n-3}) = 0$ for $n = 2$ or $4$. So we have i) for $n = 2$ or $4$. It suffices to prove $H(\gamma) \neq 0$ for $n = 3$ or $n \geq 5$ since $2\tilde{i}' = H(c\gamma)$ for $c = 1$ or $2$ implies $i_*\Delta(\eta_{4n-3}) = \Delta(2\tilde{i}') = \Delta(H(c\gamma)) = 0$.

We consider an anti-commutative diagram :

$$\begin{array}{ccccc}
\pi_{4n-3}(Y) & \xrightarrow{\Sigma} & \pi_{4n-2}(X) & \xrightarrow{H} & \pi_{4n-2}(\Sigma(Y \wedge Y)) \\
\downarrow j_* & & \downarrow j'_* & & \\
\pi_{4n-3}(Y, S^{2n-2}) & \xrightarrow{\Sigma'} & \pi_{4n-2}(X, S^{2n-1}) & & \\
\downarrow p_* & & \downarrow p'_* & & \\
\pi_{4n-3}(S^{2n-1}) & \xrightarrow{\Sigma} & \pi_{4n-2}(S^{2n}) & & .
\end{array}$$

Assume that $H(\gamma) = 0$. Then there exists an element $\beta \in \pi_{4n-3}(Y)$ such that $\gamma = \Sigma\beta$. Therefore, by (1), $[\iota_{2n-1}, \kappa] = j'_*\gamma = -\Sigma'(j_*\beta)$. By Theorem 2.1 of [3], we have an exact sequence for $n \geq 3$ :

$$\pi_{2n-1}(S^{2n-2}) \xrightarrow{Q} \pi_{4n-3}(Y, S^{2n-2}) \xrightarrow{p_*} \pi_{4n-3}(S^{2n-1}) \rightarrow 0,$$

where $Q(\ ) = [\ , \kappa']$ and $\kappa' = (\Sigma')^{-1}\kappa$ is a generator of $\pi_{2n-1}(Y, S^{2n-2}) \cong \mathbf{Z}$. By the above diagram and Lemma 3, $\Sigma(p_*j_*\beta) = -p'_*\Sigma'(j_*\beta) = 0$. So we have $p_*j_*\beta = a\Delta(\iota_{4n-1})$ for $a = 0$ or $1$ and $p_*(2j_*\beta) = 0$. Therefore we have $2j_*\beta = bQ(\eta_{2n-2})$ for $b = 0$ or $1$. By [15], $\Sigma'(2j_*\beta) = 0$ and hence we have $2[\iota_{2n-1}, \kappa] = 0$. This contradicts Lemma 3 and completes the proof of i).

By the lower square of the above diagram, $p_*$, $\Sigma$ are epimorphic and $p'_*$ is a split epimorphism. This leads us to ii) and completes the proof.

**Proposition 14.** *$H(\gamma) = \pm \tilde{\iota}'$ for $n \geq 2$.*

*Proof.* It suffices to prove that $\Sigma$: $\pi_r(Y) \rightarrow \pi_{r+1}(X)$ for $r = 4n-4$ is monomorphic. We consider the suspension homomorphism between exact sequences up to sign :

$$\begin{array}{ccccccc}
\pi_{r+1}(Y, S^{2n-2}) & \xrightarrow{\partial} & \pi_r(S^{2n-2}) & \xrightarrow{i_*} & \pi_r(Y) & \xrightarrow{j_*} & \pi_r(Y, S^{2n-2}) \\
\downarrow \Sigma' & & \downarrow \Sigma & & \downarrow \Sigma & & \downarrow \Sigma' \\
\pi_{r+2}(X, S^{2n-1}) & \xrightarrow{\partial'} & \pi_{r+1}(S^{2n-1}) & \xrightarrow{i'_*} & \pi_{r+1}(X) & \xrightarrow{j'_*} & \pi_{r+1}(X, S^{2n-1}).
\end{array}$$

By Theorem 2.1 of [3], $\pi_r(Y, S^{2n-2}) \cong \mathbf{Z}\{[\iota_{2n-2}, \kappa']\} \oplus \pi_r(S^{2n-1})$ and $\pi_{r+1}(X, S^{2n-1}) \cong \pi_{r+1}(S^{2n})$ for $n \geq 2$. Since $\pi_r(Y)$ is finite, $j_*\alpha$ for $\alpha \in \pi_r(Y)$ belongs to the second direct summand. The left $\Sigma$ has the kernel $\Delta\pi_{r+2}(S^{4n-3}) = \{\Delta(\eta_{4n-3})\}$ and $\partial'[\iota_{2n-1}, \kappa] = 2[\iota_{2n-1}, \iota_{2n-1}] = 0$. So, by chasing the diagram and using Lemma 13, we conclude that $\Sigma$: $\pi_r(Y) \rightarrow \pi_{r+1}(X)$ is monomorphic. This completes the proof.

## REFERENCES


[1] J. F. ADAMS: On the non-existence of elements of Hopf invariant one, Ann. of Math. **72** (1960), 20–104.

[2] A. L. BLAKERS and W. S. MASSEY: Products in homotopy theory, Ann. of Math. **58** (1953), 295–324.

[3] I. M. JAMES: On the homotopy groups of certain pairs and triads, Quart. J. Math. Oxford (2) **5** (1954), 260–70.

[4] I. M. JAMES: The Topology of Stiefel Manifolds, London Math. Soc. Lecture Note **24**, Cambridge, 1976.

[5] I. M. JAMES and J. H. C. WHITEHEAD: The homotopy theory of sphere bundles over spheres (I), Proc. London Math. Soc. (3) **4** (1954), 196–218.

[6] I. M. JAMES and J. H. C. WHITEHEAD: The homotopy theory of sphere bundles over spheres (II), Proc. London Math. Soc. (3) **5** (1955), 148–66.

[7] M. A. KERVAIRE: Some nonstable homotopy groups of Lie groups, Illinois J. Math. **4** (1960), 161–69.

[8] M. MAHOWALD: Some Whitehead products in $S^n$, Topology **4** (1965), 17–26.

[9] J. MUKAI: Stable homotopy of some elementary complexes, Mem. Fac. Sci. Kyushu Univ. A **20** (1966), 266–82.

[10] J. MUKAI: A remark on Toda's result about the suspension order of the stunted real projective space, Mem. Fac. Sci. Kyushu Univ. A **42** (1988), 87–94.

[11] J. MUKAI: Remarks on homotopy groups of symmetric spaces, Math. J. Okayama Univ. **32** (1990), 159–64.

[12] Y. NOMURA: Note on some Whitehead products, Proc. Japan Acad. **50** (1974), 48–52.

[13] G. F. PAECHTER: The groups $\pi_r(V_{n,m})$ (I), Quart. J. Math. Oxford (2) **7** (1956), 249–68.

[14] S. THOMEIER: Einige Ergebnisse über Homotopiegruppen von Sphären, Math. Ann. **164** (1966), 225–50.

[15] H. TODA: Generalized Whitehead products and homotopy groups of spheres, J. Inst. Poly. Osaka City Univ. **3** (1952), 43–82.

[16] H. TODA: Composition Methods in Homotopy Groups of Spheres, Ann. of Math. Studies **49**, Princeton, 1962.



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