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Abstract

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GALOIS COVERS OF DEGREE p AND SEMI-STABLE REDUCTION OF CURVES IN EQUAL CHARACTERISTIC $p > 0$

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Abstract. In this paper we study the semi-stable reduction of Galois covers of degree p above curves over a complete discrete valuation ring of equal characteristic p.

1. Introduction.

Let $p > 0$ be a prime integer. Let R be a complete discrete valuation ring, with fraction field K of characteristic p, and residue field k which we assume to be algebraically closed. Let $\mathcal{X}$ be a proper and smooth R-curve, with generic fibre $\mathcal{X}_K := \mathcal{X} \times_R K$, and special fiber $\mathcal{X}_k := \mathcal{X} \times_R k$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a finite Galois cover with Galois group G, and with $\mathcal{Y}$ normal. Let $\mathcal{Y}_K := \mathcal{Y} \times_R K$ be the generic fiber of $\mathcal{Y}$, and let $\mathcal{Y}_k := \mathcal{Y} \times_R k$ be its special fiber, which we assume to be reduced (this condition is always satisfied after a finite extension of R). If the cardinality of G is prime to p, and if the cover $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ between generic fibers is étale, then it follows from Zariski's purity theorem that $\mathcal{Y}$ is smooth (cf. [11]). If the cardinality of G is divisible by p then $\mathcal{Y}$ is not smooth in general (even if the cover f_K between generic fibres is étale). However, it follows from the theorem of semi-stable reduction of curves (cf. [2]) that $\mathcal{Y}$ admits potentially semi-stable reduction, i.e. there exists (possibly after a finite extension of R) a proper and birational morphism $\tilde{\mathcal{Y}} \to \mathcal{Y}$, where $\tilde{\mathcal{Y}}$ is a semi-stable R-curve. Moreover, there exists such a semi-stable model $\tilde{\mathcal{Y}}$ which is minimal. Our main interest is in the study of the geometry (of the special fiber) of a minimal semi-stable model $\tilde{\mathcal{Y}}$, under the assumption that p divides the cardinality of G. In this paper we study the case where $G \simeq \mathbb{Z}/p\mathbb{Z}$, and with no restriction on the ramification in the morphism f. This has been treated in the literature in the unequal characteristic case (cf. e.g. [4], and [10]).

This paper is organized as follows. In section 2, we recall the main results in [7] which describe the degeneration of étale $\mathbb{Z}/p\mathbb{Z}$-torsors in equal characteristic p. In section 3, we prove a formula comparing the dimensions of the spaces of vanishing cycles in a Galois cover $\tilde{f} : \mathcal{Y}_y \to \mathcal{X}_x$, with group $\mathbb{Z}/p\mathbb{Z}$, between formal germs of curves in equal characteristic p. This formula plays an important role in this paper. As a consequence of these results (with the same notations as above) we can determine the singular points of $\mathcal{Y}_k$, and we can compute the arithmetic genus of these singularities. More precisely, suppose that the branched points in the morphism $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ specialize in the set $B_k \subset \mathcal{X}_k$, and let $U'_k := \mathcal{X}_k - B_k$. Then f induces (by restriction to U'_k) a finite cover $f'_k : V'_k \to U'_k$, which has the structure of a torsor under a finite and flat k-group scheme of rank p. Suppose for example that this torsor is radicial (this is the most difficult case to treat), and let ω be the associated differential form (cf. [9], 1). Let Z_k be the set of zeros of ω, and let $\mathrm{Crit}(f) := Z_k \cup B_k$. If y is a singular point of $\mathcal{Y}_k$, then $f(y) \in \mathrm{Crit}(f)$. Further, let $m_y := \mathrm{ord}_{f(y)}(\omega)$. Then the arithmetic genus of y (cf. [8], 3.1) equals $(r_y + m_y)(p-1)/2$, where r_y is the number of branched points of f in the generic fiber $\mathcal{X}_K$ which specialize in $f(y)$ ($r_y = 0$, if $f(y) \in \mathrm{Crit}(f) - B_k$), (cf. 3.3.1).

In order to understand the geometry of $\tilde{\mathcal{Y}}$ one needs to understand the fiber of a singular point y of $\mathcal{Y}_k$ in the minimal semi-stable model $\tilde{\mathcal{Y}}$. This is a local problem which we study in section 4. There we consider a finite Galois cover $f_x : \mathcal{Y}_y \to \mathcal{X}_x$ of degree p between formal germs of R-curves at a closed point y (resp. x), where x is a smooth point (i.e. $\mathcal{X}_x \simeq \mathrm{Spf} R[[T]]$) and we study the geometry of a minimal semi-stable model $\tilde{\mathcal{Y}}_y$ of $\mathcal{Y}_y$. In 4.2 we exhibit what we call "simple degeneration data of rank p", comprising a tree Γ of k-projective lines which is endowed with some data of geometric and combinatorial nature, and which completely describe the geometry of $\tilde{\mathcal{Y}}_y$. More precisely, let Deg_p be the set of "isomorphism classes" of such data (cf. Definition 4.4). Then we construct a canonical specialization map Sp: $H^1_{\mathrm{et}}(\mathrm{Spec}\, L, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Deg}_p$, where L is the function field of the geometric fiber $\overline{\mathcal{X}}_x := \mathcal{X}_x \times_R \overline{R}$ of $\mathcal{X}_x$, and $\overline{R}$ is the integral closure of R in an algebraic closure of K. Our first main result in this paper is the following realization result for simple degeneration data.

Theorem.(cf. 4.6) *The specialization map* $\mathrm{Sp} : H^1_{\mathrm{et}}(\mathrm{Spec} L, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Deg_p}$ *defined in* 4.4 *is surjective.*

In other words we are able to reconstruct Galois covers of degree p above open discs (in equal characteristic p), starting from (the) degeneration data which describe the semi-stable reduction of such a cover. The proof of this result relies on the technique of formal patching initiated by Harbater and Raynaud (cf. [8], 1).

In section 5, we return to the above global situation of a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p. The results in section 4 allow us to associate with each critical point $x_i = f(y_i) \in \mathrm{Crit}(f)$, simple degeneration data $\mathrm{Deg}(x_i)$ of rank p, which describe the preimage of the singular point y_i in $\tilde{\mathcal{Y}}_k$. These simple degeneration data, plus the data given by the torsor f'_k ; $V'_k \to U'_k$, lead to the definition of "smooth degeneration data" $\mathrm{Deg}(\mathcal{X}_k)$ of rank p, which are associated with the special fiber $\mathcal{X}_k$ of $\mathcal{X}$, and which describe the geometry of the semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$. More precisely, let $\mathrm{DEG}_p(\mathcal{X}_{\mathrm{k}})$ be the set of isomorphism classes of smooth degeneration data of rank p associated with $\mathcal{X}_k$ (cf. Definition 5.3). Then we construct a canonical "specialization" map $\mathrm{Sp} : H^1_{\mathrm{et}}(\mathrm{Spec}\, L, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{DEG}_p(\mathcal{X}_{\mathrm{k}})$, where L is the function field of the geometric fiber $\overline{\mathcal{X}} := \mathcal{X} \times_R \overline{R}$, of $\mathcal{X}$, and $\overline{R}$ is the integral closure of R in an algebraic closure of K. Our second main result is the realization of smooth degeneration data associated with $\mathcal{X}_k$, if necessary after modifying the R-curve $\mathcal{X}$ into another R-curve $\mathcal{X}'$ with special fiber $\mathcal{X}'_k$ isomorphic to $\mathcal{X}_k$. More precisely, we have the following.

Theorem.(cf. 5.5) *Let* $\mathrm{Deg}(\mathcal{X}_k) \in \mathrm{DEG}_p(\mathcal{X}_{\mathrm{k}})$ *be smooth degeneration data of rank p, associated with* $\mathcal{X}_k$. *Then there exists a smooth and proper R-curve* $\mathcal{X}'$, *with special fiber isomorphic to* $\mathcal{X}_k$, *such that* $\mathrm{Deg}(\mathcal{X}_k)$ *is in the image of the specialization map* $\mathrm{Sp} : H^1_{\mathrm{et}}(\mathrm{Spec}\, L, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{DEG}_p(\mathcal{X}_{\mathrm{k}})$, *where L is the function field of the geometric fiber* $\mathcal{X}' \times_R \overline{R}$ *of* $\mathcal{X}'$, *and* $\overline{R}$ *is the integral closure of R in an algebraic closure of K.*

A similar result has been proved in [5] (cf. Remark 5.7, 2). As another application of our techniques, we prove the following result of lifting of torsors under finite and flat group schemes of rank p, in equal characteristic p.

Theorem.(cf. 5.6) *Let X be a smooth and proper k-curve, and let* $f : Y \to X$ *be a torsor under a finite and flat k-group scheme* G_k *of rank p. Then there exists a smooth and proper R-curve* $\mathcal{X}$, *with special fiber isomorphic to X, and a torsor* $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ *under an R-group scheme* G_R, *which is commutative finite and flat of rank p, such that the torsor induced on the level of special fibers* $\tilde{f}_k : \mathcal{Y}_k \to \mathcal{X}_k$ *is isomorphic to the torsor f. In other words the torsor* $\tilde{f}$ *lifts f.*

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2. Degeneration of étale $\mathbb{Z}/p\mathbb{Z}$-torsors in equal characteristic $p > 0$.

In this section we recall the degeneration of étale $\mathbb{Z}/p\mathbb{Z}$ -torsors in equal characteristic $p > 0$, as described in [7], which plays an important role in this paper. We will use the following notations: R is a complete discrete valuation ring of equal characteristic $p > 0$, with *perfect* residue field k, and fraction field $K := \mathrm{Fr}R$. We denote by π a uniformizing parameter of R.

2.1. The group schemes $\mathcal{M}_n$ (cf. also [5], 3.2). Let $n \geq 0$ be an integer, and let $\mathbb{G}_{a,R} := \text{Spec}R[X]$ be the additive group scheme over R. The map:

$$\phi_n : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$$

given by:

$$X \to X^p - \pi^{(p-1)n} X$$

is an isogeny of degree p between group schemes. The kernel of ϕ_n is denoted by $\mathcal{M}_{n,R}$, or simply $\mathcal{M}_n$ if no confusion occurs. Thus, $\mathcal{M}_n := \text{Spec}R[X]/(X^p - \pi^{(p-1)n} X)$ is a finite and flat R-group scheme of rank p. Further, the following sequence is exact in the fppf topology:

$$0 \to \mathcal{M}_n \to \mathbb{G}_{a,R} \xrightarrow{\phi_n} \mathbb{G}_{a,R} \to 0 \tag{1}$$

If $n = 0$, then the sequence (1) is the Artin-Schreier sequence (which is exact in the étale topology), and $\mathcal{M}_0$ is the étale constant group scheme $(\mathbb{Z}/p\mathbb{Z})_R$. If $n > 0$, the sequence (1) has a generic fiber which is isomorphic to the (étale) Artin-Schreier sequence, and a special fiber isomorphic to the (radicial) exact sequence:

$$0 \to \alpha_p \to \mathbb{G}_{a,k} \xrightarrow{x^p} \mathbb{G}_{a,k} \to 0 \tag{2}$$

In particular, if $n > 0$, the group scheme $\mathcal{M}_n$ has a generic fiber which is étale, isomorphic to $(\mathbb{Z}/p\mathbb{Z})_K$, and its special fiber is isomorphic to the infinitesimal group scheme $\alpha_{p,k}$. Let X be an R-scheme. The sequence (1) induces along cohomology exact sequence:

$$\begin{aligned} \mathbb{G}_{a,R}(X) \xrightarrow{\phi_n} \mathbb{G}_{a,R}(X) \to H^1_{\text{fppf}}(X, \mathcal{M}_n) \\ \to H^1_{\text{fppf}}(X, \mathbb{G}_{a,R}) \xrightarrow{\phi_n} H^1_{\text{fppf}}(X, \mathbb{G}_{a,R}) \end{aligned} \tag{3}$$

The cohomology group $H^1_{\text{fppf}}(X, \mathcal{M}_n)$ classifies the isomorphism classes of fppf- torsors, with group $\mathcal{M}_n$, above X. The exact sequence (3) allows the following description of $\mathcal{M}_n$-torsors: locally a torsor $f : Y \to X$, under the group scheme $\mathcal{M}_n$, is given by an equation $T^p - \pi^{(p-1)n} T = a$. Where T is an indeterminate, and a is a regular function on X, which is uniquely defined up to addition of elements of the form $b^p - \pi^{(p-1)n} b$ (for some regular function b). In particular, if $H^1_{\text{fppf}}(X, \mathbb{G}_{a,R}) = 0$ (e.g. if X is affine), an $\mathcal{M}_n$-torsor above X is globally defined by an equation as above.

2.2. Degeneration of étale $\mathbb{Z}/p\mathbb{Z}$-torsors. In what follows X is a formal R-scheme of finite type which is normal, connected, and flat over R. Let $X_K := X \times_R K$ (resp. $X_k := X \times_R k$) be the generic (resp. special) fiber of X. By "generic fiber" of X we mean the associated K-rigid space (cf. [1]). We further assume that the special fiber X_k is integral. Let η be the generic point of the special fiber X_k, and let $\mathcal{O}_\eta$ be the local ring of X at η, which is a discrete valuation ring with fraction field $K(X) :=$ the function field of X. Let $f_K : Y_K \to X_K$ be a nontrivial étale $\mathbb{Z}/p\mathbb{Z}$-torsor, with Y_K geometrically connected. Let $K(X) \to L$ be the corresponding extension of function fields. The following result, which describes the degeneration of f_K, is used in the next sections:

2.2.1. Theorem. *Assume that the ramification index above $\mathcal{O}_\eta$ in the extension $K(X) \to L$ equals 1. Then the torsor $f_K : Y_K \to X_K$ extends to a torsor $f : Y \to X$ under a finite and flat R-group scheme of rank p, with Y normal. Let δ be the degree of the different above η in the extension $K(X) \to L$. Then the following cases occur:*

a) $\delta = 0$. In this case f is an étale torsor under the group scheme $\mathcal{M}_0$, and $f_k : Y_k \to X_k$ is an étale $\mathbb{Z}/p\mathbb{Z}$-torsor

b) $\delta > 0$. In this case $\delta = n(p-1)$ for a certain integer $n \geq 1$, and f is a torsor under the group scheme $\mathcal{M}_n$. Further, in this case $f_k : Y_k \to X_k$ is a radicial torsor under the k group scheme α_p.

Note that starting from a torsor $f_K : Y_K \to X_K$, as in 2.2.1, the condition that the ramification index above $\mathcal{O}_\eta$ equals 1 is always satisfied after possibly a finite extension of R (cf. e.g. [3]).

Proof. cf. [7], Theorem 2.2.1.

2.2.2. It follows from 2.2.1 that an étale $\mathbb{Z}/p\mathbb{Z}$-torsor above the generic fiber X_K of X induces (canonically) a *degeneration data*, which consists of a torsor above the special fiber X_k of X, under a finite and flat k-group scheme which is either étale or of type α_p. Reciprocally, we have the following result of *lifting* of such a degeneration data.

2.2.3. Proposition. *Assume that X is affine. Let $f_k : Y_k \to X_k$ be a torsor under a finite and flat k-group scheme, which is étale (resp. of type α_p). Then f_k can be lifted to a torsor $f : Y \to X$ under a finite and flat R-group scheme of rank p, which is étale (resp. isomorphic to $\mathcal{M}_n$, for an integer $n > 0$).*

Proof. cf. [7], Proposition 2.2.3.

2.3. Degeneration of $\mathbb{Z}/p\mathbb{Z}$-torsors on the boundaries of formal fibers.

In this section we assume that the *residue field k of R is algebraically closed.* We describe the degeneration of $\mathbb{Z}/p\mathbb{Z}$-torsors on the boundary $\mathcal{X} \simeq \mathrm{Spf} R[[T]]\{T^{-1}\}$ of formal fibers of germs of formal R-curves. Here $R[[T]]\{T^{-1}\}$ denotes the ring of formal power series $\sum_{i\in\mathbb{Z}} a_i T^i$, with $\lim_{i\to-\infty} |a_i| = 0$, where $|\ |$ is an absolute value of K associated to its valuation. Note that

$R[[T]]\{T^{-1}\}$ is a complete discrete valuation ring with uniformizing parameter π, and residue field $k((t))$, where $t \equiv T \mathrm{mod} \pi$. The function T is called a *parameter* of the formal fiber $\mathcal{X}$. The following result, which describes the degeneration of $\mathbb{Z}/p\mathbb{Z}$-torsors above the formal fiber Spf $R[[T]]\{T^{-1}\}$, is used in section 3 in order to prove a formula comparing the dimensions of the spaces of vanishing cycles, in a Galois cover of degree p between formal germs of R-curves.

2.3.1. Proposition. *Let $A := R[[T]]\{T^{-1}\}$, and let $f : \mathrm{Spf} B \to \mathrm{Spf} A$ be a nontrivial Galois cover of degree p. Assume that the ramification index of the corresponding extension of discrete valuation rings equals 1. Then f is a torsor under a finite and flat R-group scheme G_R of rank p. Let δ be the degree of the different in the above extension. Then the following cases occur:*

a) $\delta = 0$. In this case f is a torsor under the étale group $(\mathbb{Z}/p\mathbb{Z})_R$. Moreover, for a suitable choice of the parameter T of A, the torsor f is given by an equation $X^p - X = T^m$ for some integer $m < 0$ which is prime to p. In this case $X^{1/m}$ is a parameter for B.

b) $0 < \delta = n(p-1)$, for some integer $n > 0$. In this case f is a torsor under the group scheme $\mathcal{M}_{n,R}$. Moreover, for a suitable choice of the parameter T, the torsor f is given by an equation $X^p - \pi^{n(p-1)}X = T^m$ with $m \in \mathbb{Z}$ prime to p. In this case $X^{1/m}$ is a parameter for B.

Proof. First it follows from 2.2.1 that we are either in the case a) or in the case b). We first start with the case a). Thus, f is an étale torsor given by an equation $X^p - X = u = \sum_{i\in\mathbb{Z}} a_i T^i \in A$. On the level of special fibers the torsor $f_k := \mathrm{Spec} B/\pi B \to \mathrm{Spec} A/\pi A$ is the étale torsor given by the equation $x^p - x = \sum_{i\geq m} \overline{a}_i t^i \in A$, where $\overline{a}_i$ is the image of a_i modulo π, and $m \in \mathbb{Z}$ is an integer. Assume that the integer $m = pm'$ is divisible by p. Then after adding $\overline{a}_m^{1/p} t^{m'} - \overline{a}_m t^m$ into the defining equation for f_k we can replace $\overline{a}_m t^m$ by $\overline{a}_m^{1/p} t^{m'}$. Repeating this process we can finally assume that the integer m is prime to p; in which case $\sum_{i\geq m} \overline{a}_i t^i = t^m \overline{v}$, and $u = T^m v$ where $v \in A$ is a unit whose image modulo π equals $\overline{v}$. Further, the integer m is necessarily negative since the residue field extension $f_k := \mathrm{Spec} B/\pi B \to \mathrm{Spec} A/\pi A$ must ramify. Finally, after extracting an m-th root of v, and replacing T by (the parameter) $Tv^{1/m}$, we obtain an equation of the form $X^p - X = T^m$. Next, assume that we are in the case b). Thus, f is a torsor under the finite and flat group scheme $\mathcal{M}_{n,R}$, for some positive integer n, given by an equation $X^p - \pi^{n(p-1)}X = u = \sum_{i\in\mathbb{Z}} a_i T^i \in A$, where u is not a p-power modulo π. On the level of special fibers the torsor $f_k := \operatorname{Spec} B/\pi B \to \operatorname{Spec} A/\pi A$ is the α_p-torsor given by the equation $x^p = \sum_{i\geq m} \overline{a}_i t^i \in A/\pi A$, where $\overline{a}_i$ is the image of a_i modulo π, and $m \in \mathbb{Z}$ is some integer. Assume that the integer $m = pm'$ is divisible by p. Then the term $\overline{a}_m t^m$ is a p-power and we can eliminate it from the defining equation for f_k (without changing the torsor f_k). Since $\sum_{i\geq m} \overline{a}_i t^i \in A/\pi A$ is not a p-power, we can repeat this process, and assume after finitely many steps that m is prime to p. In this case $\sum_{i>m} \overline{a}_i t^i = t^m \overline{v}$, and $u = T^m v$, where $v \in A$ is a unit whose image modulo π equals $\overline{v}$. Finally, after extracting an m-th root of v, and replacing T by (the parameter) $Tv^{1/m}$, we obtain an equation of the form $X^p - \pi^{n(p-1)}X = T^m$.

2.3.2. Definition. With the same notations as in 2.3.1 we define the *conductor* of the torsor f to be the integer $-m$. Further, we define the *degeneration type* of the torsor f to be $(0, m)$ in the case a) and (n, m) in the case b).

2.3.3. Remark. The above proposition implies in particular that Galois covers $f : \operatorname{Spf} B \to \operatorname{Spf} A$, as in 2.3.1, are classified by their degeneration type, as defined in 2.3.2. More precisely, given two such Galois covers which have the same degeneration type, there exists a (non canonical) Galois equivariant isomorphism between both covers.

3. Computation of vanishing cycles and examples for cyclic p-covers.

The main result of this section is Theorem 3.2.3 which gives a formula comparing the dimensions of the spaces of vanishing cycles in a Galois cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$, with group $\mathbb{Z}/p\mathbb{Z}$, between formal germs of curves in equal characteristic p. In this section we use the following notations: R is a complete discrete valuation ring of equal characteristic $p > 0$. We denote by K the fraction field of R, by π a uniformizing parameter, and k the residue field. We also denote by v_K the valuation of K which is normalized by $v_K(\pi) = 1$. We assume that the residue field k is *algebraically closed*.

3.1. By a (formal) R-curve we mean a (formal) R-scheme of finite type which is normal, flat, and whose fibers have dimension 1. For an R-scheme X we denote by $X_K := X \times_{\operatorname{Spec} R} \operatorname{Spec} K$ the *generic* fiber of X, and $X_k := X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ its *special* fiber. In what follows by a (*formal*) *germ* $\mathcal{X}$ of an R-curve we mean that $\mathcal{X} := \operatorname{Spec} \mathcal{O}_{X,x}$ is the (resp. $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ is the formal completion of the) spectrum of the local ring of an R-curve X at a closed point x. We refer to [8], 3.1, for the definition of the integers δ_x, r_x, and the arithmetic genus g_x of the point x.

3.2. The compactification process. Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ be the formal germ of an R-curve at a closed point x, with $\mathcal{X}_k$ reduced. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$, and with $\mathcal{Y}$ local. We assume that the special fiber of $\mathcal{Y}_k$ is reduced (this can always be achieved after a finite extension of R). We will construct a compactification of the above cover $\tilde{f}$ and, as an application, compute the arithmetic genus of the closed point of $\mathcal{Y}$. More precisely, we will construct a Galois cover $f : Y \to X$ of degree p between proper algebraic R-curves, a closed point $y \in Y$, and its image $x = f(y)$, such that the formal germ of X (resp. of Y) at x (resp. at y) equals $\mathcal{X}$ (resp. $\mathcal{Y}$), and such that the Galois cover $f_x : \operatorname{Spf} \hat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ (induced by f between the formal germs at y and x) is isomorphic to the above given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. The construction of such a compactification has been done in [8], 3.3.1, in the unequal characteristic case. We first start with the case where the formal germ $\mathcal{X}$ has only one boundary.

3.2.1. Proposition. *Let $D := \operatorname{Spf} R\langle 1/T \rangle$ be the formal closed disc centered at ∞ (cf. [1], 1, for the definition of $R\langle 1/T \rangle$). Let $D := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the formal boundary of D, and let $D \to D$ be the canonical morphism. Let $\tilde{f} : \mathcal{Y} \to D$ be a nontrivial torsor under a finite and flat R-group scheme of rank p, such that the special fiber of $\mathcal{Y}$ is reduced. Then there exists a Galois cover $f : Y \to D$, with group $\mathbb{Z}/p\mathbb{Z}$, whose pull back to D is isomorphic to the above given torsor $\tilde{f}$. More precisely, with the notations introduced in 2.3 we have the following possibilities:*

a) The torsor $\tilde{f}$ is étale, and has a reduction of type $(0, -m)$. In this case consider the Galois cover $f : Y \to D$ given generically by the equation $Z^p - Z = 1/T^m$. This cover is an étale torsor and its special fiber $f_k : Y_k \to X_k$ is étale. In particular, Y_k is smooth. Moreover, the genus of the smooth compactification of Y_k equals $(m-1)(p-1)/2$.

b) The cover $\tilde{f}$ is a torsor under the group scheme $\mathcal{M}_{n,R}$, for some integer $n > 0$, and has a reduction of type (n, m) for some integer m prime to p. The following two cases occur:

b-1) $m > 0$. In this case consider the Galois cover $f : Y \to D$ given generically by the equation $Z^p - \pi^{n(p-1)} Z = T^m$. This cover is ramified above ∞, with conductor m, and its special fiber $f_k : Y_k \to X_k$ is radicial. Moreover, Y_k is smooth, and its smooth compactification has genus 0.

b-2) $m < 0$. In this case consider the Galois cover $f : Y \to D$ given generically by the equation $Z^p - \pi^{n(p-1)} Z = T^m$. This cover is an étale torsor on the generic fiber, and its special fiber $f_k : Y_k \to X_k$ is radicial.

Moreover, Y_k has a unique singular point y which is above ∞ and with arithmetic genus $g_y = (-m-1)(p-1)/2$.

Proof. The proof is similar to the proof of proposition 3.3.1 in [8]. We will treat the case b-1) for the convenience of the reader. In this case consider the Galois cover $H \to \mathrm{P}^1_R$ of degree p, with H normal, above the projective R-line with parameter T, which is generically given by the equation $Z^p - \pi^{n(p-1)}Z = T^m$. This cover is ramified on the generic fiber only above the point ∞, with conductor m. Hence the genus of the generic fiber H_K of H equals $(m-1)(p-1)/2$. On the level of the special fibers the cover $H_k \to \mathrm{P}^1_k$ is an α_p torsor, outside the point ∞, defined by the equation $z^p = t^m$. The genus of the singularity above the point $t=0$ can be easily computed, it equals $(m-1)(p-1)/2$ (cf. [8] 3.3.1). From this we deduce that H_k is smooth outside $t=0$, since the arithmetic genus of H_K and that of H_k are equal.

In the next proposition we deal with the general case.

3.2.2. Proposition. *Let $\mathcal{X} := \mathrm{Spf}\,\hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x, and let $\{\mathcal{X}_i\}_{i=1}^n$ be the formal boundaries of $\mathcal{X}$. Let $\tilde{f}$ $: \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$, and with $\mathcal{Y}$ local. Assume that $\mathcal{Y}_k$ and $\mathcal{X}_k$ are reduced. Then there exists a Galois cover $f : Y \to X$ of degree p between proper algebraic R-curves Y and X, a closed point $y \in Y$ and its image $x = f(y)$, such that the formal germ of X (resp. of Y) at x (resp. at y) equals $\mathcal{X}$ (resp. equals $\mathcal{Y}$), and such that the Galois cover $\mathrm{Spf}\,\hat{\mathcal{O}}_{Y,y} \to \mathrm{Spf}\,\hat{\mathcal{O}}_{X,x}$, induced by f between the formal germs at y and x, is isomorphic to the above given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. Moreover, the formal completion of X along its special fiber has a covering which consists of n closed formal discs D_i, which are patched with $\mathcal{X}$ along the boundaries D_i, and the special fiber X_k of X consists of n smooth projective lines which intersect at the point x. In particular, the arithmetic genus of X_K equals g_x.*

Proof. Similar to the proof of proposition 3.3.2 in [8].

The next result is the main one of this section. It provides an explicit formula which compares the dimensions of the spaces of vanishing cycles in a Galois cover of degree p, between formal fibers of curves in equal characteristic $p > 0$.

3.2.3. Theorem. *Let $\mathcal{X} := \mathrm{Spf}\hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x, with $\mathcal{X}_k$ reduced. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$, with $\mathcal{Y}$ local, and $\mathcal{Y}_k$ reduced. Let $\{\wp_i\}_{i\in I}$ be the minimal prime ideals of $\hat{\mathcal{O}}_x$ which contain π, and let $\mathcal{X}_i := \mathrm{Spf}\hat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localization of $\mathcal{X}$ at $\wp_i$. For each $i \in I$, the above cover $\tilde{f}$*

*induces a torsor $\tilde{f}_i : \mathcal{Y}_i \to \mathcal{X}_i$, under a finite and flat R-group scheme of rank p, above the boundary $\mathcal{X}_i$ (cf. 2.3.1). Let (n_i, m_i) be the reduction type of $\tilde{f}_i$ (cf. 2.3.2). Let y be the closed point of $\mathcal{Y}$. Then one has the following "***local Riemann-Hurwitz formula***" :*

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s$$

where d_η is the degree of the divisor of ramification in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathcal{X}_K$ induced by $\tilde{f}$ on the generic fibers; $\mathcal{X}_K := \operatorname{Spec}(\hat{\mathcal{O}}_x \otimes_R K)$, and $\mathcal{Y}_K := \operatorname{Spec}(\hat{O}_{\mathcal{Y},y} \otimes_R K)$. Here,

$$d_s := \sum_{i \in I_{\text{et}}} (-m_i - 1)(p-1) + \sum_{i \in I_{\text{rad}}} (-m_i - 1)(p-1)$$

where I_{rad} is the subset of I consisting of those i for which $n_i \neq 0$, and I_{et} is the subset of I consisting of those i for which $n_i = 0$, and $m_i \neq 0$.

Proof. The proof is similar, using 3.2.1, to the proof of theorem 3.4 in [8] with the appropriate modifications. We briefly repeat the argument for the convenience of the reader. By Proposition 3.2.2 one can compactify the above morphism $\tilde{f}$. More precisely, we constructed in 3.2.2 a Galois cover $f : Y \to X$ of degree p between proper algebraic R-curves, a closed point $y \in Y$, and its image $x = f(y)$, such that the formal germ of X (resp. of Y) at x (resp. at y) equals $\mathcal{X}$ (resp. equals $\mathcal{Y}$), and such that the Galois cover $\operatorname{Spf} \hat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. The special fiber of $\mathcal{X}$ consists (by construction) of card(I)-distinct smooth projective lines which intersect at the closed point x. The formal completion of X along its special fiber has a covering which consists of card(I) formal closed unit discs, which are patched with the formal fiber $\mathcal{X}$ along the boundaries $\mathcal{X}_i$. The above formula follows then by comparing the arithmetic genus of the generic fiber Y_K of Y and the arithmetic genus of its special fiber Y_k. Using the precise informations given in Proposition 3.2.1 one can easily deduce that $g(Y_K) = pg_x + (1-p) + d_\eta/2 + \sum_{i \in I_>} (m_i + 1)(p-1)/2$, where $I_>$ is the subset of I consisting of those i for which the degeneration type above the boundary $\mathcal{X}_i$ is (n_i, m_i), with $n_i > 0$ and $m_i > 0$. On the other hand one has $g(Y_k) = g_y + \sum_{i \in I_<} (-m_i - 1)(p-1)/2 + \sum_{i \in I_{\text{et}}} (-m_i - 1)(p-1)$. Where $I_<$ is the subset of I consisting of those i for which the degeneration type above the boundary $\mathcal{X}_i$ is (n_i, m_i), with $n_i > 0$ and $m_i < 0$, and I_{et} is the subset of I consisting of those i for which the degeneration type above the boundary $\mathcal{X}_i$ is $(0, m_i)$. Now, since Y is flat, we have $g(Y_K) = g(Y_k)$ and the above formula directly follows.

3.3. Cyclic p-covers above germs of semi-stable curves.

In what follows, and as a direct application of theorem 3.2.3, we deduce some results in the case of a Galois cover $\mathcal{Y} \to \mathcal{X}$, above the formal germ $\mathcal{X}$ of a *semi-stable* R-curve at a closed point. These results will play an important role in sections 4 and 5 in order to exhibit and realize the degeneration data, which describe the semi-stable reduction of Galois covers of degree p, in equal characteristic p. We start with the case of a Galois cover of degree p above a germ of a *smooth* point.

3.3.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[T]]$ be the formal germ of an R-curve at a smooth point x, and let $\mathcal{X}_\eta := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the boundary of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p with $\mathcal{Y}$ local. Assume that the special fiber of $\mathcal{Y}$ is reduced. Let y be the unique closed point of $\mathcal{Y}_k$. Let d_η be the degree of the divisor of ramification in the morphism $f : \mathcal{Y}_K \to \mathcal{X}_K$. Then $d_\eta = r(p-1)$ is divisible by $p-1$. We distinguish two cases:*

1) $\mathcal{Y}_k$ is unibranche at y. Let (n, m) be the degeneration type of f above the boundary $\mathcal{X}_\eta$ (cf. 1.3.2). Then necessarily $r + m - 1 \geq 0$, and $g_y = (r + m - 1)(p-1)/2$.

2) $\mathcal{Y}_k$ has p-branches at y. Then the cover f has an étale split reduction of type $(0, 0)$ on the boundary, i.e. the induced torsor above Spf $R[[T]]\{T^{-1}\}$ is trivial, in which case $g_y = (r-2)(p-1)/2$.

As an immediate consequence of 3.3.1 one can immediately detect whether the point y is smooth or not. More precisely, we have the following:

3.3.2. Corollary. *We use the same notation as in 3.3.1. Then y is a smooth point, which is equivalent to $g_y = 0$, if and only if $r = 1 - m$ which implies that $m \leq 1$. In particular, if f has a degeneration of type (n, m) on the boundary with $n > 0$ and $m > 0$, then this only happen if $r = 0$ and $m = 1$.*

Next, we will give examples of Galois covers of degree p above the formal germ of a smooth point, which cover all the possibilities for the genus and the degeneration type on the boundary. Both in 3.3.3 and 3.3.4 we use the same notations as in 3.3.1. We first begin with examples with genus 0.

3.3.3. Examples. The following are examples given by explicit equations of the different cases, depending on the possible degeneration type above the boundary, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[T]]$, and where $g_y = 0$ (here y denotes the closed point of $\mathcal{Y}$).

1) For $m > 0$ an integer prime to p, consider the cover given generically by the equation $X^p - X = T^{-m}$. Here $r = m + 1$, and this cover has a reduction of type $(0, -m)$ on the boundary.

2) For $\tilde{m} := -m$ a negative integer prime to p, and a positive integer n, consider the cover given generically by the equation $X^p - \pi^{n(p-1)}X = T^{\tilde{m}}$. Here $r = m + 1$, and this cover has a reduction of type $(n, \tilde{m})$ on the boundary.

3) For a positive integer n consider the cover given generically by the equation $X^p - \pi^{n(p-1)}X = T$. Here $r = 0$, and this cover has a reduction of type $(n, 1)$ on the boundary.

Next, we give examples of Galois covers of degree p above formal germs of smooth points which lead to a singularity with positive genus.

3.3.4. Examples. The following are examples (given by explicit equations) of the different cases, depending on the possible reduction type, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[T]]$, and where $\mathrm{g}_y > 0$.

1) Let $m > 0$, and $m' > m$, be integers prime to p. Consider the cover given generically by the equation $X^p - X = \pi/T^{m'} + 1/T^m$. This cover has a degeneration of type $(0, -m)$ on the boundary, the point y above x is singular, and its arithmetic genus equals $(m' - m)(p - 1)/2$.

2) Let m, m', and n be positive integers with m and m' prime to p. Consider the cover given generically by the equation $X^p - X = T^m/\pi^{pn} + \pi/T^{m'}$. This cover has a degeneration of type (n, m) on the boundary, the point y above x is singular, and its arithmetic genus equals $(m'+m)(p-1)/2$.

3) Let m, m' and n be positive integers such that m and m' are prime to p, and $m' > m$. Consider the cover given generically by the equation $X^p - X = T^{-m}\pi^{-pn} + \pi/T^{m'}$. This cover has a degeneration of type $(n, -m)$ on the boundary, the point y above x is singular, and its genus equals $(m' - m)(p - 1)/2$.

Next, we examine the case of Galois covers of degree p above formal germs at *double points.*

3.3.5. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[S,T]]/(ST - \pi^e)$ be the formal germ of an R-curve at an ordinary double point x of thickness e. Let $\mathcal{X}_1 := \operatorname{Spf} R[[S]]\{S^{-1}\}$, and $\mathcal{X}_2 := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the boundaries of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$, and with $\mathcal{Y}$ local. Assume that the special fiber of $\mathcal{Y}$ is reduced. We assume that $\mathcal{Y}_k$ has two branches at the point y. Let $d_\eta := r(p - 1)$ be the degree of the divisor of ramification in the morphism $f : Y_K \to X_K$. Let (n_i, m_i) be the degeneration type on the boundaries of $\mathcal{X}$, for $i = 1, 2$. Then necessarily $r + m_1 + m_2 \geq 0$, and $g_y = (r + m_1 + m_2)(p - 1)/2$.*

3.3.6. Proposition. *We use the same notations as in Proposition 3.3.5. We consider the remaining cases:*

1) $\mathcal{Y}_k$ has $p+1$ branches at y, in which case we can assume that $\mathcal{Y}$ is completely split above $\mathcal{X}_1$. Let (n_2, m_2) be the degeneration type on the second boundary $\mathcal{X}_2$ of $\mathcal{X}$. Then necessarily $r + m_2 - 1 \geq 0$, and $g_y = (r + m_2 - 1)(p-1)/2$.

2) $\mathcal{Y}_k$ has $2p$ branches at y, in which case $\mathcal{Y}$ is completely split above the two boundaries of $\mathcal{X}$, and $g_y = (r-2)(p-2)/2$.

With the same notations as in proposition 3.3.5, and as a consequence, one can detect whether the point y is a double point or not. More precisely, we have the following:

3.3.7. Corollary. *We use the same notations as in 3.3.5. Then y is an ordinary double point, which is equivalent to $g_y = 0$, if and only if x is an ordinary double point of thickness divisible by p, and $r = m_1 + m_2$. Moreover, if $r = 0$, then $g_y = 0$ is equivalent to $m_1 + m_2 = 0$.*

Next, we give examples of Galois covers of degree p above the formal germ of a double point which lead to singularities with genus 0, i.e. double points, and such that $r = 0$. These examples will be used in sections 4 and 5 in order to realize the "degeneration data" corresponding to Galois covers of degree p in equal characteristic $p > 0$.

3.3.8. Examples. The following are examples (given by explicit equations) of the different cases, depending on the possible degeneration type on the boundaries, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[S,T]]/(ST - \pi')$, with r $= 0$, and where g$_y = 0$ for a suitable choice of e. Note that $e = pt$ must be divisible by p. In all the following examples we have $r = 0$.

1) p-Purity: if f as above has an étale reduction type on the boundaries, and $r = 0$, then f is necessarily étale and hence is completely split since $\mathcal{X}$ is strictly henselian.

2) Consider the cover given generically by the equation $X^p - X = 1/T^m = S^m/\pi^{mpt}$, where m is a positive integer prime to p, which leads to a reduction on the boundaries of type $(0, -m)$ and (mt, m).

3) Let n and m be positive integers such that m is prime to p, and $n - tm > 0$. Consider the cover given generically by the equation $X^p - X = T^m/\pi^{np} = S^{-m}/\pi^{p(n-tm)}$, which leads to a reduction on the boundaries of type (n, m) and $(n - tm, -m)$.

In fact one can describe Galois covers of degree p above formal germs of double points (in equal characteristic p), which are étale above the generic fiber and with genus 0. Namely they are all of the form given in the above examples 3.3.8. In particular, these covers are uniquely determined (up to isomorphism) by their degeneration type on the boundaries. More precisely, we have the following:

3.3.9. Proposition. Let $\mathcal{X}$ be the formal germ of an R-curve at an ordinary double point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}_k$ reduced and local, and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ étale. Let $\mathcal{X}_i$, for $i = 1, 2$, be the boundaries of $\mathcal{X}$. Let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$, and let δ_i be the corresponding degree of the different (cf. 1.3.1). Let y be the closed point of $\mathcal{Y}$, and assume that $\mathrm{g}_y = 0$. Then there exists an isomorphism $\mathcal{X} \simeq \mathrm{Spf} R[[S, T]]/(ST - \pi^{tp})$, such that if $\mathcal{X}_2$ is the boundary corresponding to the prime ideal (π, S), one of the following holds:

a) The cover f is generically given by the equation $X^p - X = 1/T^m = S^m/\pi^{mpt}$, where m is a positive integer prime to p. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(0, -m)$, and (mt, m). Here $t > 0$ can be any integer. In this case $\delta_1 = 0$, and $\delta_2 = mt(p - 1)$.

b) The cover f is generically given by an equation $X^p - X = T^m/\pi^{np} = 1/\pi^{p(n-tm)}S^m$, where $m > 0$ is an integer prime to p, and $n > 0$ is such that $n - tm > 0$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type (n, m), and $(n - tm, -m)$. In this case $\delta_1 = n(p - 1)$, and $\delta_2 = (n - tm)(p - 1)$.

Proof. The proof is similar to the proof of 4.2.5 in [8] in the unequal characteristic case.

3.3.10. Variation of the different. The following result, which is a direct consequence of Proposition 3.3.9, describes the variation of the degree of the different from one boundary to another in a cover $f : \mathcal{Y} \to \mathcal{X}$ between formal germs at double points.

3.3.11. Proposition. *Let $\mathcal{X}$ be the formal germ of an R-curve at an ordinary double point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}_k$ reduced and local, and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ étale. Let y be the closed point of $\mathcal{Y}$. Assume that $\mathrm{g}_y = 0$, which implies necessarily that the thickness $e = pt$ of the double point x is divisible by p. For each integer $0 < t' < t$, let $\mathcal{X}_{t'} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at the ideal $(\pi^{pt'}, T)$. The special fiber of $\mathcal{X}_{t'}$ consists of a projective line $P_{t'}$ which meets two germs of double points x and x'. Let η be the generic point of $P_{t'}$, and let v_η be the corresponding discrete valuation of the function field of $\mathcal{X}$. Let $f_{t'}$; $\mathcal{Y}_{t'} \to \mathcal{X}_{t'}$ be the pull back of f, which is a Galois cover of degree p, and let $\delta(t')$ be the degree of the different induced by this cover above v_η (cf. 2.2.1). Also denote by $\mathcal{X}_i$, for $i = 1, 2$, the boundaries of $\mathcal{X}$. Let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$. let (n_i, m_i) be their degeneration type, and let δ_i be the corresponding degree of the different. Say $\delta_1 = \delta(0)$, $\delta_2 = \delta(t)$, and $\delta(0) \le \delta(t)$. We have $m := -m_1 = m_2$ which we assume to be positive. Then the following holds: for $0 \le t_1 \le t_2 \le t$ we have $\delta(t_2) = \delta(t_1) + m(p - 1)(t_2 - t_1)$, and $\delta(t')$ is an increasing function of t'.*

4. Semi-stable reduction of cyclic p-covers above formal germs of curves in equal characteristic $p > 0$.

In this section we use the following notations: R is a complete discrete valuation ring, of characteristic p, with residue field k which we assume to be *algebraically closed*, and fraction field $K := \text{Fr}R$. We denote by π a uniformizing parameter of R, and v_K the valuation of K which is normalized by $v_K(\pi) = 1$. For an R-scheme (resp. a formal R-scheme) $\mathcal{X}$ we denote by $\mathcal{X}_k := \mathcal{X} \times_R k$ (resp. $\mathcal{X}_K := \mathcal{X} \times_R K$) the special fiber of $\mathcal{X}$ (resp. its generic fiber (which in the case where $\mathcal{X}$ is formal is the associated rigid space (cf. [1])).

4.1. Let $\mathcal{X}$ be either a formal semi-stable R-curve, or the formal germ of a semi-stable R-curve X at a closed point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover, with group G, such that $\mathcal{Y}$ is normal (in case $\mathcal{X}$ is a germ we also require $\mathcal{Y}$ to be local). In this paper we are mainly concerned with the case where G is cyclic of order p. It follows easily from the theorem of semi-stable reduction for curves (cf. [2]) (as well as from the compactification process, established in 3.2, in the case of a germ) that after perhaps a finite extension R' of R, with fraction field K', the formal curve (resp. germ) $\mathcal{Y}$ has semi-stable reduction over K'. More precisely, there exists a birational and proper morphism $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$, where $\mathcal{Y}'$ is the normalization of $\mathcal{Y} \times_R R'$, such that $\tilde{\mathcal{Y}}_{K'} \simeq \mathcal{Y}'_{K'}$, and the following conditions hold:

(i) The special fiber $\tilde{\mathcal{Y}}_k := \tilde{\mathcal{Y}} \times_{\text{Spec} R'} \text{Spec}\, k$ of $\tilde{\mathcal{Y}}$ is reduced.

(ii) $\tilde{\mathcal{Y}}_k$ has only ordinary double points as singularities.

Moreover, there exists such a semi-stable model $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$ which is *minimal* for the above properties. In particular, the action of G on $\mathcal{Y}'$ extends to an action on $\tilde{\mathcal{Y}}$. Let $\tilde{\mathcal{X}}$ be the quotient of $\tilde{\mathcal{Y}}$ by G, which is a semi-stable model of $\mathcal{X}' := \mathcal{X} \times_R R'$. One has the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}} & \mathcal{Y}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ \tilde{\mathcal{X}} & \xrightarrow{\tilde{g}} & \mathcal{X}' \end{array}$$

One can also choose the semi-stable models $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{X}}$ above so that the set of points $B_{K'} := \{x_{i,K'}\}_{1 \le i \le r}$, consisting of the branch locus in the morphism $f'_K : \mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$ specialize in *smooth distincts* points of $\mathcal{X}'_k$. Moreover, one can choose such $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ which are minimal for these properties. Also there exists a *minimal* extension K' as above such that these conditions are satisfied. *In what follows we always assume that $K', \tilde{\mathcal{X}}$, and $\tilde{\mathcal{Y}}$ satisfy these later properties and are minimal in the above sense.*

In the case where $\mathcal{X}$ is the germ of a formal semi-stable R-curve X, at a closed point x, the fiber $\tilde{g}^{-1}(x)$ of the closed point x in $\tilde{\mathcal{X}}$ is a tree Γ of projective lines. The tree Γ is canonically endowed with some "*degeneration data*" that we will exhibit below and in the next section, in the case where $G \simeq \mathbb{Z}/p\mathbb{Z}$, and which take into account the geometry of the special fiber $\tilde{\mathcal{Y}}_k$ of $\tilde{\mathcal{Y}}$. This will mainly follow from the results which we recalled/established in sections 2 and 3.

4.2. We use the same notation as in 4.1. We consider the case where $\mathcal{X} \simeq \operatorname{Spf} A$ is the formal germ of a semi-stable R-curve X at a smooth point x, i.e. A is (non-canonically) isomorphic to $R[[T]]$. Let R' be a finite extension of R as in 4.1, and let π' be a uniformizer of R'. Below we exhibit the degeneration data associated with the semi-stable reduction $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$.

Deg. 1. Let $\wp := (\pi')$ be the ideal of $A' := A \otimes_R R'$ generated by π', and let $\hat{A}'_\wp$ be the completion of the localization of A' at $\wp$. Let $\mathcal{X}'_\eta := \operatorname{Spf} \hat{A}'_\wp$ be the *formal boundary* of $\mathcal{X}'$, and let $\mathcal{X}'_\eta \to \mathcal{X}'$ be the canonical morphism. Consider the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}'_\eta & \xrightarrow{f_\eta} & \mathcal{X}'_\eta \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \end{array}$$

The finite cover $f_\eta : \mathcal{Y}'_\eta \to \mathcal{X}'_\eta$ is a torsor under the finite commutative and flat R'-group scheme $\mathcal{M}_{n,R'}$, for some integer $n \geq 0$, as follows from 2.3.1. Let (n, m) be the degeneration type of the torsor f_η (cf. 2.3.2), which is canonically associated with f. The arithmetic genus g_y of the point y equals $(r + m - 1)(p - 1)/2$ (cf. 3.3.1), where $d_\eta := r(p - 1)$ is the degree of the divisor of ramification in the morphism $f'_{K'} : \mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$.

Deg.2. The fiber $\tilde{g}^{-1}(x)$ of the closed point x of $\mathcal{X}'$ in $\tilde{\mathcal{X}}$ is a tree Γ of projective lines. Let Vert $(\Gamma) := \{X_i\}_{i \in I}$ be the set of irreducible components of $\tilde{g}^{-1}(x)$, which are the vertices of the tree Γ. The tree Γ is canonically endowed with an origin vertex X_{i_0}, which is the unique irreducible component of $\tilde{g}^{-1}(x)$ which meets the point x. We fix an orientation of the tree Γ starting from X_{i_0} in the direction of the ends.

Deg.3. For each $i \in I$, let $\{x_{i,j}\}_{j \in S_i}$ be the set of points of X_i in which specializes some point of $B_{K'}$ (S_i may be empty). Also let $\{z_{i,j}\}_{j \in D_i}$ be the set of double points of $\tilde{\mathcal{X}}_k$ supported by X_i. In particular, $x_{i_0,j_0} := x$ is a double point of $\tilde{\mathcal{X}}_k$. We denote by B_k the set of all points $\bigcup_{i \in I} \{x_{i,j}\}_{j \in S_i}$, which is the set of specialization of the branch locus $B_{K'}$, and by D_k the set of double points of $\tilde{\mathcal{X}}_k$. Note that by hypothesis B_k has the same cardinality as $B_{K'}$.

Deg.4. Let $\mathcal{U}$ be the formal subscheme of $\tilde{\mathcal{X}}$ obtained by removing the formal fibers of the points in $\{B_k \cup D_k\}$. Let $\{\mathcal{U}_i\}_{i\in I}$ be the set of connected components of $\mathcal{U}$. The restriction $g_i : \mathcal{V}_i \to \mathcal{U}_i$, of g to $\mathcal{U}_i$, is a torsor under the commutative finite and flat R'-group scheme $\mathcal{M}_{n_i,R'}$ of rank p, for some integer $n_i \geq 0$, as follows from 2.2.1. Further, $g_{i,k} : \mathcal{V}_{i,k} := \mathcal{V}_i \times_{R'} k \to \mathcal{U}_{i,k} := \mathcal{U}_i \times_{R'} k$ is a torsor under the k-group scheme $\mathcal{M}_{n_i,R'} \times_{R'} k$, which is either étale isomorphic to $(\mathbb{Z}/p\mathbb{Z})_k$, or radicial isomorphic to $(\alpha_p)_k$. Moreover, when we move in the graph Γ from a fixed vertex X_i in the direction of a vertex $X_{i'}$ such that $n_{i'} = 0$ (following the above fixed orientation), then the corresponding integers n_i decrease strictly (as follows from 3.3.11).

Deg.5. Each smooth point $x_{i,j} \in B_k$ is endowed via g with a degeneration data on the boundary of the formal fiber of $\tilde{\mathcal{X}}$ at $x_{i,j}$ (in the same way that we exhibited the data in Deg.l above). More precisely, for each point $x_{i,j}$ we have the reduction type $(n_{i,j} := n_i, m_{i.j})$ on the boundary of the formal fiber $\tilde{\mathcal{X}}_{i,j} \simeq \mathrm{Spf} R[[T_{i,j}]]$ of $\tilde{\mathcal{X}}$ at this point, and which is induced by g. Then $r_{i,j} = m_{i,j}+1$, where $r_{i,j}(p-1)$ is the contribution to d_η of the points which specialize into $x_{i,j}$, as follows from 2.3.2 (since the point of $\tilde{\mathcal{Y}}$ above $x_{i,j}$ is smooth). In particular, $m_{i,j} \leq 1$, since $r_{i,j} \neq 0$.

Deg.6. Each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of $\tilde{\mathcal{X}}$, with origin vertex X_i and terminal vertex $X_{i'}$, is endowed with degeneration data $(n_{i,j} := n_i, m_{i,j})$ and $(n_{i',j'} := n_{i'}, m_{i',j'})$ induced by g on the two boundaries of the formal fiber of $\tilde{\mathcal{X}}$ at this point. Also, we have $m_{i,j} + m_{i',j'} = 0$ as follows from 3.3.7 (since $r = 0$ in this case, and the point of $\tilde{\mathcal{Y}}$ above $z_{i,j}$ is a double point). Let $e_{i,j}$ be the thickness of the double point $z_{i,j}$. Then $e_{i,j} = pt_{i,j}$ is necessarily divisible by p, and we have $n_i - n_{i'} = t_{i,j}m_{i,j}$ (as follows from 3.3.11).

Deg.7. It follows, after easy calculation, that:

$$g_y = \sum_{i\in I_{\mathrm{et}}} (-2 + \sum_{j\in S_i} (m_{i,j}+1) + \sum_{j\in D_i} (m_{i,j}+1))(p-1)/2$$

where I_{et} is the subset of I consisting of those i for which the torsor f_i is étale (i.e. such that $n_i = 0$).

4.3. Remark/Example. We could also have considered the minimal semi-stable models $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{X}}$, where we assume that the branched points on the generic fiber $\tilde{\mathcal{X}} \times_{R'} K'$ specialize into smooth (non necessarily distinct) points of $\tilde{\mathcal{X}} \times_{R'} k$, and exhibit the corresponding degeneration data in this case. In what follows we give an example of a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p, where $\mathcal{X} \simeq \mathrm{Spf} R[[T]]$ is the formal germ of a smooth point, and where one can exhibit these degeneration data. More precisely, for $m > 0$ an integer, such that both m and $m+1$ are prime to p, consider the cover given generically by the equation $X^p - X = (T^{-m} + \pi T^{-m-1})$. Here $r = m+2$, and this cover has a reduction of type $(0, -m)$ on the boundary. In particular, the arithmetic genus g_y of the closed point y of $\mathcal{Y}$ equals $(p-1)/2$. The degeneration data associated to the above cover consists necessarily of a tree with only one vertex and no edges. Thus is a unique projective line X_1, with a marked point x_1, and an étale torsor $f_1 : V_1 \to U_1 := X_1 - \{x_1\}$ above U_1 with conductor 2 at the point x_1.

The above considerations lead naturally to the following abstract, geometric and combinatorial, definition of degeneration data.

4.4. Definition. K'-simple degeneration data $\mathrm{Deg}(x)$ of type $(r, (n, m))$, and rank p, where K' is a finite extension of K, consist of the following:

Deg.1. $r \geq 0$ is an integer, m is an integer prime to p such that $r + m - 1 \geq 0$, and $n \geq 0$ is an integer. Further, G_k is a commutative finite and flat k-group scheme of rank p which is either étale if $n = 0$, or radicial of type α_p otherwise.

Deg.2. $\Gamma := X_k$ is an oriented tree of k-projective lines, with set of vertices $\mathrm{Vert}(\Gamma) := \{X_i\}_{i \in I}$, which is endowed with an origin vertex X_{i_0}, and a marked point $x := x_{i_0, j_0}$ on X_{i_0}. We denote by $\{z_{i,j}\}_{j \in D_i}$ the set of double points, or (non oriented) edges of Γ, which are supported by X_i. Further, we assume that the orientation of Γ is in the direction going from X_{i_0} towards its ends.

Deg.3. For each vertex X_i of Γ there is a set (which may be empty) of smooth marked points $\{x_{i,j}\}_{j \in S_i}$ in X_i.

Deg.4. For each $i \in I$, there is a torsor $f_i : V_i \to U_i := X_i - \{\{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}\}$ under a finite commutative and flat k-group scheme $G_{i,k}$ of rank p, which is either étale or radicial of type α_p, with V_i *smooth*. Moreover, for each $i \in I$ there is an integer $n_i \geq 0$ which equals 0 if and only if f_i is étale.

Deg.5. For each $i \in I$, and $j \in S_i$, are given integers $m_{i,j}$, where $m_{i,j}$ is the conductor of the torsor f_i at the point $x_{i,j}$ (cf. [9], I), with $m_{i_0, j_0} = -m$. We further assume that $m_{i,j} \leq 1$, if $n_i > 0$.

Deg.6. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$, there is an integer $m_{i,j}$ (resp. $m_{i',j'}$) prime to p, where $m_{i,j}$ (resp. $m_{i',j'}$) is the conductor of the torsor f_i (resp. $f_{i'}$) at the point $z_{i,j}$ (resp. $z_{i',j'}$) (cf. [9], I). These data must satisfy $m_{i,j} + m_{i',j'} = 0$.

Deg.7. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of Γ, with origin vertex X_i, there is an integer $e_{i,j} = pt_{i,j}$ divisible by p such that, with the same notation as above, we have $n_i - n_{i'} = m_{i,j} t_{i,j}$. Moreover, associated with x is an integer $e = pt$ such that $n - n_{i_0} = mt$.

Deg.8. Let I_{et} be the subset of I consisting of those i for which $G_{i,k}$ is étale. Then the following equality should hold:

$$(r-m-1)(p-1)/2 = \sum_{i\in I_{\text{et}}}(-2+\sum_{j\in S_i}(m_{i,j}+1)+\sum_{j\in D_i}(m_{i,j}+1))(p-1)/2.$$

The integer $g := (r+m-1)(p-1)/2$ is called the *genus* of the degeneration data $\text{Deg}(x)$.

Note that if K' is a finite extension of K', then K'-*simple* degeneration data $\text{Deg}(x)$ can be naturally considered as K'-degeneration data, by multiplying all integers n, n_i, and $e_{i,j}$, by the ramification index of K' over K'.

There is a natural notion of isomorphism of simple degeneration data of a given type and rank p, relative to some finite extension K' of K. We will denote by Deg_p the set of isomorphism classes of K'-*simple* degeneration data of rank p, where K' runs over all finite extensions K' of K. The above discussion in 4.2 can be reinterpreted as follows:

4.5. Proposition. *Let $\mathcal{X}$ be the germ of a formal R-curve at a smooth point x, and let $f : \mathcal{Y} \to \mathcal{X}$ be a cyclic p-cover, with $\mathcal{Y}$ normal and local. Then one can associate with f, canonically, simple degeneration data* $\text{Deg}(x) \in \text{Deg}_p$ *which describes the semi-stable reduction of $\mathcal{Y}$. In other words, there exists a canonical "specialization" map* Sp: $H^1_{\text{et}}(\text{Spec}\, L, \mathbb{Z}/p\mathbb{Z}) \to \text{Deg}_p$, *where L is the function field of the geometric fiber $\overline{\mathcal{X}} := \mathcal{X} \times_R \overline{R}$ of $\mathcal{X}$, and $\overline{R}$ is the integral closure of R in an algebraic closure of K.*

Reciprocally, we have the following result of realization of degeneration data for such covers:

4.6. Theorem. *The above specialization map* $\text{Sp} : H^1_{\text{et}}(\text{Spec}L,\mathbb{Z}/p\mathbb{Z}) \to \text{Deg}_p$ *defined in* 4.4 *is surjective.*

Proof. Consider a degeneration data $\text{Deg}(x) \in \text{Deg}_p$. We have to show that $\text{Deg}(x)$ is associated to some cyclic p-cover above the formal germ of a smooth R-curve, after possibly enlarging R. We assume that the degeneration data is of type $(r,(n,m))$. We only treat the case $n = 0$, the case where $n > 0$ is treated in a similar way. The proof is done by induction on the length of the tree Γ of $\text{Deg}(x)$. Assume first that the tree Γ has minimal length. Thus Γ consists of one irreducible component $X = \mathrm{P}^1_k$, with one marked (double) point x, and r smooth distinct marked points $\{x_i\}_{i=1}^r$. Let $U := X - \{x, x_i\}_i$, and let $\overline{f} : V \to U$ be the torsor given by the data Deg.4, which is necessarily an α_p-torsor (i.e. the integer $n_i := n'$ associated to the vertex X in Deg.4 is non zero). First, for each $i \in \{1,\dots,r\}$ consider the formal germ $\mathcal{X}_i := \text{Spf}R[[T_i]]$, and the cyclic p-cover $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ given by the equation $Y_i^p - \pi^{n'} Y_i = T_i^{m_i}$, where m_i is the "conductor" associated to the point x_i in Deg.5, and n' is the positive integer associated to $\overline{f}$ in Deg.4. Let $\mathcal{X}$ be a formal projective R-line with special fiber X. Let $X' := X - \{x\}$, and let $\mathcal{X}'$ (resp $\mathcal{U}$) be the formal fiber of X' (resp. of U) in $\mathcal{X}$. The torsor $\overline{f}$ is given by an equation $t^p = \overline{u}$ where $\overline{u}$ is a regular function on U. Let u be a regular function on $\mathcal{U}$ which lifts $\overline{u}$. The cover $f : \mathcal{V} \to \mathcal{U}$, given by the equation $Y^p - \pi^{n'(p-1)} Y = u$, is a torsor under the R-group scheme $\mathcal{M}_{n'}$ which lifts the α_p-torsor $\overline{f}$. By construction the torsor f has a reduction on the formal boundary at each point x_i of type (n', m_i), which coincides with the degeneration type of the cover f_i above the formal boundary of $\mathcal{X}_i$. The technique of formal patching (cf. [8], 1) allows then one to construct a p cyclic cover $f' : \mathcal{Y}' \to \mathcal{X}'$ which restricted to $\mathcal{U}$ is isomorphic to f', and which restricted to $\mathcal{X}_i$, for each $i \in \{i, \ldots, r\}$, is isomorphic to f_i (cf. 1oc. cit.). Let $\mathcal{X}_1 \to \mathcal{X}$ be the blow up of $\mathcal{X}$ at the point x, and let X_1 be the exceptional fiber in $\mathcal{X}_1$, which meets X at the double point x. Let $e = pt$ be the integer associated to the marked double point x via Deg.7. After enlarging R we can assume that the double point x of $\mathcal{X}_1$ has thickness e. We have $-n' = mt$ by assumption. Let $\mathcal{X}_1'$ be the formal fiber of $X_1' := X_1 - \{x\}$ in $\mathcal{X}_1$. Let $f_1' : \mathcal{Y}_1' \to \mathcal{X}_1'$ be the étale $\mathbb{Z}/p\mathbb{Z}$-torsor given by the equation $Y'^p - Y' = h^m$, where h is a"parameter" on $\mathcal{X}_1'$. Further, let $\mathcal{X}_{1,x} \simeq \mathrm{Spf} R[[S,T]]/(ST - \pi^{pt})$ be the formal germ of $\mathcal{X}_1$ at the double point x. Consider the cover $f_{1,x} : \mathcal{Y}_{1,y} \to \mathcal{X}_{1,x}$ given by the equation $Y^p - Y = S^m = \pi^{ptm} T^{-m}$. Then $\mathcal{Y}_{1,y}$ is the formal germ of a double point of thickness t (cf. 3.3.8, 2)). Moreover, the cover f_1' (resp. f') has the same degeneration type (by construction), on the boundary corresponding to the double point x, as the degeneration type of the cover $f_{1,x}$ above the formal boundary with parameter T (resp. above the formal boundary with parameter S). A second application of the formal patching techniques allows one to construct a p-cyclic cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ which restricted to $\mathcal{X}'$ (resp. $\mathcal{X}_1'$ and $\mathcal{X}_{1,x}$) is isomorphic to f' (resp. to f_1' and f_x). Let $\tilde{\mathcal{X}}$ be the R-curve obtained by contracting the irreducible component X in $\mathcal{X}_1$. We denote the image of the double point x in $\tilde{\mathcal{X}}$ simply by x. The cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ induces canonically a p-cyclic cover $\tilde{f} : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ above $\tilde{\mathcal{X}}$. Let $\mathcal{X}_x \simeq R[[T]]$ be the formal germ of $\tilde{\mathcal{X}}$ at the smooth point x. Then $\tilde{f}$ induces canonically a p-cyclic cover $f_x : \mathcal{Y}_y \to \mathcal{X}_x$ where y is the closed point of $\mathcal{Y}$ above x. Now it is easy to see that the degeneration data associated to f_x, via 4.5, is isomorphic to the degeneration data $\mathrm{Deg}(x)$ we started with. Finally, the proof in the general case is very similar and is left to the reader. The only modification in the proof above is that one has to consider the p-cyclic cover $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ (above the formal germs $\mathcal{X}_i := \mathrm{Spf} R[[T_i]]$) which one obtains by the induction hypothesis as realization of the degeneration data, induced by

$\mathrm{Deg}(x)$, on the subtree Γ_i of Γ which starts from the edge x_i in the direction of the ends, and which clearly has length smaller than the length of Γ.

4.7. Remarks.

1. One can also define in the same way as in 4.4, and using the results of sections 2 and 3, the set of isomorphism classes of "double" degeneration data associated to the minimal semi-stable model of p-cyclic covers $f : \mathcal{Y} \to \mathcal{X}$ above the formal germ of a double point. Moreover, one can prove, in asimilar way as in 4.6, a result of realization for such a degeneration data.

2. It is easy to construct examples of p-cyclic covers $f : \mathcal{Y} \to \mathcal{X}$, above the formal germ of a smooth point $\mathcal{X}$, where the special fiber $\mathcal{Y}_k$ is singular and unibranche at the closed point y of $\mathcal{Y}$, and such that the configuration of the special fiber of a semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ is not a tree-like. More precisely, consider the simple degeneration data $\mathrm{Deg}(x)$ of type (n, m), with $n > 0$, $m > 0$, and $n = mt$, which consists of a graph Γ with two vertices X_1 and X_2 linked by a unique edge $\tilde{x}$ with given marked points $x = x_1$ on X_1 and x_2 on X_2. Further, X_1 is the origin of Γ. As part of the data are given étale torsors of rank p : $f_1 : V_1 \to U_1 := X_1 - \{x_1\}$, with conductor m at $x = x_1$, and $f_2 : V_2 \to U_2 := X_2 - \{x_2\}$, with conductor m' at x_2. Also is given the thickness $e = pt$ at the "double" point x with $n = tm$. Then it follows from 4.6 that there exists, after possibly a finite extension of R, a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p above the formal germ $\mathcal{X} \simeq \mathrm{Spf} R[[T]]$ at the smooth R-point x, such that the simple degeneration data associated to the above cover f is the above given one. Moreover, by construction the singularity of the closed point y of $\mathcal{Y}$ is unibranche, and the configuration of the semi-stable reduction of $\mathcal{Y}$ consists of two projective curves which meet at p-double points (the above cover will be étale in reduction above the double point $\tilde{y}$). In particular, one has $p-1$ cycles in this configuration.

5. Semi-stable reduction of cyclic p-covers above proper curves in equal characteristic p.

In what follows we use the same notation as in section 4. Our aim in this section is to describe the "degeneration data" that arise from the semi-stable reduction of Galois covers of degree p above smooth, and proper, R-curves, and prove a realization theorem for such data.

5.1. Let $\mathcal{X}$ be a formal smooth and proper R-curve, and let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $G \simeq \mathbb{Z}/p\mathbb{Z}$, such that $\mathcal{Y}$ is normal. We denote by X the special fiber $\mathcal{X}_k := \mathcal{X} \times_R k$ of $\mathcal{X}$. Let R', K', $\mathcal{X}'$, $\mathcal{Y}'$, $\tilde{\mathcal{X}}$, and $\tilde{\mathcal{Y}}$ be as in 4.1. Recall that we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}} & \mathcal{Y}' \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle f'}\big\downarrow \\ \tilde{\mathcal{X}} & \xrightarrow{\tilde{g}} & \mathcal{X}' \end{array}$$

5.2. We use the same notations as in 4.1 and 5.1. Let R' be a finite extension of R as in 4.1, and let π' be a uniformizer of R'. Below we exhibit the *degeneration data* associated with the minimal semi-stable reduction $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$.

Deg.1. Let η' be the generic point of the special fiber of $\mathcal{X}'$, and let $\mathcal{X}'_\eta$ be the formal germ of $\mathcal{X}'$ at η'. Consider the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}'_\eta & \xrightarrow{f_\eta} & \mathcal{X}'_\eta \\ \big\downarrow & & \big\downarrow \\ \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \end{array}$$

Then $f_\eta : \mathcal{Y}'_\eta \to \mathcal{X}'_\eta$ is a torsor under the finite commutative and flat R'-group scheme $\mathcal{M}_{n,R'}$ of rank p, for some integer $n \geq 0$, as follows from 2.2.1. We denote by (n,m) the degeneration type of the torsor f_η (cf. 2.3.2), which is canonically associated with f.

Deg.2. Let $\{x_i\}_{i\in I'}$ be the finite set of closed points of X in which specialize some branched points of the morphism $f'_{K'} : \mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$. We denote this set by B_k. Let $U := X - B_k$, and let $\mathcal{U}'$ be the formal fiber of U in $\mathcal{X}'$. Then the restriction $f' : \mathcal{V}' \to \mathcal{U}'$ of f' to $\mathcal{U}'$ is a torsor under a finite and flat R-group scheme of rank p (as follows from 2.2.1), which is necessarily the group scheme $\mathcal{M}_{n,R'}$. When the torsor $f'_k : \mathcal{V}'_k \to \mathcal{U}'_k = U$ is radicial, let ω be the associated differential form (cf. [9], 1), and let Z_K be the set of zeros of ω. Let $\mathrm{Crit(f)} = B_k$ if f'_k is étale (resp. $\mathrm{Crit(f)} = B_k \cup Z_k$ if f'_k is radicial), and call this the set of *critical points* of f. If $y \in \mathcal{Y}$ is a singular point, then $x = f(y) \in \mathrm{Crit}(f)$ necessarily.

Deg.3. Let $\mathrm{Crit}\ (f) = \{x_i\}_{i\in I}$. For each $i \in I$, let $\mathcal{X}'_i$ be the formal fiber of x_i in $\mathcal{X}'$, and let $f'_i : \mathcal{Y}'_i \to \mathcal{X}'_i$ be the cover induced by f'. Let (n_i, m_i) be the degeneration type of f'_i on the boundary (cf. 2.3.2). Then necessarily all the integers n_i are equal to n.

Deg.4. For each $i \in I$, the Galois cover $f'_i : \mathcal{Y}'_i \to \mathcal{X}'_i$ gives rise, via 4.5, to K'-simple degeneration data $\mathrm{Deg}(x_i)$ of type $(r_i, (n = n_i, m_i))$, where r_i is the number of branched points which specialize in x_i.

The above considerations lead naturally to the following abstract, geometric and combinatorial, definition of degeneration data.

5.3. Definition. Smooth K'-degeneration data $\text{Deg}(X)$, of rank p, consist of the following data:

Deg.1. K' is a finite extension of K. X is a proper and smooth k-curve, endowed with a finite set B_k of closed (mutually distinct) marked points. Let $U := X - B_k$.

Deg.2. $\overline{f} : V \to U$ is a torsor under a finite and flat k-group scheme G_k of rank p, and $n \geq 0$ is an integer, which equals 0 if and only if G_k is étale.

Deg.3. Let $\text{Crit}(\overline{f}) = \{x_i\}_{i\in I}$ be the set B_k if $\overline{f}$ is étale (resp. the set $\text{Crit}(\overline{f}) = B_k \cup Z_k$ if $\overline{f}$ is radicial, where Z_k is the set of zeros of the corresponding differential form). For each $i \in I$, let m_i be the conductor of the above torsor $\overline{f}$ at the point x_i (cf. [9], I). We assume that we are given K'-*simple* degeneration data $\text{Deg}(x_i)$ of type $(r_i, (n, m_i))$ (cf. 4.4).

There is a natural notion of isomorphism of smooth degeneration data of rank p, relative to a given finite extension K' of K, and associated *with* a smooth and proper k-curve X. We will denote by $\text{DEG}_p(\text{X})$ the set of isomorphism classes of smooth degeneration data of rank p, associated with X. The above discussion in 5.2 can be reinterpreted as follows:

5.4. Proposition. *Let $\mathcal{X}$ be a formal, proper, and smooth R-curve with special fiber $\mathcal{X}_k := \mathcal{X} \times_R k$, and let $f : \mathcal{Y} \to \mathcal{X}$ be a cyclic p-cover with $\mathcal{Y}$ normal. Then one can associate with f, canonically, a smooth degeneration data $\text{Deg}(\mathcal{X}_k) \in \text{DEG}_p(\mathcal{X}_k)$, which describes the semi-stable reduction of $\mathcal{Y}$. In other words there exists a canonical "specialization" map $\text{Sp} : H^1_{\text{et}}(\text{Spec}L, \mathbb{Z}/p\mathbb{Z}) \to \text{DEG}_p(\mathcal{X}_k)$, where L is the function field of the geometric fiber $\overline{\mathcal{X}} := \mathcal{X} \times_R \overline{R}$ of $\mathcal{X}$, and $\overline{R}$ is the integral closure of R in an algebraic closure of K.*

Reciprocally, we have the following result of realization of degeneration data for such covers:

5.5. Theorem. *Let $\text{Deg}(X) \in \text{DEG}_p(\text{X})$ be a smooth degeneration data of rank p, associated with the k-proper and smooth curve X. Then there exists a smooth, formal, and proper R-curve $\mathcal{X}'$, with special fiber $\mathcal{X}'_k := \mathcal{X}' \times_R k$ isomorphic to X, and such that $\text{Deg}(X)$ is in the image of the specialization map $\text{Sp} : H^1_{\text{et}}(\text{Spec}\, L, \mathbb{Z}/p\mathbb{Z}) \to \text{DEG}_p(\text{X})$ defined in 5.4, where L is the function field of the geometric fiber $\mathcal{X}' \times_R \overline{R}$ of $\mathcal{X}'$, and $\overline{R}$ is the integral closure of R in an algebraic closure of K.*

Proof. Consider smooth degeneration data $\text{Deg}(X) \in \text{DEG}_p(\text{X})$. We assume for simplicity that $\text{Deg}(X)$ is K-degeneration data. We have to show that $\text{Deg}(X)$ is associated, via the map in 5.4, with some cyclic p-cover above a formal, proper, and smooth R-curve $\mathcal{X}'$. We only treat explicitly the case where $n = 0$ (i.e. the torsor $\overline{f}$ in Deg.2 is étale), the remaining cases are treated similarly. Let $\mathcal{X}$ be a formal, smooth, and proper R-curve whose special fiber $\mathcal{X}_k$ is isomorphic to X. Let $\mathcal{U}$ be the formal fiber of U in $\mathcal{X}$, and for $i \in I$, let $\mathcal{X}_i$ be the formal fiber of x_i in $\mathcal{X}$. The étale torsor $\overline{f}$ given by Deg.2 can be lifted to an étale torsor $f : \mathcal{V} \to \mathcal{U}$, by the theorems of lifting of étale covers (one can in this specific situation write down an explicit lifting). Also, for $i \in I$, let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be a Galois cover of degree p which is a realization (as in 4.6) of the degeneration data $\mathrm{Deg}(x_i)$ given by Deg.3, and which is of type $(r_i,\ (0, m_i))$, where m_i is the Hasse-conductor of $\overline{f}$ at x_i. By construction the covers f and f_i coincide on the formal boundaries at the points x_i. The technique of formal patching (cf. [8], 1) allows one to construct a p-cyclic cover $f' : \mathcal{Y}' \to \mathcal{X}'$, where $\mathcal{X}'$ is a proper and smooth formal curve. This is obtained by gluing $\mathcal{U}$ to the $\mathcal{X}_i$ $(i \in \{1, \dots, r\})$, by identifying the boundaries corresponding to each point x_i, via some specific isomorphisms, which when restricted to $\mathcal{U}$ are isomorphic to f, and when restricted to $\mathcal{X}_i$, for each $i \in I$, are isomorphic to f_i (cf. 1oc. cit.). Note that in general $\mathcal{X}'$ is not isomorphic to $\mathcal{X}$. Indeed the identifications of formal boundaries used to construct $\mathcal{X}'$ (which are specific isomorphisms), need not be compatible with those which lead to the construction of $\mathcal{X}$ by gluing $\mathcal{U}$ to the $\mathcal{X}_i$.

The technique of formal patching used in the proof of 5.5 can be used to prove the following result.

5.6. Theorem. *Let X be a smooth and proper k-curve, and let $f : Y \to X$ be a torsor under a finite and flat k-group scheme G_k of rank p. Then there exists, a smooth and proper R-curve $\mathcal{X}$ with special fiber isomorphic to X, and a torsor $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ under an R-group scheme G_R, which is commutative finite and flat of rank p, such that the torsor $\tilde{f}_k : \mathcal{Y}_k \to \mathcal{X}_k$ induced by $\tilde{f}$ on the level of special fibers is isomorphic to f. In other words the torsor $\tilde{f}$ lifts f. In particular, G_k is isomorphic to the special fiber of G_R.*

Proof. When f is an étale torsor, 5.6 is a consequence of the theorems of lifting of étale covers (cf. [11]). Moreover, in this case one can specify the lifting $\mathcal{X}$ of the curve X. So it remains to treat the case where $G_k \simeq \alpha_p$. By the formal GAGA theorems it suffices to realize a lifting of f over a formal, proper, and smooth R-curve $\mathcal{X}$. Let $\{y_i\}_{i \in I}$ be the finite set of singular points in Y, and let $\{x_i\}_{i \in I}$ be the set of their images in X, which we call the set of *critical points* of the torsor f. For $i \in I$, let $m_i > 0$ be the *conductor* of the torsor f at the point x_i (cf. [9], 1.5). The arithmetic genus g_y of y equals $(m-1)(p-1)/2$.

Let $U := X - \{x_i\}_{i \in I}$, and let $\mathcal{U}$ be a formal affine scheme whose special fiber equals U. Also for $i \in I$, let $\mathcal{X}_i \simeq \mathrm{Spf} R[[T_i]]$ be the formal germ of an R-curve at a smooth point x_i. The restriction $f' : V \to U$ of f to U is an α_p torsor, given by an equation $y^p = \overline{u}$ where $\overline{u}$ is a regular function on U. Let ω be the differential form $d\overline{u}$. Then ω is a global differential form on X: it is the differential form associated with the torsor f. Moreover the conductor m_i equals $\text{ord}_{x_i}(\omega) + 1$ (cf. [9], 5.1).

Let $n \geq 0$ be an integer. Let $\tilde{f} : \mathcal{V} \to \mathcal{U}$ be the $\mathcal{M}_{n,R}$-torsor given by an equation $Y^p - \pi^{(n-1)p} Y = u$, where u is a regular function on $\mathcal{U}$ which lifts $\overline{u}$. Also, for $i \in I$ consider the $\mathcal{M}_{n,R}$-torsor $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ given by an equation $Y^p - \pi^{(n-1)p} Y = T_i^{m_i}$, which has a reduction on the boundary of type $(n, -m_i)$. By construction, this is the same reduction type as that of the torsor $\tilde{f}$ on the formal boundary corresponding to the point x_i. The technique of formal patching (cf. [8], 1) allows one to construct an $\mathcal{M}_{n,R}$-torsor $f' : \mathcal{Y} \to \mathcal{X}$, where $\mathcal{X}$ is a proper and smooth formal curve. This is obtained by gluing $\mathcal{U}$ to the $\mathcal{X}_i (i \in I)$, by identifying the boundaries via some specific isomorphisms, which when restricted to $\mathcal{U}$ are isomorphic to $\tilde{f}$, and when restricted to $\mathcal{X}_i$, for each $i \in I$, are isomorphic to f_i (cf. 1oc. cit.). In particular, the special fiber of $\mathcal{X}$ is isomorphic to X.

Note that, in general, if $\mathcal{X}'$ is another formal curve which lifts X, then we can not construct a lifting f' of f as above, above the curve $\mathcal{X}'$, even if the formal fiber of U in $\mathcal{X}'$ is isomorphic to $\mathcal{U}$. Indeed, the identification of formal boundaries used to construct $\mathcal{X}$ (which are specific isomorphisms), need not be compatible with those which lead to the construction of $\mathcal{X}'$ by gluing $\mathcal{U}$ to the $\mathcal{X}_i$.

5.7. Remarks.

1. Theorem 5.5 is stronger than 5.6. Indeed a translation of 5.6 in the language of 5.5 leads not only to the lifting of the torsor f, but to a lifting with specified simple degeneration data at the critical points.

2. In [5] Maugeais proved (in equal characteristic $p > 0$) in theorem 5.4 a global result of lifting for finite "admissible" covers of degree p between semi-stable curves. The methods used in the proof of 5.6 are essentially the same as he uses but more direct in the sense that the lemmas 4.2, 4.4, and corollary 4.3 he uses are avoided and we use instead our results 3.3.3, 3.3.4, and 3.3.8 which are direct consequences of the computation of vanishing cycles.

3. The same proof as in 5.6 shows that it is possible to lift, as above, α_{p^n}-torsors $f\colon Y \to X$ above a proper and smooth k-curve X.

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