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Lie Algebras Represented as a Sum of Two Subalgebras

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Abstract

Let L be a Lie algebra represented as a sum of two subalgebras A and B. We prove that if L belongs to a subcla of the cla of locally finite Lie algebras over a field of characteristic $\neq 2$ and both A and B are locally nilpotent, then L is locally soluble. We also prove that if L is a serially finite Lie algebra over a field of characteristic zero, then any common serial subalgebra of A and B is serial in L.

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LIE ALGEBRAS REPRESENTED AS A SUM OF TWO SUBALGEBRAS

MASANOBU HONDA AND TAKANORI SAKAMOTO

Abstract. Let L be a Lie algebra represented as a sum of two subalgebras A and B. We prove that if L belongs to a subclass of the class of locally finite Lie algebras over a field of characteristic $\neq 2$ and both A and B are locally nilpotent, then L is locally soluble. We also prove that if L is a serially finite Lie algebra over a field of characteristic zero, then any common serial subalgebra of A and B is serial in L.

1. Introduction

Groups G factorized by two subgroups A and B, i.e. $G = AB$, have been investigated by many authors for some decades. Among the works Kegel [8] and Wielandt [15] established the well-known theorem: if G is finite, and A and B are nilpotent, then G is soluble.

In Lie algebras there is a corresponding result: If a finite-dimensional Lie algebra L over a field $\mathfrak{k}$ of characteristic $\neq 2$ is represented as a sum of two nilpotent subalgebras A and B, then L is soluble. Goto [4] proved the case of char $\mathfrak{k} = 0$ and Panyukov [10] did the case of char $\mathfrak{k} = p > 2$. On the other hand, Aldosray [2] showed that if $L = A + B$ is an ideally finite Lie algebra over a field of characteristic zero, then any common ascendant subalgebra of both A and B is ascendant in L.

In this paper we shall generalize the result of Goto and Panyukov to a certain class of infinite-dimensional Lie algebras and extend the result of Aldosray to a wider class than that of ideally finite Lie algebras.

In Section 2 we shall show that in a locally finite Lie algebra L a common weakly serial subalgebra of each subalgebra X_i of L for $i \in I$ is always a weakly serial subalgebra of $\langle X_i \mid i \in I \rangle$ (Theorem 2). Let L be a Lie algebra represented as a sum of two subalgebras A and B. In Section 3 we shall prove that if L is a serially finite Lie algebra (resp. a hyperfinite, serially finite Lie algebra) over a field of characteristic zero, then any common serial (resp. ascendant) subalgebra of A and B is serial (resp. ascendant) in L (Theorem 8 (resp. Corollary 9)). In Section 4 we shall verify that if L belongs to the subclass $\text{L(wser)}\mathfrak{F}$ of the class of locally

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finite Lie algebras over a field of characteristic $\neq 2$ and both A and B are locally nilpotent, then L is locally soluble (Theorem 15).

2. Notation and terminology

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [3] for the use of notation and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For a totally ordered set Σ, a series (resp. a weak series) from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma \mid \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of L such that

(1) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,
(2) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,
(3) $L \backslash H = \cup_{\sigma \in \Sigma}(\Lambda_\sigma \backslash V_\sigma)$,
(4) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

H is a serial (resp. a weakly serial) subalgebra of L, which we denote by $H \text{ser} L$ (resp. $H \text{wser} L$), if there exists a series (resp. a weak series) from H to L. For an ordinal σ, H is a σ-step ascendant (resp. weakly ascendant) subalgebra of L, denoted by $H \lhd^\sigma L$ (resp. $H \leq^\sigma L$), if there exists an ascending chain $(H_\alpha)_{\alpha \leq \sigma}$ of subalgebras (resp. subspaces) of L such that

(1) $H_0 = H$ and $H_\sigma = L$,
(2) $H_\alpha \lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1}, H] \subseteq H_\alpha$) for any ordinal $\alpha < \sigma$,
(3) $H_\lambda = \cup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant (resp. a weakly ascendant) subalgebra of L, denoted by $H \text{asc} L$ (resp. $H \text{wasc} L$), if $H \lhd^\sigma L$ (resp. $H \leq^\sigma L$) for some ordinal σ. When σ is finite, H is a subideal (resp. a weak subideal) of L and denoted by $H \text{si} L$ (resp. $H \text{wsi} L$). For an ordinal α, we denote by $L^{(\alpha)}$ the α-th term of the transfinite derived series of L. A subspace H of L invariant under all derivations of L is said to be a characteristic ideal and denoted by $H \text{ch} L$.

Let $\mathfrak{X}, \mathfrak{Y}$ be classes of Lie algebras and let Δ be any of the relations $\leq, \lhd, \text{ch}, \text{si}, \text{asc}, \text{ser}, \text{wser}$. $\mathfrak{X}\mathfrak{Y}$ is the class of Lie algebras L having an ideal $I \in \mathfrak{X}$ such that $L/I \in \mathfrak{Y}$. A Lie algebra L is said to lie $\text{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra H of L such that $X \subseteq H \;\Delta\; L$. In particular we write $\text{L}\mathfrak{X}$ for $\text{L}(\leq)\mathfrak{X}$. When $L \in \text{L}\mathfrak{X}$ (resp. $\text{L}(\text{ser})\mathfrak{X}$), L is called a locally (resp. a serially) $\mathfrak{X}$-algebra. $\mathfrak{F}, \mathfrak{A}, \mathfrak{N}, \mathfrak{Z}$ and $\text{E}\mathfrak{A}$ are the classes of Lie algebras which are finite-dimensional, abelian, nilpotent, hypercentral and soluble respectively. The $\mathfrak{X}$-residual $\lambda_{\mathfrak{X}}(L)$ of L is the intersection of the ideals I of L such that $L/I \in \mathfrak{X}$. $\acute{\text{E}}_\mu(\Delta)\mathfrak{X}$ is the class of Lie algebras L having an ascending series $(L_\alpha)_{\alpha \leq \mu}$ of Δ-subalgebras such that

(1) $L_0 = 0$ and $L_\mu = L$,
(2) $L_\alpha \lhd L_{\alpha+1}$ and $L_{\alpha+1}/L_\alpha \in \mathfrak{X}$ for any ordinal $\alpha < \mu$,
(3) $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$ for any limit ordinal $\lambda \leq \mu$.

We define $\acute{\mathrm{E}}(\Delta)\mathfrak{X} = \cup_{\mu>0}\acute{\mathrm{E}}_\mu(\Delta)\mathfrak{X}$. In particular we write $\acute{\mathrm{E}}\mathfrak{X}$ for $\acute{\mathrm{E}}(\leq)\mathfrak{X}$. When $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{X}$, L is called a hyper $\mathfrak{X}$-algebra. The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L. For a locally finite Lie algebra L the locally soluble radical $\sigma(L)$ of L is the unique maximal locally soluble ideal of L. The set of left Engel elements of L is denoted by $\mathfrak{e}(L)$.

3. Common weakly serial subalgebras

Before considering a common ascendant subalgebra of two permutable subalgebras in Section 4, we shall state more general forms in the following interesting theorem, which is a generalization of [12, Theorem 7]. To do this we need the following useful result.

Lemma 1 ([5, Theorem 2.12]). *Let H be a subalgebra of a locally finite Lie algebra L. Then $H \,\mathrm{wser}\, L$ if and only if $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$ and $H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \subseteq \mathfrak{e}(L/\lambda_{\mathrm{L}\mathfrak{N}}(H))$.*

Theorem 2. *Let L be a locally finite Lie algebra over any field and let $\{X_i\}_{i\in I}$ be a collection of subalgebras of L. If H is a common weakly serial subalgebra of X_i for any $i \in I$, then H is a weakly serial subalgebra of $\langle X_i \mid i \in I\rangle$.*

Proof. We may put $L = \langle X_i \mid i \in I\rangle$. Using Lemma 1 we have $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd X_i$ for any $i \in I$, and so $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$. We may also assume that $\lambda_{\mathrm{L}\mathfrak{N}}(H) = 0$ by $\lambda_{\mathrm{L}\mathfrak{N}}(H/\lambda_{\mathrm{L}\mathfrak{N}}(H)) = 0$ and [5, Proposition 2.5]. Then we get $H \subseteq \mathfrak{e}(X_i)$ for all $i \in I$ by using Lemma 1.

On the other hand, L is spanned by the elements of a form $[x_1, x_2, \ldots, x_n]$, where each x_k belongs to $\cup_{i\in I} X_i$. For any $h \in H$, there is an $m \in \mathbb{N}$ such that $x_k(\mathrm{ad}\ h)^m = 0$ for $1 \leq k \leq n$. Then we can show that

$$[x_1, x_2, \ldots, x_n](\mathrm{ad}\ h)^{nm} = 0$$

by induction on n, using Leibniz formula. Therefore we have $H \subseteq \mathfrak{e}(L)$. Thus it follows from Lemma 1 that $H \,\mathrm{wser}\, L$. □

As a direct result of Theorem 2, we have the following:

Corollary 3. *Let L be a Lie algebra over any field and let $\{X_i\}_{i\in I}$ be a collection of subalgebras of L.*

(1) *If $L \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$ and $H \,\mathrm{asc}\, X_i$ for any $i \in I$, then $H \,\mathrm{asc}\langle X_i \mid i \in I\rangle$.*
(2) *If $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$ and $H \,\mathrm{wasc}\, X_i$ for any $i \in I$, then $H \,\mathrm{wasc}\langle X_i \mid i \in I\rangle$.*
(3) *If $L \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ and $H \,\mathrm{ser}\, X_i$ for any $i \in I$, then $H \,\mathrm{ser}\langle X_i \mid i \in I\rangle$.*

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Proof. (1) Since $\acute{\mathrm{E}}(\triangleleft)\mathfrak{F} \le \mathrm{L}\mathfrak{F}$ by [7, Corollary 3.3] we obtain $L \in \mathrm{L}\mathfrak{F}$. Hence Theorem 2 implies that $H\mathrm{wser}\langle X_i \mid i \in I\rangle$. Because $\langle X_i \mid i \in I\rangle \in \acute{\mathrm{E}}(\triangleleft)(\mathfrak{A} \cap \mathfrak{F})$ we conclude from [6, Proposition 2] that $H\mathrm{asc}\langle X_i \mid i \in I\rangle$.

(2) and (3) follow from [6, Theorem 1] and [5, Theorem 2.7] respectively as in the proof of (1). □

4. Common ascendant subalgebras

Let L be a Lie algebra and let A, B be subalgebras of L. As in groups we say that L is *factorized* by A and B if $L = A + B$.

Let L be factorized by A and B, and let $H \le A \cap B$. In this section we shall consider some conditions under which $H\mathrm{asc}A$ and $H\mathrm{asc}B$ implies $H\mathrm{asc}L$. First we easily see the following:

Lemma 4. *Let $L \in \acute{\mathrm{E}}(\triangleleft)\mathfrak{A}$ and let $L = A+B$ be the sum of two subalgebras A and B. If $H\mathrm{asc}A$ and $H\mathrm{asc}B$, then $H\mathrm{asc}L$.*

Proof. Since $H\mathrm{wasc}A$ and $H\mathrm{wasc}B$, it is evident that $H\mathrm{wasc}L$. Therefore [12, Corollary to Theorem 2] indicates $H\mathrm{asc}L$. □

Remark. As in the proof of Lemma 4, we can show the following, which is a generalization of [9, Corollary to Proposition 2] : Let $L \in \acute{\mathrm{E}}(\triangleleft)\mathfrak{A}$ and let $H \le X_i$ $(i = 1, 2, \dots, n)$ be subalgebras of L such that $\langle X_1, \dots, X_n\rangle = X_1 + \cdots + X_n$. If $H\mathrm{asc}X_i$ for any i, then $H\mathrm{asc}\langle X_1, \dots, X_n\rangle$.

The following is originally due to Tôgô and is a generalization of [9, Remark to Lemma 4], for it is clear that $\mathrm{E}\mathfrak{A} \cup \mathfrak{Z} \le \acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A}$ over any field.

Lemma 5. *Let L be a Lie algebra such that $L = H+K$ with $H \le L, K \triangleleft L$ and $K \in \acute{\mathrm{E}}_\mu(\mathrm{ch})\mathfrak{A}$. If $H \le^\lambda L$, then $H \triangleleft^{\lambda\mu} L$.*

Proof. Let $(K_\alpha)_{\alpha\le\mu}$ be an ascending abelian series of characteristic ideals of K. We note that $K_\alpha \triangleleft L$ by [3, Lemma 1.4.4]. Therefore $K_\alpha \triangleleft H+K_\alpha \le L$ for all $\alpha \le \mu$. Now for any $\alpha < \mu$ we put $\overline{H} = (H + K_\alpha)/K_\alpha, \overline{K}_{\alpha+1} = K_{\alpha+1}/K_\alpha$. Then

$$\overline{K}_{\alpha+1} \triangleleft \overline{H} + \overline{K}_{\alpha+1} \quad \text{and} \quad \overline{K}_{\alpha+1} \in \mathfrak{A}.$$

On the other hand, we have $\overline{H} \le^\lambda \overline{H} + \overline{K}_{\alpha+1}$ as $H \le^\lambda H + K_{\alpha+1}$. By virtue of [12, Lemma 3], we obtain $\overline{H} \triangleleft^\lambda \overline{H} + \overline{K}_{\alpha+1}$. Hence $H + K_\alpha \triangleleft^\lambda H + K_{\alpha+1}$ for all $\alpha < \mu$. For any limit ordinal $\beta \le \mu$ it is trivial that $H + K_\beta = \cup_{\alpha<\beta}(H + K_\alpha)$. Therefore it follows that $H \triangleleft^{\lambda\mu} L$. □

Now using Lemma 5 we can generalize [9, Proposition 5] to the following:

Proposition 6. *Let L be a Lie algebra such that $L = A + B = H + K$ with $A, B, H \leq L, K \lhd L$ and $K \in \acute{\mathrm{E}}_\mu(\mathrm{ch})\mathfrak{A}$. If $H \leq^\lambda A$ and $H \leq^\lambda B$, then $H \lhd^{\lambda\mu} L$.*

Proof. It is evident that $H \leq^\lambda L$. Therefore we conclude the assertion from Lemma 5. □

The following corresponds to [9, Theorem 6].

Proposition 7. *Let $\mathfrak{X}$ be a class of Lie algebras and suppose that $L = A+B \in \mathfrak{X}$ with $A, B \leq L$, $H\mathrm{asc}A$ and $H\mathrm{asc}B$, always implies that $H\mathrm{asc}L$. Then $L = A+B \in (\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})\mathfrak{X}$ with $H\mathrm{asc}A$ and $H\mathrm{asc}B$ always implies that $H\mathrm{asc}L$.*

Proof. Let $L = A + B \in (\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})\mathfrak{X}$ with $H \lhd^\lambda A$ and $H \lhd^\lambda B$. Then there exists an ideal K of L such that $K \in \acute{\mathrm{E}}_\mu(\mathrm{ch})\mathfrak{A}$ and $L/K \in \mathfrak{X}$. Here we denote images under the natural map $L \longrightarrow L/K$ by bars. Then

$$\overline{L} = \overline{A} + \overline{B} \in \mathfrak{X}, \ \overline{H} \lhd^\lambda \overline{A} \ \text{ and } \ \overline{H} \lhd^\lambda \overline{B}.$$

By the hypothesis, there exists an ordinal $\alpha = \alpha(H, \lambda)$ such that $\overline{H} \lhd^\alpha \overline{L}$, so $H + K \lhd^\alpha L$. On the other hand, $H \leq^\lambda L$ since $H \leq^\lambda A$ and $H \leq^\lambda B$. Hence $H \leq^\lambda H+K$. On account of Lemma 5, it follows that $H \lhd^{\lambda\mu} H+K$. Thus we can reach that $H \lhd^{\lambda\mu+\alpha} L$. □

Let L be factorized by A and B over a field of characteristic zero and let $H\mathrm{asc}A$ and $H\mathrm{asc}B$. Then Aldosray proved that if $L \in \mathrm{L}(\lhd)\mathfrak{F}$ then $H\mathrm{asc}L$ ([2, Theorem 6]). We know the facts that $\mathrm{L}(\lhd)\mathfrak{F} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}$ ([14, Lemma 4.1]) and that if $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$, then the notion of serial subalgebras of L coincides with that of ascendant subalgebras of L ([6, Theorem 1]). Now we shall prove the main theorem in this section, which generalize the result of Aldosray.

Theorem 8. *Let L be a serially finite Lie algebra over a field of characteristic zero and let H, A, B be subalgebras of L such that $L = A + B$ and $H \leq A \cap B$. If H is a common serial subalgebra of both A and B, then H is serial in L.*

Proof. From [11, Theorem 5 and Corollary 6] it follows that

$$\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd A \quad \text{and} \quad H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \leq \rho(A/\lambda_{\mathrm{L}\mathfrak{N}}(H)),$$

$$\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd B \quad \text{and} \quad H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \leq \rho(B/\lambda_{\mathrm{L}\mathfrak{N}}(H)).$$

Hence we have $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$. Therefore it is enough to show that $H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \leq \rho(L/\lambda_{\mathrm{L}\mathfrak{N}}(H))$. Now since $H\mathrm{wser}L$ by Theorem 2, Lemma 1 indicates

$$H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \subseteq \mathfrak{e}(L/\lambda_{\mathrm{L}\mathfrak{N}}(H)).$$

Here we denote images under the natural map $L \longrightarrow L/\lambda_{\mathrm{L}\mathfrak{N}}(H)$ by bars. Then

$$\overline{L} = \overline{A} + \overline{B} \in \mathrm{L}(\mathrm{ser})\mathfrak{F},\ \overline{H}\mathrm{ser}\overline{A},\ \overline{H}\mathrm{ser}\overline{B},$$

$$\overline{H} \leq \rho(\overline{A}) \cap \rho(\overline{B}),\ \overline{H} \subseteq \mathfrak{e}(\overline{L}),$$

because of [3, Proposition 13.2.4]. Hence we may replace $\overline{L}, \overline{H}, \overline{A}, \overline{B}$ by L, H, A, B.

Then by [13, Theorem 2] L is, so-called, a neoclassical Lie algebra. That is to say, $L = \sigma(L)\dot{+}\Lambda$, where Λ is a direct sum of finite-dimensional, non-abelian simple subalgebras (see [3, Chapter 13]). As the first paragraph of the proof we set $\overline{L} = L/\sigma(L) = \overline{A} + \overline{B}$. Then

$$\overline{L} \cong \Lambda \in \mathrm{L}(\lhd)\mathfrak{F},\ \overline{H}\mathrm{ser}\overline{A},\ \overline{H}\mathrm{ser}\overline{B}.$$

Moreover $\overline{A}, \overline{B} \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$ owing to [14, Lemma 4.1]. Hence we have $\overline{H}\mathrm{asc}\overline{A}$, $\overline{H}\mathrm{asc}\overline{B}$ using [6, Theorem 1(1)]. Now we can derive from [2, Theorem 6] that $\overline{H}\mathrm{asc}\overline{L}$, so $H + \sigma(L)\mathrm{asc}L$. Furthermore $H + \rho(L) \lhd H + \sigma(L)$ owing to [3, Corollary 13.3.13]. Hence $H + \rho(L)\mathrm{asc}L$. On the other hand we obtain $H \in \mathrm{L}\mathfrak{N}$ by $H \leq \rho(A) \cap \rho(B)$. As $H \subseteq \mathfrak{e}(L)$, H acts on $\rho(L)$ by nil derivations, which indicates $H + \rho(L) \in \mathrm{L}\mathfrak{N}$ by [3, Theorem 16.3.8(b)]. Thus we can reach $H + \rho(L) \leq \rho(L)$ by using [3, Theorem 13.3.7], that is, $H \leq \rho(L)$. This completes the theorem. □

By making use of Theorem 8 and [6, Theorem 1(1)], we can obtain a better result than [2, Theorem 6].

Corollary 9. *Let L be a hyperfinite, serially finite Lie algebra over a field of characteristic zero and be factorized by A and B. If H is a common ascendant subalgebra of both A and B, then H is ascendant in L.*

Remark. Over any field, $\mathrm{L}(\lhd)\mathfrak{F} < \acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{F}$. For, let X be an abelian Lie algebra with basis $\{x_0, x_1, \dots\}$ and let σ be the derivation of X defined by $x_0\sigma = 0$ and $x_{i+1}\sigma = x_i$ $(i \geq 0)$. Form the split extension $L = X\dot{+}\langle\sigma\rangle$. Then $L \in \mathfrak{Z} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{F}$ but $L \notin \mathrm{L}(\lhd)\mathfrak{F}$ (see [6, Remark 1]).

Proposition 7 and Corollary 9 directly lead the following:

Corollary 10. *Let L be a Lie algebra belonging to $(\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})(\acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{F})$ over a field of characteristic zero and be factorized by A and B. If $H\mathrm{asc}A$ and $H\mathrm{asc}B$, then $H\mathrm{asc}L$.*

Using Lemma 5 and Corollary 10, we can easily prove the following corollary, which is a generalization of [1, Corollaries 1 and 2].

Corollary 11. *Let L be a Lie algebra belonging to* $(\acute{\mathrm{E}}(\mathrm{ch})\mathfrak{A})(\acute{\mathrm{E}}(\lhd)\mathfrak{F}\cap \mathrm{L}(\mathrm{ser})\mathfrak{F})$ *over a field of characteristic zero and let X_i $(i=1,2,...,n)$ be subalgebras of L such that $L = X_1 + X_2 + \cdots + X_n$ and $\langle X_i, X_j\rangle = X_i + X_j$ for all $i, j = 1, 2, ..., n$.*

(1) *If $H\mathrm{asc}X_i$ for all i, then $H\mathrm{asc}L$.*
(2) *For each i, if $X_i\mathrm{asc}\langle X_i, X_j\rangle$ for all j, then $X_i\mathrm{asc}L$.*

5. A generalization for the result of Goto and Panyukov

In this section we shall generalize the following result.

Lemma 12 (Goto, Panyukov)**.** *Let L be a finite-dimensional Lie algebra over a field of characteristic $\neq 2$. If L is represented as a sum of two nilpotent subalgebras A and B, then L is soluble.*

For our purpose we need the following two lemmas.

Lemma 13. *Let H be a finitely generated subalgebra of a Lie algebra L.*

(1) *If $H\mathrm{wasc}L$, then $H^{(\omega)}\mathrm{ch}L$.*
(2) *Assume that $L \in \mathrm{L}\mathfrak{F}$. If $H\mathrm{wser}L$, then $H^{(\omega)} \lhd L$.*

Proof. (1) Using [12, Theorem 4] we have $H \leq^{\omega} L$. Hence [5, Lemma 2.10] leads $H^{(\omega)} \lhd L$. Next form the split extension $M = L\dot{+}\mathrm{Der}L$. Then $H\mathrm{wasc}M$. The argument above indicates that $H^{(\omega)} \lhd M$, so $H^{(\omega)}\mathrm{ch}L$.

(2) For any $I \lhd H$ such that $H/I \in \mathrm{LE}\mathfrak{A}$, we have $H^{(\omega)} \leq I$ since $H/I \in \mathrm{E}\mathfrak{A}$. Therefore $H^{(\omega)} \leq \lambda_{\mathrm{LE}\mathfrak{A}}(H)$. Since, in general, $\lambda_{\mathrm{LE}\mathfrak{A}}(H) \leq H^{(\omega)}$, it follows from [5, Proposition 2.11] that $H^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(H) \lhd L$. □

Lemma 14. *Let L be a Lie algebra over a field of characteristic $\neq 2$ and let $L = A + B$ be a sum of Engel subalgebras A and B.*

(1) *If $H \in \mathfrak{F}$ and $H\mathrm{wasc}L$, then $H \in \mathrm{E}\mathfrak{A}$.*
(2) *Assume that $L \in \mathrm{L}\mathfrak{F}$. If $H \in \mathfrak{F}$ and $H\mathrm{wser}L$, then $H \in \mathrm{E}\mathfrak{A}$.*

Proof. (1) Because $H^{(\omega)}$ is a finite-dimensional ideal of L by Lemma 13, it follows from [3, Corollary 1.4.3] that

$$C_L(H^{(\omega)}) \lhd L \quad \text{and} \quad L/C_L(H^{(\omega)}) \in \mathfrak{F}.$$

Now we denote images under the natural map $L \longrightarrow L/C_L(H^{(\omega)})$ by bars. Then we have $\overline{L} \in \mathfrak{F}$ and $\overline{L} = \overline{A} + \overline{B}$ is a sum of nilpotent subalgebras $\overline{A}$ and $\overline{B}$. Therefore Lemma 12 shows $\overline{L} \in \mathrm{E}\mathfrak{A}$. In particular, $\overline{H} \in \mathrm{E}\mathfrak{A}$, so $H^{(\omega)} \subseteq C_L(H^{(\omega)})$. Hence $H^{(\omega+1)} = [H^{(\omega)}, H^{(\omega)}] = 0$. This concludes that $H \in \mathrm{E}\mathfrak{A}$.

(2) Since $H^{(\omega)} \lhd L$ by Lemma 13, we can show that $H \in \mathrm{E}\mathfrak{A}$ as in the proof of (1). □

Now we shall prove the main theorem in the section, which is a generalization of Lemma 12.

Theorem 15. *Let L be a Lie algebra over a field of characteristic $\neq 2$. If $L \in \mathrm{L}(\mathrm{wser})\mathfrak{F}$ and L is represented as a sum of two locally nilpotent subalgebras A and B, then L is locally soluble.*

Proof. Let X be a finite subset of L. Then there exists a subalgebra H of L such that $X \subseteq H\mathrm{wser}L$ and $H \in \mathfrak{F}$. Therefore it follows from Lemma 14(2) that $H \in \mathrm{E}\mathfrak{A}$. Thus $L \in \mathrm{LE}\mathfrak{A}$. □

Finally we shall state about any subalgebra of the intersection of permutable two locally nilpotent subalgebras.

Corollary 16. *Let L be a Lie algebra over a field of characteristic $\neq 2$ and let L be factorized by two locally nilpotent subalgebras A and B.*

(1) *If $L \in \mathrm{L}(\mathrm{wser})\mathfrak{F}$, then $H\mathrm{ser}L$ for any subalgebra H of $A \cap B$.*

(2) *If $L \in \mathrm{L}(\lhd)\mathfrak{F}$, then $H \lhd^{\omega} L$ for any subalgebra H of $A \cap B$.*

Proof. (1) Using [3, Proposition 13.2.4] we obtain $H\mathrm{ser}A$ and $H\mathrm{ser}B$. Since $L \in \mathrm{LE}\mathfrak{A}$ by Theorem 15 we conclude from Corollary 3 that $H\mathrm{ser}L$.

(2) From (1) we have $H\mathrm{ser}L \in \mathrm{LE}\mathfrak{A} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Therefore $H \lhd^{\omega} L$ in virtue of [5, Theorem 3.3]. □

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