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Toshihiro Hadano*

*Meijo University

ON THE DIOPHANTINE EQUATION $a^x=b^y+c^z$

TOSHIHIRO HADANO

0. Introduction We consider the Diophantine equation $a^x=b^y+c^z$ for given, distinct integers a, b, c that are larger than 2. It is well known that there are only a finite number of non-negative integral solutions (x, y, z) for every such equation (cf. [2; Chap. 23, Theorem 4]). T. Nagell [3] has found all the solutions of non-trivial equations of this type in which a, b and c are distinct primes $\leqq 7$. In this note, by means of elementary congruencial methods in many cases, following Nagell, we shall find all the solutions of the non-trivial equations $a^x=b^y+c^z$ when a, b and c, with $b<c$, are distinct primes $\leqq 17$. Of these equations, A. Makowski [1] and S. Uchiyama [4] have solved for $(a, b, c)=(5, 2, 11)$ and $(2, 3, 13)$ respectively.

1. Results*)

Case	Equation	Solutions (x, y, z)
1	$2^x=3^y+11^z$	(1, 0, 0), (2, 1, 0)
2	$3^x=2^y+11^z$	(1, 1, 0), (2, 3, 0), (3, 4, 1)
3	$11^x=2^y+3^z$	(1, 1, 2), (1, 3, 1)
4	$2^x=5^y+11^z$	(1, 0, 0), (4, 1, 1)
5	$5^x=2^y+11^z$	(1, 2, 0), (3, 2, 2)
6	$11^x=2^y+5^z$	none
7	$2^x=7^y+11^z$	(1, 0, 0), (3, 1, 0), (7, 1, 2)
8	$7^x=2^y+11^z$	none
9	$11^x=2^y+7^z$	(1, 2, 1)
10	$2^x=3^y+13^z$	(1, 0, 0), (2, 1, 0), (4, 1, 1), (8, 5, 1)
11	$3^x=2^y+13^z$	(1, 1, 0), (2, 3, 0)
12	$13^x=2^y+3^z$	(1, 2, 2)
13	$2^x=5^y+13^z$	(1, 0, 0)
14	$5^x=2^y+13^z$	(1, 2, 0)
15	$13^x=2^y+5^z$	(1, 3, 1)
16	$2^x=7^y+13^z$	(1, 0, 0), (3, 1, 0), (9, 3, 2)
17	$7^x=2^y+13^z$	none
18	$13^x=2^y+7^z$	none
19	$2^x=11^y+13^z$	(1, 0, 0)

*) Here, we have excluded in the list the results obtained by Nagell [3].

20	$11^x = 2^y + 13^z$	none
21	$13^x = 2^y + 11^z$	(1, 1, 1)
22	$2^x = 3^y + 17^z$	(1, 0, 0), (2, 1, 0)
23	$3^x = 2^y + 17^z$	(1, 1, 0), (2, 3, 0), (4, 6, 1)
24	$17^x = 2^y + 3^z$	(1, 4, 0), (1, 3, 2)
25	$2^x = 5^y + 17^z$	(1, 0, 0)
26	$5^x = 2^y + 17^z$	(1, 2, 0), (2, 3, 1)
27	$17^x = 2^y + 5^z$	(1, 4, 0)
28	$2^x = 7^y + 17^z$	(1, 0, 0), (3, 1, 0)
29	$7^x = 2^y + 17^z$	(2, 5, 1)
30	$17^x = 2^y + 7^z$	(1, 4, 0)
31	$2^x = 11^y + 17^z$	(1, 0, 0)
32	$11^x = 2^y + 17^z$	none
33	$17^x = 2^y + 11^z$	(1, 4, 0)

2. Proof *Cases* 1, 8, 13, 17, 18, 19, 22, 25, 28 *and* 31. These equations can easily be treated by considering them modulo 3, modulo 8 and modulo 16 (the modulus 16 is used only in the case 28).

Case 6. The equation is found to be impossible if we take it modulo 8 if $y=0$, and modulo 5 if $y=1$ or 2. If $y\geqq 3$ then x and z are even, as is readily seen when the equation is taken modulo 8. Put then $x=2s$, $z=2t$ to get $(11^s)^2-(5^t)^2=2^y$, whence we have $11^s+5^t=2^{y-1}$ and $11^s-5^t=2$. The latter equation is, however, impossible as we have seen above.

Case 20. This is found to be impossible just as in the case 6.

Case 4. On taking the equation modulo 5, modulo 17 and modulo 2^5, we find that it has no solutions with $x>4$.

Case 5. We get $y\equiv 2 \pmod 4$ by taking the equation modulo 5. If $y\geqq 6$, then x and y are found to be even, when we take the equation modulo 8, and so we can make use of the case 6. Thus, on being taken with the modulus 8, the equation will be reduced to the one

$$5^x = 4 + 11^z \quad \text{with} \quad 2\nmid x,\ 2\mid z.$$

By considering the above equation to the moduli 7 and 13, we get either $x\equiv 1 \pmod{12}$ and $z\equiv 0 \pmod{12}$, or $x\equiv 3 \pmod{12}$ and $z\equiv 2 \pmod{12}$. Taking modulo 11 and modulo 41 if $z>0$, we find that the former choice is impossible. Thus $x\equiv 0 \pmod 3$. Put $5^{x/3}=X$ and $11^{z/2}=Y$ to get $Y^2=X^3-4$. This last equation has only two solutions $(X, Y)=(2, 2)$, $(5, 11)$ (see e. g. [2 ; p. 123]). So we get $x=1$, $z=0$ and $x=3$, $z=2$.

Cases 9, 11, 12, 14, 21, 23 *and* 27. By an argument similar to the one used in the case 5, these equations are reduced respectively to the following ones :

(9)	$11^x = 4 + 7^z$	with	$2 \nmid x$, $2 \nmid z$,
(11)	$3^x = 2 + 13^z$	with	$2 \nmid x$, $2 \mid z$,
(12)	$13^x = 4 + 3^z$	with	$2 \nmid x$, $2 \mid z$,
(14)	$5^x = 4 + 13^z$	with	$2 \nmid x$, $2 \mid z$,
(21)	$13^x = 2 + 11^z$	with	$2 \nmid x$, $2 \nmid z$,
(23)	$3^x = 2 + 17^z$	with	$2 \nmid x$, $2 \mid z$,
and			
(27)	$17^x = 1 + 2^y$	with	$2 \nmid x$, $2 \mid y$.

(Note that in the case 23 the equation is reduced to (23) and (27).) So we shall treat these equations instead.

(9): Taking modulo 16 and modulo 25 we get $x-1 \equiv 0 \pmod{20}$, $z-1 \equiv 0 \pmod{20}$. Suppose $x > 1$, $z > 1$. Then $11(11^{x-1}-1) = 7(7^{z-1}-1)$, where $11^{x-1}-1 \equiv 0 \pmod{11^5+1}$. Hence $7^{z-1}-1 \equiv 0 \pmod{13421}$ by the fact that $11^5+1 \equiv 0 \pmod{13421}$ for the prime 13421, and so $z-1 \equiv 0 \pmod{13420}$. Thus $z-1 \equiv 0 \pmod{110}$. However, we have $7^{110}-1 \equiv 0 \pmod{11^2}$. This is a contradiction.

(11): If $z > 0$, then $3^x \equiv 2 \pmod{13}$. This is impossible.

(12): If $z > 2$, then $13^x \equiv 4 \pmod{27}$. This is impossible.

(14): If $z > 0$, then $5^x \equiv 4 \pmod{13}$. This is impossible.

(21): Taking modulo 11 and modulo 13 we get $x-1 \equiv 0 \pmod{10}$, $z-1 \equiv 0 \pmod{12}$. Suppose $x > 1$, $z > 1$. Then $13(13^{x-1}-1) = 11(11^{z-1}-1)$, where $13^{x-1}-1 \equiv 0 \pmod{13^{10}-1}$. Hence $11^{z-1}-1 \equiv 0 \pmod{30941}$ by $13^{10}-1 \equiv 0 \pmod{30941}$, where 30941 is a prime, and so $z-1 \equiv 0 \pmod{30940}$. Thus $z-1 \equiv 0 \pmod{13}$. Therefore $z-1 \equiv 0 \pmod{12 \cdot 13}$. This is, however, a contradiction since $11^{12 \cdot 13}-1 \equiv 0 \pmod{13^2}$.

(23): If $x > 1$, then $2 + 17^z \equiv 0 \pmod 9$. This is impossible.

(27): Suppose $x > 1$, $y > 4$. Then $17(17^{x-1}-1) = 2^4(2^{y-4}-1)$. Since x is odd and $17^2-1 \equiv 0 \pmod{2^5}$, we have a contradiction.

Case 2. Taking the equation modulo 8 and modulo 11 we get either $y = 1$ and $z = 0$, or $y \geqq 3$ and $x \equiv z \pmod 2$. We may argue as before if x and z are even. If x and z are odd, then we get $y = 4$ on taking the equation to the moduli 17 and 2^6. In such a way as this we readily find that the equations in the cases 2, 3, 15, 24, 26, 29 and 32 are reduced to the following equations :

(2)	$3^x = 2^4 + 11^z$	with	$2 \nmid x$, $2 \nmid z$,
(3)	$11^x = 2^3 + 5^z$	with	$2 \nmid x$, $2 \nmid z$,
(15)	$13^x = 2^3 + 5^z$	with	$2 \nmid x$, $2 \nmid z$,
(24)	$17^x = 2^3 + 3^z$	with	$2 \nmid x$, $2 \mid z$,
$(26)_1$	$5^x = 2^2 + 17^z$	with	$2 \nmid z$, $2 \mid z$,

(26)$_2$ $5^x = 2^3 + 17^z$ with $2 \mid x$, $2 \nmid z$,
(29) $7^x = 2^5 + 17^z$ with $2 \mid x$, $2 \nmid z$,
and
(32) $11^x = 2^3 + 17^z$ with $2 \mid x$, $2 \nmid z$,

respectively. So we treat these equations in turn.
(2): Taking modulo 11 we get $3^x \equiv 5 \pmod{11}$, and so $x-3 \equiv 0 \pmod 5$. Suppose $x>3$, $z>1$. Then $3^3(3^{x-3}-1) = 11(11^{z-1}-1)$. Therefore $3^{x-3}-1 \equiv 0 \pmod{11^2}$, since $3^5-1 \equiv 0 \pmod{11^2}$. This is a contradiction.
(3): Taking modulo 7 we get $x-1 \equiv 0 \pmod 6$. Suppose $x>1$, $z>1$. Then $11(11^{x-1}-1) = 3(3^{z-1}-1)$. Therefore $11^{x-1}-1 \equiv 0 \pmod{3^2}$, since $11^6-1 \equiv 0 \pmod{3^2}$. This is a contradiction.
(15): Taking modulo 5 and modulo 13 we get $x-1 \equiv 0 \pmod 4$, $z-1 \equiv 0 \pmod 4$. Suppose $x>1$, $z>1$. Then $13(13^{x-1}-1) = 5(5^{z-1}-1)$. Therefore $13^{x-1}-1 \equiv 3 \pmod{31}$, $5^{z-1}-1 \equiv 0 \pmod 7$, since $13^4-1 \equiv 0 \pmod 7$ and $5^6-1 \equiv 0 \pmod{31}$. Hence $x-1 \equiv 0 \pmod{60}$, and so $13^{x-1}-1 \equiv 0 \pmod{13^{60}-1}$. Thus $13^{x-1}-1 \equiv 0 \pmod{5^2}$, since $13^{60}-1 \equiv 0 \pmod{5^2}$. This is a contradiction.
(24): Taking modulo 5 and modulo 13 we get $x-1 \equiv 0 \pmod 6$. Suppose $x>1$, $z>2$. Then $17(17^{x-1}-1) = 3^2(3^{z-2}-1)$. Therefore $17^{x-1}-1 \equiv 0 \pmod{3^3}$, since $17^6-1 \equiv 0 \pmod{3^3}$. This is a contradiction.
(26)$_1$: If $z>0$, then $5^x \equiv 4 \pmod{17}$. This is impossible since x is odd.
(26)$_2$: Suppose $x>2$, $z>1$. Then $5^2(5^{x-2}-1) = 17(17^{z-1}-1)$. Hence $17^{z-1}-1 \equiv 0 \pmod{5^2}$, so that $z-1 \equiv 0 \pmod{20}$. Therefore we have a contradiction since $17^{10}-1 \equiv 0 \pmod{5^3}$.
(29): Taking modulo 7 and modulo 17 we get $x-2 \equiv 0 \pmod{16}$, $z-1 \equiv 0 \pmod 6$. Suppose $z>1$. Then $7^2(7^{x-2}-1) = 17(17^{z-1}-1)$. Therefore $7^{x-2}-1 \equiv 0 \pmod{307}$, since $17^6-1 \equiv 0 \pmod{307}$, and so $x-2 \equiv 0 \pmod{306}$. Thus $x-2 \equiv 0 \pmod{16 \cdot 17}$ and $7^{16\cdot 17}-1 \equiv 0 \pmod{17^2}$. This is a contradiction.
(32): Taking modulo 16 and modulo 13 we get $x \equiv 0 \pmod 3$. Hence $17^z = (11^{x/3})^3 - 8 = (11^{x/3}-2)\{(11^{x/3}-2)^2 + 6 \cdot 11^{x/3}\}$, and so we must have $11^{x/3} - 2 = 17^s$ for some integer $s \geqq 0$, but this is impossible.

Case 7: We find that x and y are odd and z is even, except in the trivial solution, on taking the equation modulo 3 and modulo 8. And so we can put $x-2=n$, where n is odd and $\geqq 3$, $y=2b+1$ and $z=2a$. We then consider the decomposition of 2^n in the quadratic number field $\mathbf{Q}(\sqrt{-7})$, which has class number one. Thus we have $b=0$ by making use of the same method as in Nagell [3; pp. 580-581]. Hence $y=1$ in the non-trivial solutions.

Case 16. In this case also, by Nagell's method, we get $n \equiv 0$ (mod

7^2), where $x-2=n$ as before. Put $y=2b+1$, $z=2a$. If $b>1$, then we have $A_{49} \mid A_n$, where in Nagell's notation $(1+\sqrt{-7})^n = A_n + B_n\sqrt{-7}$ (A_n, B_n: rational integers). But this contradicts $\pm 2^{n-1} \cdot 13^a = A_n$, as is easily verified. Thus we get either $a=1$, $b=1$, $n=1$, or $b=0$. Now, if $b=0$, i.e. $y=1$, then the equations in the cases 7 and 16 will reduce to the famous Ramanujan equation $2^x = 7 + Y^2$ (cf. e.g. [2; Chap. 23]), and proof for the cases 7 and 16 is completed.

Cases 30 *and* 33. In these cases x is odd and y and z are even, as can be seen by taking the equations modulo 3 and modulo 8. So we have $U^2+V^2=17^x$ with $U=2^{y/2}$, $V=7^{z/2}$ or $11^{z/2}$, that is,

$$(U+iV)\,(U-iV) = \varepsilon(4+i)^x\,(4-i)^x$$

with $\varepsilon=\pm 1$, $\pm i$ ($i=\sqrt{-1}$). Put $(4+i)^x = a_x + ib_x$. Then $a_{x+1} = 4a_x - b_x$, $b_{x+1} = a_x + 4b_x$, $a_1 = 4$, $b_1 = 1$. By induction we get $4 \parallel a_x$, $2 \nmid b_x$ since x is odd. We take $\varepsilon = 1$, as we may, to get $U+iV=(4+i)^x$, and so $U=a_x$ is a 2-power integer if, and only if, $a_x = U = 4$, $V=1$. We therefore obtain $x=1$, $y=4$, $z=0$.

Thus we have completed the proof of all the cases proposed, except one case, the case 10, which was treated by Uchiyama [4] by using the decomposition in the field $\mathbf{Q}(\sqrt{-39})$.

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DEPARTMENT OF MATHEMATICS,
MEIJŌ UNIVERSITY

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