Mathematical Journal of Okayama University

Volume 23, *Issue* 1 1981 *Article* 4

JUNE 1981

Note on Azumaya algebras and H-separable extensions

Shûichi Ikehata*

*Okayama University

Math. J. Okayama Univ. **23** (1981), 17—18

NOTE ON AZUMAYA ALGEBRAS AND *H*-SEPARABLE EXTENSIONS

SHÛICHI IKEHATA

Let A/B be a ring extension with common identity 1, and C be the center of A. If $A\otimes_B A$ is A-A-isomorphic to an A-A-direct summand of a finite direct sum A^n then A/B is called to be H-separable. As is well known, A/C is H-separable if and only if A is an Azumaya C-algebra. The purpose of this note is to prove the following theorem, which has an application (Th. 2).

Theorem 1. *Let A be an Azumaya C-algebra, and $A\supset B\supset C$. If A_B is projectvie then A/B is H-separable.*

Proof. Since A/C is separable, there exists an element $\sum_i r_i\otimes s_i$ in $A\otimes_C A$ such that $\sum_i r_i s_i=1$ and $\sum_i ar_i\otimes s_i=\sum_i r_i\otimes s_i a$ for all $a\in A$. Further, since A_B is f. g. projective, there exists a finite number of elements $t_j\in A$ and $f_j\in \mathrm{Hom}(A_B,\ B_B)$ such that $\sum_j t_j f_j(a)=a$ for all $a\in A$. Then, the mapping $\theta: u\otimes v\rightarrow\sum_j ut_j\otimes f_j(v)$ of $A\otimes_C A$ into itself is an endomorphism, and

$$\begin{aligned}\textstyle\sum_{i,j} r_i t_j\otimes f_j(s_i ax)y &= \theta(\textstyle\sum_i r_i\otimes s_i ax)y=\theta(\textstyle\sum_i ar_i\otimes s_i x)y\\ &= \textstyle\sum_{i,j} ar_i t_j\otimes f_j(s_i x)y\end{aligned}$$

where $a,\ x,\ y\in A$. This implies that the map $\phi: A\otimes_B A\rightarrow A\otimes_C A$ defined by $x\otimes y\rightarrow\sum_{i,j} r_i t_j\otimes f_j(s_i x)\,y$ is an A-A-homomorphism. Obviously, the canonical map $\psi: A\otimes_C A\rightarrow A\otimes_B A$ is an A-A-homomorphism and $\psi\phi$ is the identity map of $A\otimes_B A$. Hence ${}_A A\otimes_B A_A <\oplus_A A\otimes_C A_A$. Since A/C is H-separable, it follows that A/B is H-separable.

Next, we need the following

Lemma. *Let A/B be H-separable, and ${}_A M$ a unital A-module. If ${}_B M$ is a generator then so is ${}_A M$.*

Proof. Since ${}_B M$ is a generator, ${}_B B<\oplus\,{}_B M^n$ for some integer $n>0$. Further, since A/B is H-separable, ${}_A A\otimes_B A_A<\oplus\,{}_A A^m_A$ for some integer $m>0$. Then, we obtain ${}_A A\simeq {}_A A\otimes_B B<\oplus\,{}_A A\otimes_B M^n\simeq {}_A(A\otimes_B M)^n <\oplus\,{}_A M^{mn}$.

Now, let B be a commutative ring, G a finite group of automorphisms of B, and $R = B^G$ (the fixed ring of G in B). Moreover, $\Delta(B; G)$ denotes the trivial crossed product $\bigoplus_{\sigma\in G} Bu_\sigma$ with $u_\sigma u_\tau = u_{\sigma\tau}$ and $u_\sigma b = \sigma(b)u_\sigma$ (σ, $\tau \in G$, $b \in B$). Obviously, the map $j : \Delta(B; G) \to \mathrm{Hom}(B_R, B_R)$ defined by $j(bu_\sigma)(x) = b\sigma(x)$ (b, $x \in B$, $\sigma \in G$) is a ring homomorphism. If j is an isomorphism and B_R is f. g. projective then B/R is called to be G-Galois (cf. [1], [2]). Under this situation, we shall prove the following theorem which contain some characterizations of Galois extensions of commutative rings.

Theorem 2. *Let B be a commutative ring, G a finite group of automorphisms of B, $R = B^G$,* and *$\Delta = \Delta(B; G)$. Then the following conditions are equivalent.*

(1) *B/R is G-Galois.*

(2) *Δ is an Azumaya R-algebra.*

(3) *Δ/B is H-separable.*

When this is the case, B is a maximal commutative R-subalgebra of Δ with $\Delta\otimes_R B \simeq M_m(B)$ and $B\otimes_R \Delta \simeq M_m(B)$, where m is the order of G.

Proof. (1)$\Longrightarrow$(2). It is well known ([2, Prop. 3.1.2 and Prop. 2.4.1]). (2)$\Longrightarrow$(3). Since ${}_B\Delta$ is free, it follows from Th. 1. (3)$\Longrightarrow$(1). By Lemma, ${}_\Delta B$ is a generator. Hence B/R is G-Galois by [1, Prop. A. 1]. Finally, if B/R is G-Galois then B coincides with the centralizer of B in Δ, and hence the last assertion follows immediatly from [3, Lemma 1 (3)].

REFERENCES

[1] M. AUSLANDER and O. GOLDMAN: The Brauer group of a commutative ring, Trans. Amer. Math. Soc. **97**(1960), 367—409.

[2] F. DEMEYER and E. INGRAHAM: Separable Algebras over Commutative Rings, Lecture Notes in Math. **181**, Springer, Berlin, 1971.

[3] T. NAKAMOTO: On QF-extensions in an H-separable extensions, Proc. Japan Acad. **50** (1974), 440—443.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(*Received June 17, 1980*)