Mathematical Journal of Okayama University

Volume 18, *Issue* 1 | 1975 | *Article* 7

DECEMBER 1975

A note on quotient rings over a quasi-Frobenius extension

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A NOTE ON QUOTIENT RINGS OVER A QUASI-FROBENIUS EXTENSION

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Let A/B be a ring extension. When A/B is a Quasi-Frobenius (QF) (resp. Frobenius) extension, it seems to be natural to ask where the maximal right quotient ring $Q_{\max}(A)$ of A in the sense of Utumi-Lambek is a QF (resp. Frobenius) extension of the maximal right quotient ring $Q_{\max}(B)$ of B or not. In the previous paper [4], we showed that the answer is affirmative if A/B is a Frobenius extension such that

$(*)_0$ A is finitely generated as a B-module by elements which centralize B.

In the present paper, we shall further investigate the above question under a suitable hypothesis. Our main result of this paper is the following:

If A/B is a QF (resp. Frobenius) extension satisfying $(*)_0$ or the condition:

$(**)_B$ *the class of right B-modules with zero duals is closed under taking submodules*

then $Q_{\max}(A)$ is a QF (resp. Frobenius) extension of $Q_{\max}(B)$ such that $Q_{\max}(A) \cong Q_{\max}(B) \otimes {}_B A \cong \otimes {}_B Q_{\max}(B)$ canonically. Here, by the dual of a right B-module X, we mean $\mathrm{Hom}(X_B, B_B)$.

It is interesting to observe that when A/B is QF, A satisfies $(**)_A$ if and only if B does $(**)_B$ (Proposition 1. 4).

Throughout this paper, A will denote a ring with identity element and B a subring of A with the common identity element. The notation and terminology are same as [4] unless otherwise specified.

1. Let R be a ring with identity element. A non-empty set $\mathfrak{F}$ of right ideals of R is called an idempotent filter of R if the following conditions are satisfied:

(i) If $I \in \mathfrak{F}$ and $I \subset J$, then $J \in \mathfrak{F}$.

(ii) If $I, J \in \mathfrak{F}$, then $I \cap J \in \mathfrak{F}$.

(iii) If $I \in \mathfrak{F}$, $a \in R$, then $a^{-1}I = \{x \in R \mid ax \in I\} \in \mathfrak{F}$.

(iv) If $I \in \mathfrak{F}$, $a^{-1}J \in \mathfrak{F}$ for every $a \in I$, then $J \in \mathfrak{F}$.

Let $\mathfrak{F}$ be an idempotent filter of R. Then $\mathfrak{F}$ is obviously a directed set by inverse inclusion and $Q_{\mathfrak{F}}(R) = \varinjlim_{I \in \mathfrak{F}} \mathrm{Hom}_R(I, R/T_{\mathfrak{F}}(R))$ has a ring structure in a natural way, where $T_{\mathfrak{F}}(R) = \{a \in R \mid aI = 0$ for

Mathematical Journal of Okayama University, Vol. 18 [1975], Iss. 1, Art. 7

some $I \in \mathfrak{F}\}$, called the right quotient ring of R with respect to $\mathfrak{F}$, and moreover there is a canonical ring homomorphism $R \longrightarrow Q_{\mathfrak{F}}(R)$ whose kernel is $T_{\mathfrak{F}}(R)$. Thus the last homomorphism is monic if and only if the following condition is enjoyed:

(v) The left annihilator of every member of $\mathfrak{F}$ is zero.

The largest set of right ideals of R satisfying (i)—(v) is the filter consisting of all dense right ideals of R, where a right ideal I of R is said to be dense if for every x, $y \in R$, $y \neq 0$, there exists some $a \in R$ such that $xa \in I$ and $ya \neq 0$, or equivalently, if $\mathrm{Hom}(R/I_R, E(R_R)_R) = 0$, $E(R_R)$ the injective hull of R_R. The right quotient ring with respect to this filter is called the maximal right quotient ring of R and denoted by $Q_{\max}(R)$ (see for details, [6] and [8]).

Now we shall turn to a ring extension A/B.

Proposition 1.1. *Let* $\mathfrak{F}$ *be a non-empty set of right ideals of* B *and* $\tilde{\mathfrak{F}}$ *the set of right ideals of* A *which contain some member in* $\mathfrak{F}$. *If* $\mathfrak{F}$ *satisfies* (i), (ii) *and* (iv), *then* $\tilde{\mathfrak{F}}$ *satisfies* (i), (ii) *and* (iv). *Morever if*
(*) *A is generated as a B-module by elements which centralize B,*
then $\tilde{\mathfrak{F}}$ *satisfies* (iii) *whenever so does* $\mathfrak{F}$.

Proof. Clearly, $\tilde{\mathfrak{F}}$ satisfies (i) and (ii). Let I be a member in $\tilde{\mathfrak{F}}$ with $I \supset J$ for some $J \in \mathfrak{F}$. Let K be a right ideal of A such that $x^{-1}K \in \tilde{\mathfrak{F}}$ for every $x \in I$. Since $b^{-1}(K \cap B) = b^{-1}K \cap B \in \mathfrak{F}$ for every $b \in J$, we obtain $K \cap B \in \mathfrak{F}$, and so, $K \in \tilde{\mathfrak{F}}$, proving (iv). Next assume further (*) and (iii) for $\mathfrak{F}$. Then every element a of A can be written as $a = \sum b_i a_i$ with some $a_i \in V_A(B) = \{a \in A \mid ab = ba$ for all $b \in B\}$ and $b_i \in B$ $(i = 1, \cdots, t)$. Since $b_i^{-1}(I \cap B) \in \mathfrak{F}$ for all i, we have $\cap\, b_i^{-1}(I \cap B) \in \mathfrak{F}$, and so, $a^{-1}I \in \tilde{\mathfrak{F}}$ because of $a^{-1}I \supset \cap\, b_i^{-1}(I \cap B)$.

To contrast with the above proposition, we shall prove the next.

Proposition 1.2. *Let* $\mathfrak{F}$ *be a non-empty set of right ideals of* A *satisfying* (i), (ii) *and* (iii), *and* $\mathfrak{F}'$ *the set of right ideals* J *of* B *with* $J \supset I'$ *for some* $I \in \mathfrak{F}$, *where* I' *denotes the right ideal of* B *generated by the elements of the form* $f(x)$, $x \in I$, $f \in \mathrm{Hom}({}_BA, {}_BB)$. *Then* $\mathfrak{F}'$ *satisfies* (i), (ii) *and* (iii). *Moreover, if* ${}_BA$ *satisfies* (*) *and is projective then* $\mathfrak{F}'$ *satisfies* (iv) *whenever so does* $\mathfrak{F}$.

Proof. Obviously, $\mathfrak{F}'$ satisfies (i) and (ii). Let I and b be elements in $\mathfrak{F}$ and B respectively. Since $(b^{-1}I)' \subset b^{-1}I'$, we have $b^{-1}I' \in \mathfrak{F}'$, proving (iii). Assume further that ${}_BA$ is projective and (*) is satisfied. Let $\{a_i\}_i \subset V_A(B)$ be a generating set for the left B-module A. Then, by the projectivity of ${}_BA$, there exists a subset $\{f_i\}_i \subset$

$\mathrm{Hom}(A_B, B_B)$ such that, for each $x \in A$, $f_i(x) = 0$ for almost all i and $x = \sum f_i(x) a_i$. (Such a system $(f_i, a_i)_i$ is called a dual basis for ${}_BA$). Let J be a right ideal of B such that $y^{-1}J \in \mathfrak{F}'$ for every $y \in I'$. For every element $x = \sum f_i(x) a_i$ in I, $\cap f_i(x)^{-1}J$ is contained in $x^{-1}JA$ and contains some I_0' $(I_0 \in \mathfrak{F})$ as a member of $\mathfrak{F}'$, and so, $I_0 \subset I_0' \cdot A \subset x^{-1}JA$. Therefore $JA \in \mathfrak{F}$ by (iv), which implies $(JA)' \subset J \in \mathfrak{F}'$ as desired.

We now consider the following conditions on a ring R which have been investigated in [2]:

$(**)_R$ The class of right R-modules with zero duals is closed under taking submodules.

$(***)_R$ The class of torsionless right R-modules is closed under taking extensions.

As was mentioned in [2], if the injective hull $E(R_R)_R$ of R_R is torsionless then R satisfies $(**)_R$ and $(***)_R$. We shall show in the following proposition that these conditions are inherited for QF extensions. Although, the next lemma was cited in the foot note of [2, p. 450], for the sake of completeness, we give here a proof.

Lemma 1.3. *A ring R satisfies $(**)_R$ if and only if* $\mathrm{Hom}(X_R, E(R_R)_R) = 0$ *for every X_R with zero dual.*

Proof. It is enough to prove the only if part. Let X_R be a right R-module with zero dual. If $f(X) \neq 0$ for some $f \in \mathrm{Hom}(X_R, E(R_R)_R)$, then $f(X) \cap R \neq 0$, which contradicts $(**)_R$.

Proposition 1.4. *Suppose A/B is a QF extension. Then B satisfies $(**)_B$ (resp. $(***)_B$) if and only if so does A.*

Proof. Let $(\alpha_i : \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ be a right QF system for A/B[1]. First, we claim the followings:

a) *If Y_B is torsionless, then* $\mathrm{Hom}(A_B, Y_B)_A$ *is torsionless and a mapping $\alpha_Y : Y \longrightarrow \mathrm{Hom}(A_B, Y_B)^p$ defined by $\alpha_Y(y) = (\lambda_y \cdot \alpha_i)_i$ for $y \in Y$ is monic, where λ_y denotes a mapping $B \longrightarrow Y$ given by $\lambda_y(b) = yb$ for $b \in B$.* In fact, the first assertion is clear, because $Y_B \subsetneq \Pi B_B$ implies $\mathrm{Hom}(A_B, Y_B)_A \subsetneq \Pi\ \mathrm{Hom}(A_B, B_B)_A \subsetneq \Pi A_A$. To see the second one, let us assume $\lambda_y \cdot \alpha_i = 0$ for all i. Then $\alpha_i \cdot f(y) = 0$ in $\mathrm{Hom}(A_B, B_B)$ for every $f \in \mathrm{Hom}(Y_B, B_B)$, and so, $f(y) = 0$. As Y_B is torsion-

1) A *right QF system* for A/B is defined as a system $(\alpha_i ; \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ with $\sum_{i,j} \alpha_i(x_{ij}) y_{ij} = 1$, where every α_i is a B-B-homomorphism of A to B and every $\sum_j x_{ij} \otimes y_{ij}$ is a casimir element of $A \otimes_B A$, that is, an element which commutes with all elements of A. A left QF system for A/B is defined similarly (see [5]).

less, this means $y = 0$.

b) *X_A is torsionless if and only if so is it as a B-module.* Noting $X_A \subset \mathrm{Hom}(A_B, X_B)_A$, this is a consequence of a).

c) ${}_B\mathrm{Hom}(X_A, A_A) \sim {}_B\mathrm{Hom}(X_B, B_B)$ *for every* X_A. In fact, ${}_B\mathrm{Hom}(X_A, A_A) \sim {}_B\mathrm{Hom}(X_A, \mathrm{Hom}(A_B, B)_B) \cong {}_L\mathrm{Hom}(X \otimes {}_A A_B, B_B) \cong {}_B\mathrm{Hom}(X_B, B_B)$.

Now assume the validity of $(^{**})_A$, and let Y be a right B-module with zero dual and Y' an arbitrary submodule of Y. Since $\mathrm{Hom}(A_B, Y'_B)_A \subset \mathrm{Hom}(A_B, Y_B)_A$ and $\mathrm{Hom}(\mathrm{Hom}(A_B, Y_B)_A, A_A) \sim \mathrm{Hom}(Y \otimes {}_B A_A, A_A) \cong \mathrm{Hom}(Y_B, A_B) \mid \mathrm{Hom}(Y_B, B_B)$ (as abelian groups), the validity of $(^{**})_A$ yields $\mathrm{Hom}(Y'_B, A_B) = 0$, and so, $\mathrm{Hom}(Y'_B, B_B) = 0$ as desired. The converse is clear by c). Next assume the validity of $(^{***})_A$. Let $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ be a short exact sequence of tight B-modules with Y', Y'' torsionless. We have the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{Y'}} & & \downarrow{\scriptstyle \alpha_{Y}} & & \downarrow{\scriptstyle \alpha_{Y''}} & & \\
0 & \rightarrow & \mathrm{Hom}(A_B, Y'_B)^p & \rightarrow & \mathrm{Hom}(A_B, Y_B)^p & \rightarrow & \mathrm{Hom}(A_B, Y''_B)^p & \rightarrow & 0
\end{array}$$

Therefore $\mathrm{Hom}(A_B, Y_B)$ is torsionless as an A and hence as a B-module by a) and $(^{***})_A$. Recalling here that $\alpha_{Y'}$ and $\alpha_{Y''}$ are both monic by a), we can see α_Y to be monic and Y_B to be torsionless. Conversely, A satisfies $(^{***})_A$ whenever B dose $(^{***})_B$ from b).

Proposition 1.5. *Suppose that A/B is QF and that either $(^*)$ or $(^{**})_B$ is satisfied. If I (resp. J) is a dense right ideal of A (resp. B), then so is $I \cap B$ (resp. JA) in B (resp. A).*

Proof. First, we note the following fact that the canonical injection $B/I \cap B \longrightarrow A/I$ induces a surjection $\mathrm{Hom}\,(A/I_B, E(B_B)_B) \longrightarrow \mathrm{Hom}\,(B/I \cap B_B, E(B_B)_B)$. Let us now assume $(^*)$. Then we have $E(A_A)_A \sim \mathrm{Hom}(A_B, E(B_B)_B)_A$ by the proof of [4, Theorem]. Thus we have $\mathrm{Hom}\,(A/I_B, E(B_B)_B) \cong \mathrm{Hom}(A/I_A, \mathrm{Hom}(A_B, E(B_B)_B)_A) \sim \mathrm{Hom}(A/I_A, E(A_A)_A) = 0$ as abelian groups. Therefore, recalling the above mention, $I \cap B$ is dense in B. Further, noting $\mathrm{Hom}(A/JA_A, E(A_A)_A) \sim \mathrm{Hom}(B/J \otimes {}_B A_A, \mathrm{Hom}(A_B, E(B_B)_B)_A) \sim \mathrm{Hom}(B/J_B, E(B_B)_B) = 0$, it follows that JA is dense in A.

Next assume $(^{**})_B$ for B. Thus A satisfies $(^{**})_A$ by Proposition 1.4. Since $\mathrm{Hom}(A/JA_A, A_A) \cong \mathrm{Hom}(B/J_B, A_B) \mid \mathrm{Hom}(B/J_B, B_B)$, we have $\mathrm{Hom}(A/JA_A, A_A) = 0$. It follows that JA is dense in A by Lemma 1.3. Moreover we have $\mathrm{Hom}(A/I_B, B_B) \sim \mathrm{Hom}(A/I_A, A_A)$ by c) in the proof of Proposition 1. 4, and so, $\mathrm{Hom}(A/I_B, E(B_B)_B)=0$ by Lemma 1. 3. Therefore, recalling the mention at the begining of the proof, $I\cap B$ is dense in B.

Remark. Assume ${}_BA$ to be torsionless. If I is a dense right ideal of A, then I' defined in Proposition 1. 2 is a dense right ideal of B. In fact, let x and y be in B, with $y\neq 0$. Since I is dense in A, there exists $a\in A$ such that $xa\in I$ and $ya\neq 0$. Our assumption yields $f(ya)\neq 0$ for some $f\in\mathrm{Hom}({}_BA, {}_BB)$. Therefore we have $xf(a)=f(xa)\in I'$ and $yf(a)\neq 0$.

Let $\mathfrak{F}$ be now an idempotent filter of B such that $\tilde{\mathfrak{F}}$ defined in Proposition 1. 1 is also an idempotent filter of A. Then we can construct the right quotient rings $Q_{\mathfrak{F}}(B)$ and $Q_{\tilde{\mathfrak{F}}}(A)$ of B and A with respect to $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$, respectively. Let us set $\bar{B}=B/T_{\mathfrak{F}}(B)$, $\bar{A}=A/T_{\tilde{\mathfrak{F}}}(A)$, $\tilde{B}=Q_{\mathfrak{F}}(B)$ and $\tilde{A}=Q_{\tilde{\mathfrak{F}}}(A)$. First suppose A/B to be a QF extension with a right QF system $(\alpha_i : \sum_j x_{ij}\otimes y_{ij})_{1\leq i\leq p}$ and with a left QF system $({}_i\beta_k : \sum_l w_{kl}\otimes z_{kl})_{1\leq k\leq q}$. Since $T_{\tilde{\mathfrak{F}}}(A)\cap B=T_{\mathfrak{F}}(B)$, we can and shall regard $\bar{B}$ as a subring of $\bar{A}$. Let φ be an arbitrary right B-homomorphism of A to B. It is easy to see $\varphi(T_{\tilde{\mathfrak{F}}}(A))\subset T_{\mathfrak{F}}(B)$. Therefore, φ induces a right $\bar{B}$-homomorphism $\bar{\varphi}:\bar{A}\longrightarrow\bar{B}$ in a natural way. Furthermore, we have $T_{\mathfrak{F}}(B)A=AT_{\mathfrak{F}}(B)=T_{\tilde{\mathfrak{F}}}(A)$, since for $a\in T_{\tilde{\mathfrak{F}}}(A)$, $a=\sum_{i,j}\alpha_i(ax_{ij})y_{ij}=\sum_{k,l}w_{kl}\,{}_l\beta_k(z_{kl}a)$ with $\alpha_i(ax_{ij})$, ${}_l\beta_k(z_{kl}a)\in T_{\mathfrak{F}}(B)$. Next, let g be a right B-homomorphism of $J(J\in\mathfrak{F})$ to $\bar{B}$. Since ${}_BA$ is finitely generated (f. g.) projective, we have an unique extension $\hat{g}: JA\longrightarrow\bar{A}$ of g so that the diagram

$$\begin{array}{ccc} J\otimes_B A & \xrightarrow{g\otimes \mathrm{Id}_A} & B/T_{\mathfrak{F}}(B)\otimes_B A \\ \wr\| & & \wr\| \\ JA & \xrightarrow{\hat{g}} & A/T_{\mathfrak{F}}(B)A=\bar{A} \end{array}$$

is commutative, where the vertical mappings are the natural isomorphisms. Thus we can obtain a ring homomorphism $\rho:\tilde{B}\longrightarrow\tilde{A}$ defined by

$$\rho([g])=[\hat{g}]$$

for $[g]\in\tilde{B}$. It is obviously monic. Therefore, we can and shall regard $\tilde{B}$ as a subring of $\tilde{A}$ by ρ. Moreover we shall denote by $\tilde{a}$ the image of $a\in A$ under the canonical mapping $A\longrightarrow\tilde{A}$, and by f' the restriction of f to J for a representative $f: JA\longrightarrow\bar{A}$ of $[f]\in\tilde{A}(J\in\mathfrak{F})$.

Then we claim the followings.

a) *Every right B-homomorphism* φ *of* A *to* B *induces a right* $\widetilde{B}$-*homomorphism* $\tilde{\varphi}$ *of* $\widetilde{A}$ *to* $\widetilde{B}$ *defined by*

$$\tilde{\varphi}([f])=[\bar{\varphi}\circ f']$$

for $[f]\in\widetilde{A}$. *In particular, if* $(\varphi_s, a_s)_{1\leq s\leq s_0}$ $(\varphi_s\in \mathrm{Hom}(A_B, B_B), a_s\in A)$ *is a dual basis for* A_B, *then* $(\tilde{\varphi}_s, \tilde{a}_s)_{1\leq s\leq s_0}$ *is also a dual basis for* $\widetilde{A}_{\widetilde{B}}$.

b) *For* $\psi\in \mathrm{Hom}({}_BA_B, {}_BB_B)$, $\tilde{\psi}$ *which can be defined by* a) *regarding* ψ *as a right B-homomorphism is contained in* $\mathrm{Hom}({}_{\widetilde{B}}\widetilde{A}_{\widetilde{B}}, {}_{\widetilde{B}}\widetilde{B}_{\widetilde{B}})$.

c) *If* $\sum x_t\otimes y_t\in A\otimes_B A$ *is a casimir element, then* $\sum \tilde{x}_t\otimes\tilde{y}_t\in\widetilde{A}\otimes_{\widetilde{B}}\widetilde{A}$ *is also a casimir element.*

d) $\sum_{i,j}\tilde{a}_i(\tilde{x}_{ij})\tilde{y}_{ij}=\tilde{1}$ and $\sum_{k,l}\widetilde{w}_{kl}\widetilde{\beta}_k(\tilde{z}_{kl})=\tilde{1}$.

The above assertions a), b) and d) can be easily verified. Since $\widetilde{A}_{\widetilde{B}}$ is f. g. projective by a), a mapping $\widetilde{A}\otimes_{\widetilde{B}}\widetilde{A}\ni u\otimes v\longmapsto(\eta\longmapsto\eta(u)v)\in \mathrm{Hom}({}_{\widetilde{B}}\mathrm{Hom}(\widetilde{A}_{\widetilde{B}}, \widetilde{B}_{\widetilde{B}}), {}_{\widetilde{B}}\widetilde{A})$ is bijective. Hence, to see c), it is enough to show the following equation

$$\sum_t\tilde{\varphi}_s(\tilde{x}_t)\tilde{y}_t\cdot[f]=\sum_t\tilde{\varphi}_s([f]\cdot\tilde{x}_t)\tilde{y}_t \quad\text{for } [f]\in\widetilde{A},$$

where $(\tilde{\varphi}_s, \tilde{a}_s)_s$ is a dual basis for $\widetilde{A}_{\widetilde{B}}$ induced by a dual basis $(\varphi_s, a_s)_s$ for A_B. So, let $f: JA\longrightarrow\bar{A}(J\in\mathfrak{F})$ be a representative of $[f]\in\widetilde{A}$. Then we have

$$\tilde{y}_t\cdot[f]=[JA\ni bx\longmapsto\bar{y}_t f(b)x\in\bar{A}],$$
$$\tilde{\varphi}_s(\tilde{x}_t)=[A\ni x\longmapsto\bar{\varphi}_s(\bar{x}_t)x\in\bar{A}],$$

where, for $a\in A$, $\bar{a}$ denotes the image of $a\in A$ under the canonical mapping $A\longrightarrow\bar{A}$ and $[X\ni x\longmapsto y\in Y]$ denotes a class to which a mapping $X\ni x\longmapsto y\in Y$ belongs. Therefore the left hand of the above equation is equal to $[JA\ni bx\longmapsto\sum_t\bar{\varphi}_s(\bar{x}_t)\bar{y}_t f(b)x\in\bar{A}]$. On the other hand, $\bigcap_t x_t^{-1}(JA)$ contains some member J_1 in $\mathfrak{F}$, and so, $\bigcap_t y_t^{-1}(J_1A)$ contains some member J_2 in $\mathfrak{F}$. We have then

$$\tilde{y}_t=[A\ni x\longmapsto\bar{y}_t x\in\bar{A}],$$
$$\tilde{\varphi}_s([f]\cdot\tilde{x}_t)=[J_1A\ni bx\longmapsto\bar{\bar{\varphi}}_s(f(x_tb))\cdot x\in\bar{A}].$$

Furthermore, for any b in $J_2\cap J\in\mathfrak{F}$, we have $y_tb\in J_1A$, and so, in an equation $y_tb=\sum_{i,j}\alpha_i(y_tbx_{ij})y_{ij}$, every $\alpha_i(y_tbx_{ij})$ is contained in J_1. It follows that the right hand of the above equation is equal to

$$[(J_2\cap J)A\ni bx\longmapsto\sum_l\sum_{i,j}\bar{\varphi}_s(f(x_l\alpha_i(y_lbx_{ij})))y_{ij}x\in\bar{A}].$$

However, we have

$$\begin{aligned}\sum_l\sum_{i,j}\bar{\varphi}_s(f(x_l\alpha_i(y_lbx_{ij})))y_{ij}&=\sum_{i,j}\bar{\varphi}_s(f(b\sum_l x_l\alpha_i(y_lx_{ij})))y_{ij}\\&=\sum_{i,j}\bar{\varphi}_s(f(b)\sum_l x_l\alpha_i(y_lx_{ij}))y_{ij}\\&=\sum_l\bar{\varphi}_s(f(b)x_l)\sum_{i,j}\alpha_i(y_lx_{ij})y_{ij}\\&=\sum_l\bar{\varphi}_s(f(b)x_l)y_l\\&=\sum_l\bar{\varphi}_s(\bar{x}_l)\bar{y}_lf(b),\end{aligned}$$

where the first and the last equalities are followed by the fact that $\sum x_l\otimes y_l\in A\otimes_BA$ is a casimir element. Therefore we have the desired equation. Thus we have shown that $\tilde{A}/\tilde{B}$ is a QF extension with a right QF system $(\tilde{\alpha}_i;\ \sum_j\tilde{x}_{ij}\otimes\tilde{y}_{ij})_i$ and with a left QF system $(\tilde{\beta}_k;\ \sum_l\tilde{w}_{kl}\otimes\tilde{z}_{kl})_k$. As $\tilde{\alpha}_i(\tilde{a})=\widetilde{\alpha_i(a)}$ and $\tilde{\beta}_k(\tilde{a})=\widetilde{\beta_k(a)}$ for $a\in A$, mappings

$$\tilde{B}\otimes_BA\ni v\otimes a\longmapsto v\tilde{a}\in\tilde{A}$$

and

$$A\otimes_B\tilde{B}\ni a\otimes v\longmapsto\tilde{a}v\in\tilde{A}$$

are isomorphisms whose inverses are given respectively by

$$\tilde{A}\ni u\longmapsto\sum_{i,j}\tilde{\alpha}_i(ux_{ij})\otimes y_{ij}\in\tilde{B}\otimes_BA$$

and

$$\tilde{A}\ni u\longmapsto\sum_{k,l}w_{kl}\otimes\tilde{\beta}_k(z_{kl}u)\in A\otimes_B\tilde{B}.$$

Similarly, we can show that if A/B is a Frobenius extension with a Frobenius system $(h;\ r_i,\ l_i)_i$, then $\tilde{A}/\tilde{B}$ is a Frobenius extension with a Frobenius system $(\tilde{h};\ \tilde{r}_i,\ \tilde{l}_i)_i$. To summarize, we have the following proposition.

Proposition 1.6. *Let $\mathfrak{F}$ be an idempotent filter of B such that $\tilde{\mathfrak{F}}$ defined in Proposition 1. 1 is also an idempotent filter of A. If A/B is a QF (resp. Frobenius) extension then $Q_{\tilde{\mathfrak{F}}}(A)$ is a QF (resp. Frobenius) extension of $Q_{\mathfrak{F}}(B)$ such that the canonical mappings*

$$Q_{\mathfrak{F}}(B)\otimes_BA\ni x\otimes a\longmapsto xa\in Q_{\tilde{\mathfrak{F}}}(A)$$

and

$$A\otimes_BQ_{\mathfrak{F}}(B)\ni a\otimes x\longmapsto ax\in Q_{\tilde{\mathfrak{F}}}(A)$$

Mathematical Journal of Okayama University, Vol. 18 [1975], Iss. 1, Art. 7

are both isomorphisms. In particular, if $Q_{\mathfrak{F}}(B)$ is flat as a left (resp. right) B-module then $Q_{\tilde{\mathfrak{F}}}(A)$ is flat as a left (resp. right) A-module.

We are now ready for proving the following main theorem.

Theorem 1.8. *If A/B is a QF (resp. Frobenius) extension satisfying the condition $(*)$ or $(**)_A$ (or $(**)_B$), then $Q_{\max}(A)$ is a QF (resp. Frobenius) extension of $Q_{\max}(B)$ such that the canonical mappings*

$$Q_{\max}(B) \otimes {}_B A \ni x \otimes a \longmapsto xa \in Q_{\max}(A)$$

and

$$A \otimes {}_B Q_{\max}(B) \ni a \otimes x \longmapsto ax \in Q_{\max}(A)$$

are both isomorphisms. In particular, if $Q_{\max}(B)$ is flat as a left (resp. right) B-module, then $Q_{\max}(A)$ is flat as a left (resp. right) A-module.

Proof. Let $\mathfrak{F}$ be the filter of dense right ideals of B. Then $\tilde{\mathfrak{F}}$ defined in Proposition 1. 1 coincides with the filter of dense right ideals of A by Propositions 1. 4 and 1. 5. It follows that the theorem is a direct consequence of the above proposition.

If the injective hull of B_B is torsionless, then the condition $(**)_B$ is enjoyed by Lemma 1. 3. Thus we have the following as a consequence of the theorem.

Corollary. *If A is a QF (resp. Frobenius) extension of B such that the injective hull of B_B is torsionless, then the same conclusion as in the theorem holds.*

As will be seen from the following examples, the conditions $(*)$ and $(**)_B$ for a ring extension A/B are independent even when A/B is a Frobenius extension.

Example 1. Let $\mathbf{Z}$ be the ring of integers. A group ring $\mathbf{Z}[G]$ of a finite group G over $\mathbf{Z}$ is a Frobenius extension of $\mathbf{Z}$ with a Frobenius homomorphism $h: \mathbf{Z}[G] \ni \sum a_g \cdot g \longmapsto a_e \in \mathbf{Z}$ (e the identity of G). Obviously, $\mathbf{Z}[G]/\mathbf{Z}$ satisfies $(*)$. However $\mathbf{Z}$ does not satisfy $(**)_{\mathbf{Z}}$.

Example 2. Let R be a triangular matrix ring $\begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$ over a field K and A the 2×2 matrix ring $(R)_2$ over the ring R. As is well known, R satisfies the condition $(**)_R$ (in fact, R is a QF-3 ring), and so, A does $(**)_A$. Let us denote by σ the inner automorphism of the ring A induced by the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in A, by G the group generated by σ, and by B the subring of A consisting of elements of A left fixed by every element of G. Put $x_1=y_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $x_2=y_2=\begin{pmatrix}0&0\\0&1\end{pmatrix}\in A$. It is then easy to see that

$$\sigma\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}d&c\\b&a\end{pmatrix} \text{ for } \begin{pmatrix}a&b\\c&d\end{pmatrix}\in A,$$

$$x_1\cdot y_1+x_2\cdot y_2=1,\ x_1\cdot\sigma(y_1)+x_2\cdot\sigma(y_2)=0.$$

Therefore, A/B is a G-Galois extension. Hence it is a Frobenius extension with a Frobenius system $(h;\ x_i,\ y_i)_{1<i<2}$, where h is the trace homomorphism of A to B defined by

$$h(x)=x+\sigma(x) \text{ for } x\in A.$$

It follows that B satisfies $(^{**})_B$ by Proposition 1.4. However, to be easily seen, A/B never enjoies the condition $(^*)$.

Now assume that ${}_BA_B|{}_BB_B$, that is, there exist f_k: ${}_BA_B\longrightarrow{}_BB_B$ and $a_k\in V_A(B)$ $(k=1,\cdots,t)$ such that $\sum a_k\cdot f_k(a)=a$ for all $a\in A$. Since each f_k sends $V_A(B)$ into the center C of B, $V_A(B)$ is an f. g. projective, faithful C-module. Hence we have $V_A(B)\sim C$ as C-modules. Moreover mappings $V_A(B)\otimes_C B\longrightarrow A$, $x\otimes y\longmapsto xy$, and $A\longrightarrow V_A(B)\otimes_C B$, $x\longmapsto\sum a_k\otimes f_k(x)$, are the mutually inverse isomorphisms. Similarly we have $B\otimes_C V_A(B)\cong A$ by the correspondence $y\otimes x\longmapsto yx$, and so, ${}_BA_B\sim{}_BB_B$.

Proposition 1.8. *Suppose that A/B is QF (resp. Frobenius) such that B is right artinian with ${}_BA_B|{}_BB_B$. Let M be an f. g. right B-module which is a generator and a cogenerator. Put $N=\mathrm{Hom}(A_B, M_B)$, $A'=\mathrm{End}(N_A)$ and $B'=\mathrm{End}(N_B)$. Then there holds the following:*

a) *B'/A' is QF (resp. Frobenius).*

b) *B' and A' are both semi-primary QF-3 rings.*

c) *$A=\mathrm{End}({}_{A'}N)$ and $B=\mathrm{End}({}_{B'}N)$.*

Proof. As is mentioned above, ${}_BA_B\sim{}_BB_B$, and so, $N_B=\mathrm{Hom}(A_B, M_B)_B\sim\mathrm{Hom}(B_B, M_B)_B\cong M_B$; $N\otimes{}_BA_A\sim N_A$. Thus a) is a consequence of [5, Theorem 1. 1]. Next, to be easily seen, N_A and N_B are f. g. generators and cogenerators. Hence b) is a direct consequence of [7, Theorem 3. 1]. Finally c) is well known.

Remark. If B is a commutative artinian ring then such a module M does always exist. Indeed, the injective hull $E(B/J_B)$ is an f. g. cogenerator, where J is the radical of B (see [1, Proposition 10. 5]). Hence $M=B(B/J_B)\oplus B$ is a required one.

As an immediate consequence of the above proposition, we have the following.

Corollary. *Every QF (resp. Frobenius) algebra over a commutative artinian ring can be obtained as an endomorphism ring of a semiprimary QF-3 ring which is a QF (resp, Frobenius) extension of a semiprimary QF-3 ring.*

2. Supplements. A right R-module whose f. g. submodules are torsionless is said to be locally torsionless. In connection with [3, Proposition 2. 12], we shall show the following.

Proposition 2.1. *Suppose A/B to be QF extension, Then $E(A_A)_A$ is locally torsionless if and only if so is $E(B_B)_B$.*

Proof. In general, to be easily seen, if M_R is locally torsionless then so is every submodule of M^n_R. This remark will be used freely in the sequel. First we assume $E(A_A)_A$ locally torsionless. Since $E(A_A)_B$ is injective, we can consider as $E(B_B)\subset E(A_A)$. If Y_B is an arbitrary f. g. submodule of $E(B_B)$, then $Y\subset YA\subset E(A_A)$. Thus, by our assumption, YA_A, and so, YA_B is torsionless. Hence Y_B is torsionless. Conversely, assume $E(B_B)_B$ locally torsionless. Since $A_A|\mathrm{Hom}(A_B, B_B)_A \subsetneq \mathrm{Hom}(A_B, E(B_B)_B)_A$, we have $E(A_A)_A|\mathrm{Hom}(A_B, E(B_B)_B)_A$, and so, we have only to show $\mathrm{Hom}(A_B, E(B_B)_B)_A$ locally torsionless. To this end, take an arbitrary f. g. submodule X_A of $\mathrm{Hom}(A_B, E(B_B)_B)_A$. Then, recalling that A_B is f. g., we have $X_A\subsetneq \mathrm{Hom}(A_B, Y_B)_A$ for some f. g. (torsionless) submodule Y_B of $E(B_B)$. Hence X_A is torsionless as a submodule of the torsionless module $\mathrm{Hom}(A_B, Y_B)_A$.

Proposition 2.2. *Suppose A/B a right QF extension. Then ${}_BA$ is a generator if and only if there exists a right A-module which is a cogenerator as a B-module.*

Proof. Let $(\alpha_i;\ \sum_j x_{ij}\otimes y_{ij})_{1\leq i\leq p}$ be a right QF system for A/B. Then the trace ideal $\mathrm{Hom}({}_BA, {}_BB)(A)$ of ${}_BA$ coincides with $\sum\alpha_i(A)$. Accordingly, ${}_BA$ is a generator if and only if $\sum\alpha_i(A)=B$. First assume ${}_BA$ a generator. Then, for every right B-module Y, α_Y: $Y\longrightarrow \mathrm{Hom}(A_B, Y_B)^p$ defined as in the proof of Proposition 1. 4 is monic. Hence every right A-module which is a cogenerator (such a module exists always) is a cogenerator as a B-module. Conversely, assume that there exists a right A-module V which is a cogenerator as a B-module. If ${}_BA$ is not a generator, then there exists a maximal right ideal J of B

containing $\sum \alpha_i(A)$. Since V_B is a cogenerator, there exists some $f \in \mathrm{Hom}(B/J_B, V_B)$ with $f(1+J) \neq 0$. However, $\mathrm{Ann}_A(f(1+J)) = \{a \in A \mid f(1+J)a = 0\} \supset JA \supset \sum \alpha_i(A) \cdot A = A$, which yields a contradiction $f(1+J)=0$.

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(*Received March 12, 1974*)
(*Revised January 31, 1975*)