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We shall assume throughout that all rings and algebras are commutative with identity and that all homomorphisms are unital. Recall that a ring-homomorphism $R \to T$ is said to be universally going-down in case $S \to S \otimes_R T$ satisfies going-down (henceforth abbreviated GD) for each change of base $R \to S$. This concept was introduced in [7] and studied extensively in [4]. The most natural examples of universally going-down homomorphisms $R \to T$ arise when dim $(R) = 0$ or T is R-flat [4, Proposition 3.3]. It is interesting that, in some cases, universally going-down reduces to one of these two archetypes. For zero-dimensionality, this is essentially well known and summarized in Proposition 1 below. Our main result, Theorem 3, is that any universally going-down overring extension of an integrally closed domain must be flat. This will follow easily from the main results of [4] and [5], with which we assume familiarity.

Proposition 1. *For a ring R, the following five conditions are equivalent:*

(1) *$R \to T$ is universally going-down for each R-algebra T;*

(2) *The canonical map $R \to R/P$ is universally going-down for each nonminimal* $P \in$ *Spec* (R)*;*

(3) *The canonical map $R \to R/P$ satisfies GD for each non-minimal $P \in$ Spec (R);*

(4) *R/P is R-flat for each nonminimal $P \in$ Spec (R);*

(5) *$dim(R) = 0$.*

Proof. The equivalence of (3), (4), and (5) was observed in [3, Proposition 2.1]. As noted above, [4, Proposition 3.3] yields that (5) ⇒ (1); and (1) ⇒ (2) ⇒ (3) trivially, to complete the proof.

The above theme that (universally) GD-behavior often entails flatness was also noted in [3, Remark 2.6 (c)]. It was shown there that if R is a reduced ring, then the canonical map $R \to R/P$ satisfies (universally) GD for each nonmaximal $P \in$ Spec (R) (if and) only if each such R/P is R-flat. As this result is false without the "reduced" hypothesis [3, Remark 2.6 (b)], we are motivated to consider the "reduced" case of Proposition 1.

Corollary 2. *For a ring R, the following six conditions are equivalent:*

(1) *R is reduced and $R \to T$ is universally going-down for each R-algebra T;*

(2) *Each R-algebra is R-flat;*

(3) *R is reduced and the canonical map $R \to R/P$ is universally going-down for each maximal ideal P of R;*

(4) *R is reduced and the canonical map $R \to R/P$ satisfies GD for each maximal ideal P of R;*

(5) *R/P is R-flat for each maximal ideal P of R;*

(6) *R is von Neumann regular (i.e., absolutely flat).*

Proof. As cited in [3], the equivalence (2)$\Leftrightarrow$(6) is in well known work of Harada and Auslander. Also, Akiba [1, Corollary 4] (and, much later, [3, Remark 2.6 (e)]) established (6)$\Leftrightarrow$(5). Moreover, since (6) is well known to be equivalent to the condition that R be reduced and zero-dimensional, Proposition 1 yields (1)$\Leftrightarrow$(6). Similarly, (4) $\Rightarrow$ (6), as it is easy to see that the GD condition in (4) implies $\dim(R) = 0$. Finally, (1) $\Rightarrow$ (3) $\Rightarrow$ (4) trivially, completing the proof.

Before stating our main result, we recall a definition from [4]. A ring-homomorphism $f: R \to T$ is said to be quasi-going-up (in short, QGU) if, for each pair of primes $P_1 \subset P_2$ of R such that $f(P_2)\,T \neq T$ and each $Q_1 \in \operatorname{Spec}(T)$ such that $f^{-1}(Q_1) = P_1$, there exists $Q_2 \in \operatorname{Spec}(T)$ such that $Q_1 \subset Q_2$ and $f^{-1}(Q_2) = P_2$. The key fact used in the next proof is that universally going-down overring extensions of domains satisfy this weak form of going-up, even after change of base.

Theorem 3. *Let T be an overring of a domain R such that R is integrally closed in T. Then the inclusion map $R \to T$ is universally going-down (if and) only if T is R-flat.*

Proof. The parenthetic assertion holds since flat implies universally going-down. Conversely, to show T is R-flat, a criterion of Richman (cf. proof of [8, Theorem 2]) reduces us to verifying the following: if $P \in \operatorname{Spec}(R)$ and $PT \neq T$, then $T_P = R_P$. (As usual, T_P denotes the ring of fractions $T_{R \setminus P}$.) For any such P, the hypothesis yields that R_P is integrally closed in T_P. Thus, it suffices to verify that, for each $P \in \operatorname{Spec}(R)$ such that $PT \neq T$, one has that T_P is integral over R_P. In the terminology of [5], our task is thus to show that the inclusion map $f: R \to T$ is quasiintegral. By the main result of [5], namely [5, Theorem 3.2], this is equivalent to showing that f is universally QGU, in the sense that $S \to S \otimes_R T$ is QGU for each change of base $R \to S$. However, the main result of [4], namely [4, Theorem 3.17], assures that each universally going-down overring extension of a domain *is* universally QGU. The proof is complete.

Remark 4. (a) A result of Papick (cf. [6, (3.14)]) asserts that if T is an overring of a coherent domain R such that R is integrally closed in T, then $R \subset T$ satisfies GD (if and) only if T is R-flat. One may regard the assertion of Theorem 3 in the same vein, where the finiteness hypothesis of coherence has been eliminated, at the expense of enhancing the GD hypothesis to universally going-down.

(b) One way to motivate the "integrally closed in" hypothesis in Theorem 3 is via Corollary 2, for any von Neumann regular ring is trivially integrally closed. Another way is to note that the "dual" situation, that of an *integral* overring extension of domains that is universally going-down, has been extensively characterized (cf. [4, Corollaries 3.19 and 3.20]).

(c) The above "flat" impact of universally going-down should be contrasted with the effect of another type of "enhanced GD" condition considered in some of our recent work. Let R be a domain such that $A \subset B$ satisfies GD for all pairs $A \subset B$ of subrings of R. Then by [2, Theorem 2.1 and Proposition 2.5], $\dim(A) \leq 1$ and $\dim(B) \leq 1$ for all subrings $A \subset B$ of R, but it need *not* follow that B is A-flat.

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