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## Sums of reciprocals of some multiplicative functions

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## SUMS OF RECIPROCAL OF SOME MULTIPLICATIVE FUNCTIONS

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**1. Introduction.** Throughout this paper,  $m$  denotes a positive integral variable,  $p$  denotes a prime and  $x$  denotes a real variable  $\geq 3$ . In 1900 Landau [5] established that

$$(1.1) \quad \sum_{m \leq x} \frac{1}{\varphi(m)} = \frac{315 \zeta(3)}{2\pi^4} \left( \log x + \gamma - \sum_p \frac{\log p}{p^2 - p + 1} \right) + O(x^{-1} \log x),$$

where  $\varphi(m)$  is the Euler-totient function,  $\zeta(s)$  is the Riemann zeta function and  $\gamma$  is the Euler constant. In 1916 Ramanujan [6] established that

$$(1.2) \quad \sum_{m \leq x} \frac{1}{\tau(m)} = x \left\{ \frac{A_1}{(\log x)^{\frac{1}{2}}} + \frac{A_2}{(\log x)^{\frac{3}{2}}} + \cdots + \frac{A_r}{(\log x)^{r-\frac{1}{2}}} + O((\log x)^{-(r-\frac{1}{2}}) \right\},$$

where  $\tau(m)$  is the number of divisors of  $m$ ,  $r$  is any positive integer,  $A_1 = \pi^{-\frac{1}{2}} \prod_p \left\{ (p^2 - p)^{\frac{1}{2}} \log \left( \frac{p}{p-1} \right) \right\}$ , and  $A_2, \dots, A_r$  are more complicated constants.

In this paper, we establish the asymptotic formulae for the sums  $\sum_{m \leq x} \frac{1}{\sigma(m)}$  and  $\sum_{m \leq x} \frac{1}{\psi(m)}$ , where  $\sigma(m)$  is the sum of the divisors of  $m$  and  $\psi(m)$  is Dedekind's  $\psi$ -function (cf. [2], p. 123) which has the following arithmetical form :

$$(1.3) \quad \psi(m) = \sum_{d|m} \mu^2(d) \delta = m \prod_{p|m} \left( 1 + \frac{1}{p} \right),$$

$\mu$  being the Möbius function. In fact, we prove the following general result and then deduce (see §4) asymptotic expressions for  $\sum_{m \leq x} \frac{1}{\sigma(m)}$  and  $\sum_{m \leq x} \frac{1}{\psi(m)}$  :

**Theorem.** *Suppose  $g$  is a multiplicative function satisfying*

$$(1.4) \quad g(p) = \frac{1}{p+1}, \quad \text{for all primes } p,$$

*and for each  $\varepsilon > 0$ ,*

$$(1.5) \quad p^j(g(p^j) - g(p^{j-1})) = O(p^{je}),$$

for all primes  $p$  and positive integers  $j$ . Then we have

$$(1.6) \quad \sum_{m \leq x} \frac{g(m)}{m} = A \log x + B + O(x^{-1} \log^{\frac{2}{3}} x (\log \log x)^{\frac{4}{3}}),$$

where

$$(1.7) \quad A \equiv A(g) = \sum_{m=1}^{\infty} \frac{g^*(m)}{m},$$

and

$$(1.8) \quad B = A\gamma - \sum_{m=1}^{\infty} \frac{g^*(m) \log m}{m}.$$

In the above,  $g^*$  is the multiplicative function defined by

$$(1.9) \quad g^*(m) = \sum_{d|m} \mu(d) g\left(\frac{m}{d}\right).$$

**2. Prerequisites.** In this section, we state some known results and prove some lemmas which are needed in the present discussion. Let  $[x]$  denote, as usual, the largest integer  $\leq x$ . We need the following best known result of its kind which is due to Arnold Walfisz [8]:

**Lemma 2.1** (cf. [8], (36), p. 144).

$$\sum_{m \leq x} \frac{\mu(m)}{m} \rho\left(\frac{x}{m}\right) = O(\lambda(x)),$$

where

$$(2.1) \quad \rho(x) = x - [x] - \frac{1}{2},$$

and

$$(2.2) \quad \lambda(x) = \begin{cases} \log^{\frac{2}{3}} x (\log \log x)^{\frac{4}{3}}, & \text{if } x \geq 3, \\ 1, & \text{if } 0 < x < 3. \end{cases}$$

**Remark 2.1.** It is clear that  $\lambda(x)$  is increasing for  $x \geq 3$ . Using this, it can be shown that if  $x > 0$ , then

$$\lambda(x) \leq H\lambda(y), \quad \text{for all } y \leq x,$$

where  $H$  is an absolute positive constant.

**Lemma 2.2.** Let  $f$  be any multiplicative function satisfying

$$(2.3) \quad f(m) = O(m^\varepsilon), \quad \text{for every } \varepsilon > 0,$$

and

$$(2.4) \quad f(p) + 1 = O(1/\sqrt{p}) \text{ for all primes } p.$$

Further, let  $h$  be the arithmetic function defined by

$$(2.5) \quad h(m) = \sum_{d|m} f(d).$$

Then the series  $\sum_{m=1}^{\infty} h(m) m^{-s}$  converges absolutely for any  $s > \frac{1}{2}$ .

*Proof.* Since  $f$  is multiplicative, it follows (cf. [4], Theorem 265, p. 235) that  $h$  is also multiplicative. It is known (cf. [3], Theorem 41) that if  $h$  is multiplicative and the product  $\prod_p \left\{ 1 + \sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}} \right\}$  converges then the series  $\sum_{m=1}^{\infty} h(m) m^{-s}$  converges absolutely. Hence, in the present context, it suffices to prove that  $\prod_p \left\{ 1 + \sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}} \right\}$  converges for  $s > \frac{1}{2}$ . Let  $s > \frac{1}{2}$  and  $0 < \epsilon < s - \frac{1}{2}$ . From (2.3) and (2.5) it follows that  $h(m) = O(m^\epsilon)$ . Since  $h(p) = 1 + f(p)$ , by (2.4) we have

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}} &= \frac{|1+f(p)|}{p^s} + \sum_{m=2}^{\infty} \frac{|h(p^m)|}{p^{ms}} \\ &= O(p^{-s-\frac{1}{2}}) + O\left(\sum_{m=2}^{\infty} p^{-m(s-\epsilon)}\right) \\ &= O(p^{-s-\frac{1}{2}}) + O(p^{-2(\epsilon-\epsilon)}(1 - p^{-(\epsilon-\epsilon)})^{-1}) \\ &= O(p^{-s-\frac{1}{2}}) + O(p^{-2(\epsilon-\epsilon)}), \text{ for large } p. \end{aligned}$$

Now,

$$\begin{aligned} \sum_p \sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}} &= O\left(\sum_p p^{-s-\frac{1}{2}}\right) + O\left(\sum_p p^{-2(\epsilon-\epsilon)}\right) \\ &= O(1) + O(1) = O(1), \end{aligned}$$

since  $s > \frac{1}{2}$  and  $2(s - \epsilon) > 1$ . Hence Lemma 2.2 follows.

**Lemma 2.3.** *Let  $f$  be as in Lemma 2.2. Then we have*

$$\sum_{m \leq x} \frac{f(m)}{m} = O(1).$$

*Proof.* By (2.5) and the Möbius inversion formula ([4], Theorem

266, p. 236) we have

$$(2.6) \quad f(m) = \sum_{d|m} \mu(d) h\left(\frac{m}{d}\right).$$

Since  $\sum_{m \leq x} \mu(m) m^{-1} = O(1)$ , we have by (2.6),

$$\begin{aligned} \sum_{m \leq x} \frac{f(m)}{m} &= \sum_{d \leq x} \frac{\mu(d) h(\delta)}{d\delta} = \sum_{\delta \leq x} \frac{h(\delta)}{\delta} \sum_{d \leq x/\delta} \frac{\mu(d)}{d} \\ &= O\left(\sum_{\delta \leq x} \frac{|h(\delta)|}{\delta}\right) = O(1), \text{ by Lemma 2.2.} \end{aligned}$$

Hence Lemma 2.3 follows.

**Lemma 2.4.** *Under the hypothesis of Lemma 2.2, we have*

$$(2.7) \quad F(x) \equiv \sum_{m \leq x} \frac{f(m)}{m} \rho\left(\frac{x}{m}\right) = O(\lambda(x)),$$

where  $\rho(x)$  and  $\lambda(x)$  are given by (2.1) and (2.2) respectively.

*Proof.* By (2.6), Lemma 2.1, Remark 2.1 and Lemma 2.2, we have

$$\begin{aligned} F(x) &= \sum_{d \leq x} \frac{\mu(d) h(\delta)}{d\delta} \rho\left(\frac{x}{d\delta}\right) = \sum_{\delta \leq x} \frac{h(\delta)}{\delta} \sum_{d \leq x/\delta} \frac{\mu(d)}{d} \rho\left(\frac{x}{d\delta}\right) \\ &= O\left(\sum_{\delta \leq x} \frac{|h(\delta)| \lambda\left(\frac{x}{\delta}\right)}{\delta}\right) = O(\lambda)(x) \sum_{\delta \leq x} \frac{|h(\delta)|}{\delta} = O(\lambda(x)). \end{aligned}$$

Hence Lemma 2.4 follows.

**Lemma 2.5.** *Let  $g^*$  be given by (1.9). Then we have*

$$(2.8) \quad G^*(x) = \sum_{m \leq x} g^*(m) = O(1).$$

*Proof.* From (1.9) it follows that for  $j \geq 1$ ,

$$(2.9) \quad g^*(p^j) = g(p^j) - g(p^{j-1}).$$

Since  $g^*$  is multiplicative, it follows from (1.5) and (2.9) that

$$(2.10) \quad mg^*(m) = O(m^\epsilon), \text{ for every } \epsilon > 0.$$

Further, by (2.9) and (1.4), we have

$$pg^*(p) = p(g(p) - 1) = p\left(\frac{p}{p+1} - p\right) = -\frac{p}{p+1},$$

so that

$$(2.11) \quad 1 + pg^*(p) = 1 - \frac{p}{p+1} = \frac{1}{p+1} = O\left(\frac{1}{p}\right) = O\left(\frac{1}{p^{\frac{1}{2}}}\right).$$

Hence if we take  $f(m) = mg^*(m)$ , from (2.10) and (2.11), it is clear that the conditions (2.3) and (2.4) are satisfied. Now, Lemma 2.5 follows from Lemma 2.3.

**Lemma 2.6.** *We have*

$$\sum_{m \leq x} g^*(m) \rho\left(\frac{x}{m}\right) = O(\lambda(x)),$$

where  $g^*$  is given by (1.9).

*Proof.* Taking  $f(m) = mg^*(m)$  in Lemma 2.4, we obtain Lemma 2.6, in virtue of (2.10) and (2.11).

**Lemma 2.7.** *We have*

$$\sum_{m \leq x} \frac{g^*(m)}{m} = A + O(x^{-1}),$$

where  $A$  is given by (1.7).

*Proof.* The series  $\sum_{m=1}^{\infty} \frac{g^*(m)}{m}$  converges absolutely by (2.10). If  $G^*(x)$  is given by (2.8), then by Lemma 2.5 and partial summation we have

$$\begin{aligned} \sum_{m > x} \frac{g^*(m)}{m} &= -\frac{G^*(x)}{([x]+1)} + \sum_{m > x} G^*(m) \left(\frac{1}{m} - \frac{1}{m+1}\right) \\ &= O(x^{-1}) + O\left(\sum_{m > x} \frac{1}{m^2}\right) = O(x^{-1}) + O(x^{-1}) = O(x^{-1}). \end{aligned}$$

Since by (1.7),  $\sum_{m \leq x} \frac{g^*(m)}{m} = A - \sum_{m \leq x} \frac{g^*(m)}{m}$ , we obtain Lemma 2.7.

**Lemma 2.8.** *We have*

$$\sum_{m \leq x} \frac{g^*(m) \log m}{m} = \sum_{m=1}^{\infty} \frac{g^*(m) \log m}{m} + O(x^{-1+\epsilon}), \text{ for every } \epsilon > 0.$$

*Proof.* By (2.10) we have

$$\sum_{m \leq x} \frac{g^*(m) \log m}{m} = O\left(\sum_{m > x} \frac{1}{m^{2-\epsilon}}\right) = O(x^{-1+\epsilon}).$$

Hence Lemma 2.8 follows.

Now, we are in a position to prove the following important

**Lemma 2.9.** *Let  $g$  be as given in the statement of Theorem. Then we have*

$$(2.12) \quad \sum_{m \leq x} g(m) = Ax + O(\lambda(x)),$$

where  $A$  is given by (1.7) and  $\lambda(x)$  as defined in (2.2).

*Proof.* From (1.9) and the converse of the Möbius inversion formula (cf. [4], Theorem 267, p. 236)

$$(2.13) \quad g(m) = \sum_{d|m} g^*(d).$$

Now, by (2.13), (2.1), Lemmas 2.6 and 2.7, and (2.8), we have

$$\begin{aligned} \sum_{m \leq x} g(m) &= \sum_{d \leq x} g^*(d) = \sum_{d \leq x} g^*(d) \left[ \frac{x}{d} \right] \\ &= x \sum_{d \leq x} \frac{g^*(d)}{d} - \sum_{d \leq x} g^*(d) \rho \left( \frac{x}{d} \right) - \frac{1}{2} \sum_{d \leq x} g^*(d) \\ &= Ax + O(1) + O(\lambda(x)) + O(1) = Ax + O(\lambda(x)). \end{aligned}$$

Hence Lemma 2.9 follows.

**Lemma 2.10** (cf. [4], Theorem 422, p. 347). *For  $x \geq 2$ ,*

$$\sum_{m \leq x} \frac{1}{m} = \log x + \gamma + O(x^{-1}).$$

**3. Proof of Theorem.** Let

$$G(x) = \sum_{m \leq x} g(m) \text{ and } \Delta(x) = G(x) - Ax.$$

Then by (2.12) we have  $\Delta(x) = O(\lambda(x))$ . Now, by partial summation, we have

$$\begin{aligned} \sum_{m \leq x} \frac{g(m)}{m} &= \frac{G(x)}{x} + \int_1^x \frac{G(t)}{t^2} dt \\ &= A + \frac{\Delta(x)}{x} + \int_1^x \frac{1}{t^2} \{At + \Delta(t)\} dt \\ &= A + \frac{\Delta(x)}{x} + A \int_1^x \frac{dt}{t} + \int_1^x \frac{\Delta(t)}{t^2} dt \\ &= A + \frac{\Delta(x)}{x} + A \log x + \int_1^\infty \frac{\Delta(t)}{t^2} dt - \int_x^\infty \frac{\Delta(t)}{t^2} dt \\ &= A(\log x + C) - \int_x^\infty \frac{\Delta(t)}{t^2} dt + O(x^{-1} \lambda(x)), \end{aligned}$$

where  $C = 1 + \frac{1}{A} \int_1^\infty \frac{J(t)}{t^2} dt$ , is a constant. (Here, of course, we assume that  $A \neq 0$ ). Also, we have

$$\begin{aligned} \int_x^\infty \frac{J(t)}{t^2} dt &= O\left(\int_x^\infty \frac{\lambda(t)}{t^2} dt\right) = O\left(x^{-\varepsilon} \lambda(x) \int_x^\infty \frac{dt}{t^{2-\varepsilon}}\right) \\ &= O(x^{-\varepsilon} \lambda(x) x^{-1+\varepsilon}) = O(x^{-1} \lambda(x)), \end{aligned}$$

where we used that  $x^{-\varepsilon} \lambda(x)$  is decreasing for every  $\varepsilon > 0$ . Hence we obtain

$$(3.1) \quad \sum_{m \leq x} \frac{g(m)}{m} = A(\log x + C) + O(x^{-1} \lambda(x)).$$

On the other hand, we have by (2.13), Lemmas 2.10, 2.7, 2.8 and (1.8),

$$\begin{aligned} \sum_{m \leq x} \frac{g(m)}{m} &= \sum_{d \leq x} \frac{g^*(d)}{d} = \sum_{d \leq x} \frac{g^*(d)}{d} \sum_{d \leq x/d} \frac{1}{d} \\ &= \sum_{d \leq x} \frac{g^*(d)}{d} \left\{ \log x - \log d + \gamma + O\left(\frac{d}{x}\right) \right\} \\ &= (\log x + \gamma) \sum_{d \leq x} \frac{g^*(d)}{d} - \sum_{d \leq x} \frac{g^*(d) \log d}{d} + O(x^{-1} \sum_{d \leq x} |g^*(d)|) \\ &= (\log x + \gamma) (A + O(x^{-1})) - \sum_{d=1}^\infty \frac{g^*(d) \log d}{d} \\ &\quad + O(x^{-1+\varepsilon}) + O(x^{-1} \sum_{d \leq x} |g^*(d)|) \\ &= A \log x + B + O(x^{-1+\varepsilon}) + O(x^{-1} \sum_{d \leq x} |g^*(d)|). \end{aligned}$$

Now, by (2.10), we have

$$\sum_{d \leq x} |g^*(d)| = O\left(x^\varepsilon \sum_{d \leq x} \frac{1}{d}\right) = O(x^\varepsilon \log x) = O(x^{2\varepsilon}),$$

for every  $\varepsilon > 0$ . Hence we have

$$(3.2) \quad \sum_{m \leq x} \frac{g(m)}{m} = A \log x + B + O(x^{-1+2\varepsilon}),$$

for every  $\varepsilon > 0$ . Now, comparing (3.1) and (3.2), we find that  $AC = B$ , from which the theorem follows.

**4. Applications.** First we have

**Corollary 4.1.** For  $x \geq 3$

$$(4.1) \quad \begin{aligned} \sum_{m \leq x} \frac{1}{\sigma(m)} &= a \left( \log x + \gamma + \sum \frac{(p-1)^2 \beta(p) \log p}{p \alpha(p)} \right) \\ &\quad + O\left(x^{-1} \log^{\frac{2}{3}} x (\log \log x)^{\frac{4}{3}}\right), \end{aligned}$$



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where

$$(4.2) \quad a = \prod_p \alpha(p),$$

$$(4.3) \quad \alpha(p) = 1 - \frac{(p-1)^2}{p} \sum_{j=1}^{\infty} \frac{1}{(p^j-1)(p^{j+1}-1)},$$

and

$$(4.4) \quad \beta(p) = \sum_{j=1}^{\infty} \frac{j}{(p^j-1)(p^{j+1}-1)}.$$

*Proof.* Taking  $g(m) = \frac{m}{\sigma(m)}$  in Theorem, we see that  $g$  satisfies

$$(1.4). \quad \text{Since } \sigma(p^j) = \frac{p^{j+1}-1}{p-1}, \text{ we have for } j \geq 1,$$

$$(4.5) \quad \begin{aligned} g^*(p^j) &= g(p^j) - g(p^{j-1}) \\ &= \frac{p^j(p-1)}{p^{j+1}-1} - \frac{p^{j-1}(p-1)}{p^j-1} = -\frac{p^{j-1}(p-1)^2}{(p^{j+1}-1)(p^j-1)}, \end{aligned}$$

so that

$$\begin{aligned} p^j |g(p^j) - g(p^{j-1})| &= \frac{p^j}{p^j-1} \cdot \left(\frac{p-1}{p}\right)^2 \cdot \frac{p^{j+1}}{p^{j+1}-1} \\ &\leq \frac{p^j}{p^j-1} \cdot \frac{p^{j+1}}{p^{j+1}-1} \leq 2 \cdot 2 = 4. \end{aligned}$$

Thus  $g$  satisfies (1.5) also. Further by (4.5) and the Euler infinite product theorem (cf. [4], Theorem 286) for  $s > 0$  we have

$$(4.6) \quad \begin{aligned} \sum_{m=1}^{\infty} \frac{g^*(m)}{m^s} &= \prod_p \left(1 + \sum_{j=2}^{\infty} \frac{g^*(p^j)}{p^{js}}\right) \\ &= \prod_p \left(1 - (p-1)^2 \sum_{j=1}^{\infty} \left(\frac{p^{j-1}}{(p^{j+1}-1)(p^j-1)p^{js}}\right)\right). \end{aligned}$$

From (4.6) ( $s = 1$ ), (1.7), (4.2) and (4.3), we have

$$(4.7) \quad A = a.$$

Now, differentiating the series in (4.6) with respect to  $s$  termwise, and then putting  $s = 1$ , we obtain

$$(4.8) \quad \sum_{m=1}^{\infty} \frac{g^*(m) \log m}{m} = -a \sum_p \frac{(p-1)^2 \beta(p) \log p}{p \alpha(p)},$$

where  $a$ ,  $\alpha(p)$  and  $\beta(p)$  are respectively given by (4.2), (4.3) and (4.4).

Now, (4. 1) follows from (4. 7) and (4. 8), by taking  $g(m) = \frac{m}{\sigma(m)}$  in (1. 6).

**Remark 4.1.** Asymptotic formula for the sum  $\sum_{z \leq m \leq x} \frac{1}{\log \sigma(m)}$  has been established by J.-M. De Koninck and J. Galambos (cf. [1], § 3, Theorem).

**Corollary 4.2.** For  $x \geq 3$ , we have

$$(4. 9) \quad \sum_{m \leq x} \frac{1}{\phi(m)} = \alpha \left( \log x + \gamma + \sum_p \frac{\log p}{p^2 + p - 1} \right) + O\left(x^{-1} \log^{\frac{2}{3}} x (\log \log x)^{\frac{3}{4}}\right),$$

where

$$(4. 10) \quad \alpha = \prod_p \left( 1 - \frac{1}{p(p-1)} \right).$$

*Proof.* Taking  $g(m) = \frac{m}{\phi(m)}$  in Theorem, we see that  $g$  satisfies

(1. 4). By (1. 3) we have for  $j \geq 1$ ,

$$(4. 11) \quad g^*(p^j) = g(p^j) - g(p^{j-1}) = \begin{cases} -\frac{1}{p+1}, & \text{if } j = 1, \\ 0, & \text{if } j \geq 2. \end{cases}$$

Thus  $g$  satisfies (1. 5) also. Now, by the Euler infinite product theorem and (4. 11), we have for  $s > 0$ ,

$$(4. 12) \quad \sum_{m=1}^{\infty} \frac{g^*(m)}{m^s} = \prod \left( 1 - \frac{1}{p^s(p+1)} \right).$$

From (4. 12) ( $s = 1$ ), (1. 7) and (4. 10), we see that

$$(4. 13) \quad A = \alpha.$$

Now, differentiating the series in (4. 12) with respect to  $s$  termwise, and then putting  $s = 1$ , we obtain

$$(4. 14) \quad \sum_{m=1}^{\infty} \frac{g^*(m) \log m}{m} = -\alpha \sum_p \frac{\log p}{p^2 + p - 1},$$

where  $\alpha$  is given by (4. 10). Now, (4. 9) follows from (4. 13) and (4. 14), by taking  $g(m) = \frac{m}{\phi(m)}$  in (1. 6).

**Remark 4.2.** Formula (4. 9) has been established by Suryanarayana (cf. [7], Lemma 2. 10, p. 12) with a weaker  $O$ -estimate for the error term,

namely  $O(x^{-1} \log x)$ .

**Remark 4.3.** It may be interesting to improve the  $O$ -estimate of the error term in (1.1), which we could not do.

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