Set containment characterization and mathematical programming

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***Abstract*—Recently, many researchers studied set containment characterizations. In this paper, we introduce some set containment characterizations for quasiconvex programming. Furthermore, we show a duality theorem for quasiconvex programming by using set containment characterizations. Notions of quasiconjugate for quasiconvex functions, especially 1, -1-quasiconjugate, 1-semiconjugate, H-quasiconjugate and R-quasiconjugate, play important roles to derive characterizations of the set containments.**

I. Introduction

Motivated by general nonpolyhedral knowledge-based data classification, the containment problem which consists of characterizing the inclusion $A \subset B$, where $A = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in I\}$ and $B = \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, j \in J\}$, was studied by many researchers. The first characterizations were given by Mangasarian [5] for linear systems and for systems involving differentiable convex functions, and key to this approach was Farkas' Lemma and the duality theorems of convex programming, respectively. Jeyakumar [4] established dual characterizations of the set containment, assuming the convexity of f_i, $i \in I$, and the convexity (the concavity) of h_j, $j \in J$, so that A is a closed convex set and B is a closed convex set (a reverse convex set, respectively).

In this paper, we introduce some set containment characterizations for quasiconvex programming in [9], [10], that is, we show set containment characterizations, assuming that all f_i are quasiconvex, all h_j are linear, or all f_i are quasiconvex and all h_j are quasiconcave. These dual characterizations are provided in terms of level sets of H-quasiconjugate, R-quasiconjugate, 1, -1-quasiconjugate and 1-semiconjugate functions. Furthermore, we show a duality theorem for quasiconvex programming by using set containment characterizations. In [12], Thach established the duality theorem for quasiconvex programming by using R-quasiconjugate, but did not give any specific conditions. In this paper, we give another proof of this duality theorem and give a specific condition for the storong duality by using reverse convex set containment characterization in [10].

II. Notation and Preliminaries

Throughout this paper, let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Remember that f is said to be quasiconvex if, for all x_1, $x_2 \in \mathbb{R}^n$ and $\alpha \in (0,1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Define

$$L(f, \diamond, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. Symbol $\diamond$ represents any binary relation. Then f is quasiconvex if and only if for any $\alpha \in \mathbb{R}$, $L(f, \leq, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$, $L(f, <, \alpha) = \{x \in \mathbb{R}^n \mid f(x) < \alpha\}$ is a convex set. We know that any convex function is quasiconvex, but the converse is not true. A subset S of $\mathbb{R}^n$ is said to be evenly convex if it is the intersection of some family of open halfspaces. A subset S of $\mathbb{R}^n$ is said to be H-evenly convex if it is the intersection of some family of open halfspaces, and each open halfspace containing 0. Note that the whole space and the empty set are H-evenly convex. Also, any open convex set and any closed convex set are evenly convex. Clearly, every evenly convex set is convex and a nonempty subset S of $\mathbb{R}^n$ is H-evenly convex if and only if S is an evenly convex set which contains 0. A function f is said to be evenly quasiconvex if $L(f, \leq, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$. A function f is said to be strictly evenly quasiconvex if $L(f, <, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$. A function f is said to be H-evenly quasiconvex if $L(f, \leq, \alpha)$ is H-evenly convex for all $\alpha \in \mathbb{R}$. A function f is said to be strictly H-evenly quasiconvex if $L(f, <, \alpha)$ is H-evenly convex for all $\alpha \in \mathbb{R}$. Clearly, every evenly quasiconvex function is quasiconvex, every lower semicontinuous (lsc) quasiconvex function is evenly quasiconvex, and every upper semicontinuous (usc) quasiconvex function is strictly evenly quasiconvex. It is easy to show that every strictly evenly quasiconvex function is evenly quasiconvex, but the converse is not generally true, see [9]. A function f is said to achieve the minimum value at the origin if $f(x_k) \to \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$ for any sequence $\{x_k\} \subset \mathbb{R}^n \setminus \{0\}$ with $x_k \to 0$. Let γ^0 be the set of all functions that that achieve the minimum value at the origin.

Next, we introduce notions of quasiconjugates.

Definition 1 ([3])**.** The λ-quasiconjugate of f is the function $f^{\nu}_{\lambda} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^{\nu}_{\lambda}(u) = \lambda - \inf\{f(x) \mid \langle u, x\rangle \geq \lambda\}, \forall u \in \mathbb{R}^n.$$

Definition 2 ([7])**.** The λ-semiconjugate of f is the function $f^{\theta}_{\lambda} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^{\theta}_{\lambda}(u) = \lambda - \inf\{f(x) \mid \langle u, x\rangle > \lambda\}, \forall u \in \mathbb{R}^n.$$

Singer [7] defined the λ-semiconjugate in the following form,

$$f^{\theta}_{\lambda}(u) = \lambda - 1 - \inf\{f(x) \mid \langle u, x\rangle > \lambda - 1\}, \forall u \in \mathbb{R}^n,$$

but we redefine the λ-semiconjugate to unify these above conjugates.

Definition 3 ([11])**.** H-quasiconjugate of f is the function $f^H : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^H(\xi) = \begin{cases} -\inf\{f(x) \mid \langle \xi, x\rangle \geq 1\} & \text{if} \quad \xi \neq 0 \\ -\sup\{f(x) \mid x \in \mathbb{R}^n\} & \text{if} \quad \xi = 0. \end{cases}$$

Definition 4 ([12])**.** R-quasiconjugate of f is the function $f^R : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that, for any $\xi \in \mathbb{R}^n$,

$$f^R(\xi) = -\inf\{f(x) \mid \langle \xi, x\rangle \geq -1\}.$$

Clearly, $f^H + 1 = f^{\nu}_1$ on $\mathbb{R}^n \setminus \{0\}$ and $f^R - 1 = f^{\nu}_{-1}$ on $\mathbb{R}^n$. So, H-quasiconjugate and R-quasiconjugate are special cases of λ-quasiconjugate.

Given a set $S \subset \mathbb{R}^n$, we shall denote by int$S$, cl$S$ and co$S$ the interior, the closure, and the convex hull generated by S, respectively. The evenly convex hull of S, denoted by ec$S$, is the smallest evenly convex set which contains S. The H-evenly convex hull of S, denoted by Hec$S$, is the smallest H-evenly convex set which contains S. Note that co$S \subset$ ec$S \subset$ clco$S$, and these differences are slight because clco$S =$ clec$S$. Moreover if S is nonempty, then Hec$S =$ ec$(S \cup \{0\})$.

III. Set Containment Characterizations

In this section, we mention about set containment characterizations by the authors. At first, we introduce characterizations of the containment of a convex set, defined by infinite quasiconvex constraints, in an evenly convex set, i.e., let I, J, S, W be arbitrary sets, f_i and g_j be quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$, $v_s \in \mathbb{R}^n$ and $\alpha_s \in \mathbb{R}$ for each $s \in S$, $u_w \in \mathbb{R}^n$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Then, we show the characterization of $A \subset B$, where

$$\begin{aligned} A &= \{x \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}, \\ B &= \{x \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s, \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}. \end{aligned}$$

In [9], we show the following set containment characterization by using H-quasiconjugate.

Theorem 1. [9] Let J be a finite set, S be an arbitrary set, g_j be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and included in γ^0 for each $i \in I$, $v_s \in \mathbb{R}^n \setminus \{0\}$ and $\alpha_s \in (0, \infty)$ for each $s \in\ S$. If for all $x \in \mathbb{R}^n \setminus \{0\} \quad \sup_{j\in J} g_j(x) > \sup_{j\in J} g_j(0)$ then the following conditions (i) and (ii) are equivalent:
(i) $\{x \mid \forall i \in J, g_j(x) < \beta\} \subset \{x \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s\}$;
(ii) $\forall s \in S, \ \dfrac{v_s}{\alpha_s} \in \text{Hec} \bigcup_{j\in J} L(g^H_j, \leq, -\beta)$.

In [10], we show the following set containment characterization by using 1-quasiconjugate.

Theorem 2. [10] Let I, J, S and W be arbitrary sets, $\beta \in \mathbb{R}$, f_i be a evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, g_j be a strictly evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, $v_s \in \mathbb{R}^n$ and $\alpha_s \in (0, \infty)$ for each $s \in S$, and $u_w \in \mathbb{R}^n$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $f_i(0) \leq \beta$ for each $i \in I$, $g_j(0) < \beta$ for each $j \in J$ and int$\{x \in \mathbb{R}^n \mid f_i(x) \leq \beta, i \in I, g_j(x) < \beta, j \in J\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent.
(i) $A \subset B$,
(ii) $\forall s \in S$,
$\dfrac{v_s}{\alpha_s} \in \text{Hec}\Big[\cup_{i\in I} L((f_i)^{\nu}_1, <, 1-\beta) \bigcup \cup_{j\in J} L((g_j)^{\nu}_1, \leq, 1-\beta)\Big]$,
$\forall w \in W$,
$\dfrac{u_w}{\gamma_w} \in \text{clHec}\Big[\cup_{i\in I} L((f_i)^{\theta}_1, <, 1-\beta) \bigcup \cup_{j\in J} L((g_j)^{\theta}_1, \leq, 1-\beta)\Big]$.

where

$$\begin{aligned} A &= \{x \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}, \\ B &= \{x \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s, \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}. \end{aligned}$$

Next, we present characterizations of the containment of a convex set, defined by infinite quasiconvex constraints, in a reverse convex set, defined by infinite quasiconvex constraint, i.e., let I, J, W be arbitrary sets, f_i and g_j be quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$ and for each $j \in J$, k_w be a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Then, we show the characterization of $A \subset B$, where

$$\begin{aligned} A &= \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}, \\ B &= \{x \in \mathbb{R}^n \mid \forall w \in W, k_w(x) \geq \gamma_w\}. \end{aligned}$$

In [9], we show the following set containment characterization by using H-quasiconjugate and R-quasiconjugate.

Theorem 3. [9] Let J and W be arbitrary sets, g_j be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ included in γ^0 for each $j \in J$, k_w be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that for all $x \in \mathbb{R}^n \setminus \{0\} \quad \sup_{j\in J} g_j(x) > \sup_{j\in J} g_j(0)$ and $L(k_w, <, \gamma_w) \neq \emptyset$ for each $w \in W$ and $\sup_{j\in J} g_j(0) < \beta$ for some $\beta \in \mathbb{R}$. Then, the following conditions are equivalent.
(i) $\{x \mid \forall j \in J, g_j(x) < \beta\} \subset \{x \mid \forall w \in W, k_w(x) \geq \gamma_w\}$,
(ii) $\forall w \in W$,
$0 \in \text{Hec} \bigcup_{j\in J} L(g^H_j, \leq, -\beta) \setminus \{0\} + L(k^R_w, \leq, -\gamma_w)$.

In [10], we show the following set containment characterization by using 1-semiconjugate and -1-quasiconjugate.

Theorem 4. [10] Let I, J and W be arbitrary sets, f_i and g_j be quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$, k_w be a usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Assume that $0 \in \text{int}[(\cap_{i\in I} L(f_i, \leq, \beta)) \bigcap (\cap_{j\in J} L(g_j, <, \beta))]$ and $\cap_{w\in W} L(k_w, <, \gamma_w)$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $A \subset B$,
(ii) $\forall w \in W$,
$0 \in \text{clHec}\{\cup_{i\in I} L((f_i)^\theta_1, <, 1-\beta) \bigcup \cup_{j\in J} L((g_j)^\theta_1, \leq, 1-\beta)\}$
$+L((k_w)^\nu_{-1}, \leq, -1-\gamma_w) \setminus \{0\}$.

where

$$A = \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\},$$
$$B = \{x \in \mathbb{R}^n \mid \forall w \in W, k_w(x) \geq \gamma_w\}.$$

IV. A Duality Theorem for Quasiconvex Programming

In [10], we show that set containment characterizations are useful to consider quasiconvex minimization problem. We consider the following quasiconvex programming problem. Let I be an arbitrary set, f_i be a lsc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, $A = \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq 0\}$, and k be a usc quasiconvex function. Assume that $0 \in \text{int}A$, and consider the following problem (P),

$$(P) \begin{cases} \text{minimize } k(x), \\ \text{subject to } x \in A. \end{cases}$$

In [12], a dual problem of (P) is defined by

$$(D) \begin{cases} \text{minimize } k^R(z), \\ \text{subject to } z \in -A^*, \end{cases}$$

and conditions for the strong duality are discussed, but specific conditions are not given. In this paper, we show another proof of this duality theorem and give a specific condition by using reverse convex set containment characterization.

Theorem 5. Let I be an arbitrary set, f_i be a lsc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, $A = \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq 0\}$, and k be a usc quasiconvex function. Assume that $0 \in \text{int}A$. Then, the following equality holds.

$$\inf_{x\in A} k(x) = -\inf_{z\in -A^*} k^R(z),$$

where $A^* = \{z \in \mathbb{R}^n \mid \forall x \in A, \langle z, x\rangle \leq 1\}$.

Proof: By using Theorem 4, for each $\gamma \in \mathbb{R}$, following conditions (i) and (ii) are equivalent.

(i) $\cap_{i\in I} L(f_i, \leq, 0) \subset L(k, \geq, \gamma)$,
(ii) $0 \in \text{clHec}\cup_{i\in I} L((f_i)^\theta_1, <, 1) + L(k^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}$.

Clearly, $\inf_{x\in A} k(x) = \sup\{\gamma \in \mathbb{R} \mid \cap_{i\in I} L(f_i, \leq, 0) \subset L(k, \geq, \gamma)\}$. So, we can prove that $\inf_{x\in A} k(x) = \sup\{\gamma \mid 0 \in \text{clHec}\cup_{i\in I} L((f_i)^\theta_1, <, 1) + L(k^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}$. Also, the value in the right hand-side is equal to $-\inf_{z\in T}(k^\nu_{-1}(z) + 1)$, where $T = -\text{clHec} \cup_{i\in I} L((f_i)^\theta_1, <, 1)$. Furthermore, $A^* = -T$ and $k^\nu_{-1} + 1 = k^R$. Since, $\inf_{x\in A} k(x) = -\inf_{z\in -A^*} k^R(z)$. ■

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