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High Accuracy Homography Computation without Iterations

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We present highly accurate least-squares (LS) alternatives to the theoretically optimal maximum likelihood (ML) estimator for homographies between two images. Unlike ML, our estimators are non-iterative and yield solutions even in the presence of large noise. By rigorous error analysis, we derive a "hyperaccurate" estimator which is unbiased up to second order noise terms. Then, we introduce a computational simplification, which we call "Taubin approximation", without incurring a loss in accuracy. We experimentally demonstrate that our estimators have accuracy surpassing the traditional LS estimator and comparable to the ML estimator.

1. INTRODUCTION

Computing a homography between two images is the first step in many computer vision applications including panoramic image generation, camera calibration using reference planes, 3-D reconstruction of objects that have planar faces, and detecting obstacles on a planar surface.

The simplest and most widely used method for estimating homographies is the least squares (LS), which minimizes the algebraic distance [3], but it has limited accuracy in the presence of noise. A more accurate solution can be obtained by maximum likelihood (ML), which under independent and isotropic Gaussian noise reduces to minimization of the reprojection error subject to the homography constraint (Gold standard [3]). However, all ML-based estimators are iterative and may not converge for very large noise levels. In addition, an appropriate initial guess is needed to start the iterations. Thus, an accurate algebraic estimator which yields analytical solutions is desired.

Similar circumstances also arise in other problems including fitting a circle/ellipse to a noisy point sequence and estimating fundamental matrices from noisy point correspondences. For such problems, the Taubin estimator [16] has emerged as an accurate algebraic alternative with accuracy comparable to ML [6, 9]. However, the Taubin estimator is defined only for a single constraint equation, while a homography is described by multiple equations. It was only recently that Rangarajan and Papamichalis [14] revealed the existence of a "Taubin-like" estimator for homographies, but they failed to rigorously analyze the accuracy of their estimator.

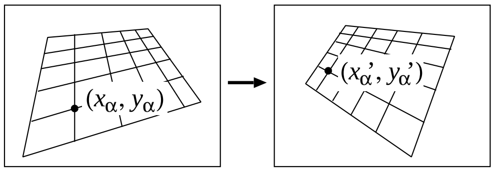

Figure 1: Computing a homography between two images.

Recently, Al-Sharadqah and Chernov [1] and Rangarajan and Kanatani [13] proposed a highly accurate LS estimator for circle fitting based on the perturbation theory of Kanatani [6]. Their "hyperaccurate" estimator improves the accuracy by eliminating the bias of the fitted circle up to second order noise terms. The contribution of this paper is to extend their hyperaccurate circle fitting to homographies. The big difference between circle fitting and homography estimation is that a circle is represented by a quadratic polynomial, while a homography is represented by a set of bilinear polynomials. Consequently, as we show later, the bias due to the nonlinearity of the constraint is smaller for homographies than for circles.

Our task is to compute a 9-D vector $\boldsymbol{h}$ that encodes the homography for given N point correspondences (x_α, y_α) and (x'_α, y'_α), $\alpha = 1, ..., N$. We will show

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$$\boldsymbol{N}=\frac{1}{N}\sum_{\alpha=1}^{N}\begin{pmatrix} x_\alpha^2+y_\alpha'^2+f_0^2 & x_\alpha y_\alpha & f_0x_\alpha & -x_\alpha'y_\alpha' & 0 & 0 & -f_0x_\alpha' & 0 & 0 \\ x_\alpha y_\alpha & y_\alpha^2+y_\alpha'^2+f_0^2 & f_0y_\alpha & 0 & -x_\alpha'y_\alpha' & 0 & 0 & -f_0x_\alpha' & 0 \\ f_0x_\alpha & f_0y_\alpha & f_0^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_\alpha'y_\alpha' & 0 & 0 & x_\alpha^2+x_\alpha'^2+f_0^2 & x_\alpha y_\alpha & f_0x_\alpha & -f_0y_\alpha' & 0 & 0 \\ 0 & -x_\alpha'y_\alpha' & 0 & x_\alpha y_\alpha & y_\alpha^2+x_\alpha'^2+f_0^2 & f_0y_\alpha & 0 & -f_0y_\alpha' & 0 \\ 0 & 0 & 0 & f_0x_\alpha & f_0y_\alpha & f_0^2 & 0 & 0 & 0 \\ -f_0x_\alpha' & 0 & 0 & -f_0y_\alpha' & 0 & 0 & x_\alpha^2+x_\alpha'^2+y_\alpha'^2 & 2x_\alpha y_\alpha & 2f_0x_\alpha \\ 0 & -f_0x_\alpha' & 0 & 0 & -f_0x_\alpha' & 0 & 2x_\alpha y_\alpha & y_\alpha^2+x_\alpha'^2+y_\alpha'^2 & 2f_0y_\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 2f_0x_\alpha & 2f_0y_\alpha & 2f_0^2 \end{pmatrix}, \tag{1}$$

$$\boldsymbol{M}=\frac{1}{N}\sum_{\alpha=1}^{N}\begin{pmatrix} x_\alpha^2(f_0^2+y_\alpha'^2) & x_\alpha y_\alpha(f_0^2+y_\alpha'^2) & f_0x_\alpha(f_0^2+y_\alpha'^2) & -x_\alpha^2x_\alpha'y_\alpha' & -x_\alpha y_\alpha x_\alpha'y_\alpha' & -f_0x_\alpha x_\alpha'y_\alpha' & -f_0x_\alpha^2x_\alpha' & -f_0x_\alpha y_\alpha x_\alpha' & -f_0^2x_\alpha x_\alpha' \\ x_\alpha y_\alpha(f_0^2+y_\alpha'^2) & y_\alpha^2(f_0^2+y_\alpha'^2) & f_0y_\alpha(f_0^2+y_\alpha'^2) & -x_\alpha y_\alpha x_\alpha'y_\alpha' & -y_\alpha^2x_\alpha'y_\alpha' & -f_0y_\alpha x_\alpha'y_\alpha' & -f_0x_\alpha y_\alpha x_\alpha' & -f_0y_\alpha^2x_\alpha' & -f_0^2y_\alpha x_\alpha' \\ f_0x_\alpha(f_0^2+y_\alpha'^2) & f_0y_\alpha(f_0^2+y_\alpha'^2) & f_0^2(f_0^2+y_\alpha'^2) & -f_0x_\alpha x_\alpha'y_\alpha' & -f_0y_\alpha x_\alpha'y_\alpha' & -f_0^2x_\alpha'y_\alpha' & -f_0^2x_\alpha x_\alpha' & -f_0^2y_\alpha x_\alpha' & -f_0^3x_\alpha' \\ -x_\alpha^2x_\alpha'y_\alpha' & -x_\alpha y_\alpha x_\alpha'y_\alpha' & -f_0x_\alpha x_\alpha'y_\alpha' & x_\alpha^2(f_0^2+x_\alpha'^2) & x_\alpha y_\alpha(f_0^2+x_\alpha'^2) & f_0x_\alpha(f_0^2+x_\alpha'^2) & -f_0x_\alpha^2y_\alpha' & -f_0x_\alpha y_\alpha y_\alpha' & -f_0^2x_\alpha y_\alpha' \\ -f_0x_\alpha x_\alpha'y_\alpha' & -f_0y_\alpha x_\alpha'y_\alpha' & -f_0^2x_\alpha'y_\alpha' & x_\alpha y_\alpha(f_0^2+x_\alpha'^2) & y_\alpha^2(f_0^2+x_\alpha'^2) & f_0y_\alpha(f_0^2+x_\alpha'^2) & -f_0x_\alpha y_\alpha y_\alpha' & -f_0y_\alpha^2y_\alpha' & -f_0^3y_\alpha y_\alpha' \\ -f_0x_\alpha x_\alpha'y_\alpha' & -f_0y_\alpha x_\alpha'y_\alpha' & -f_0^2x_\alpha'y_\alpha' & f_0x_\alpha(f_0^2+x_\alpha'^2) & f_0y_\alpha(f_0^2+x_\alpha'^2) & f_0^2(f_0^2+x_\alpha'^2) & -f_0^2x_\alpha y_\alpha' & -f_0^2y_\alpha y_\alpha' & -f_0^3y_\alpha' \\ -f_0x_\alpha^2x_\alpha' & -f_0x_\alpha y_\alpha x_\alpha' & -f_0^2x_\alpha x_\alpha' & -f_0x_\alpha^2y_\alpha' & -f_0x_\alpha y_\alpha y_\alpha' & -f_0^2x_\alpha y_\alpha' & x_\alpha^2(x_\alpha'^2+y_\alpha'^2) & x_\alpha y_\alpha(x_\alpha'^2+y_\alpha'^2) & f_0x_\alpha(x_\alpha'^2+y_\alpha'^2) \\ -f_0x_\alpha y_\alpha x_\alpha' & -f_0y_\alpha^2x_\alpha' & -f_0^2y_\alpha x_\alpha' & -f_0x_\alpha y_\alpha y_\alpha' & -f_0y_\alpha^2y_\alpha' & -f_0^2y_\alpha y_\alpha' & x_\alpha y_\alpha(x_\alpha'^2+y_\alpha'^2) & y_\alpha^2(x_\alpha'^2+y_\alpha'^2) & f_0y_\alpha(x_\alpha'^2+y_\alpha'^2) \\ -f_0^2x_\alpha x_\alpha' & -f_0^2y_\alpha x_\alpha' & -f_0^3x_\alpha' & -f_0^2x_\alpha y_\alpha' & -f_0^2y_\alpha y_\alpha' & -f_0^3y_\alpha' & f_0x_\alpha(x_\alpha'^2+y_\alpha'^2) & f_0y_\alpha(x_\alpha'^2+y_\alpha'^2) & f_0^2(x_\alpha'^2+y_\alpha'^2) \end{pmatrix}. \tag{2}$$

that an accurate value of $\boldsymbol{h}$ is obtained by solving the generalized eigenvalue problem

$$\boldsymbol{N}\boldsymbol{h} = \mu\boldsymbol{M}\boldsymbol{h}, \tag{3}$$

for the generalized eigenvalue μ with the *largest absolute value*, where $\boldsymbol{N}$ and $\boldsymbol{M}$ are 9×9 symmetric matrices in Eqs. (4) and (5), respectively, and f_0 is a scale constant to be explained later. The key to the accuracy improvement is the *choice of the matrix* $\boldsymbol{N}$. We show in Sec. 7 how to choose $\boldsymbol{N}$ to achieve "hyperaccuracy". Eq. (4) is its "Taubin approximation". In Sec. 8, we experimentally demonstrate that our hyperaccurate estimator and its Taubin approximation have accuracy surpassing the traditional LS estimator and comparable to the ML estimator.

2. HOMOGRAPHY

A *homography* is an image mapping in the form

$$x'=f_0\frac{h_{11}x+h_{12}y+h_{13}f_0}{h_{31}x+h_{32}y+h_{33}f_0},\ y'=f_0\frac{h_{21}x+h_{22}y+h_{23}f_0}{h_{31}x+h_{32}y+h_{33}f_0}, \tag{4}$$

where f_0 is a scale constant chosen so that all terms have nearly equal magnitude; its absence would incur serious accuracy loss in finite precision numerical computation. If we define 3-D homogeneous coordinate vectors

$$\boldsymbol{x} = \begin{pmatrix} x/f_0 \\ y/f_0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{x}' = \begin{pmatrix} x'/f_0 \\ y'/f_0 \\ 1 \end{pmatrix}, \tag{5}$$

Eqs. (4) can be equivalently written as

$$\boldsymbol{x}' \cong \boldsymbol{H}\boldsymbol{x}, \tag{6}$$

where $\boldsymbol{H}$ is a 3×3 matrix with elements h_{ij}, and $\cong$ denotes equality up to a nonzero constant. Equation (6) states that vectors $\boldsymbol{x}'$ and $\boldsymbol{H}\boldsymbol{x}$ are parallel, so we can equivalently write this as

$$\boldsymbol{x}' \times \boldsymbol{H}\boldsymbol{x} = \boldsymbol{0}. \tag{7}$$

If we define 9-D vectors

$$\begin{aligned} \boldsymbol{\xi}^{(1)} &= \begin{pmatrix} 0 & 0 & 0 & -f_0x & -f_0y & -f_0^2 & xy' & yy' & f_0y' \end{pmatrix}^\top, \\ \boldsymbol{\xi}^{(2)} &= \begin{pmatrix} f_0x & f_0y & f_0^2 & 0 & 0 & 0 & -xx' & -yx' & -f_0x' \end{pmatrix}^\top, \\ \boldsymbol{\xi}^{(3)} &= \begin{pmatrix} -xy' & -yy' & -f_0y' & xx' & yx' & f_0x' & 0 & 0 & 0 \end{pmatrix}^\top, \end{aligned} \tag{8}$$

the three components of Eq. (7) are, after multiplication of f_0^2,

$$(\boldsymbol{\xi}^{(1)}, \boldsymbol{h}) = 0, \quad (\boldsymbol{\xi}^{(2)}, \boldsymbol{h}) = 0, \quad (\boldsymbol{\xi}^{(3)}, \boldsymbol{h}) = 0, \tag{9}$$

where $\boldsymbol{h}$ is a 9-D vector with components h_{11}, h_{12}, ..., h_{99}. Throughout this paper, we denote the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by $(\boldsymbol{a}, \boldsymbol{b})$.

3. ALGEBRAIC SOLUTION

Let $\boldsymbol{\xi}_\alpha^{(k)}$ be the value of $\boldsymbol{\xi}^{(k)}$, $k = 1, 2, 3$, for $\{(x_\alpha, y_\alpha), (x'_\alpha, y'_\alpha)\}$, $\alpha = 1, ..., N$. Our task is to estimate an $\boldsymbol{h}$ such that $(\boldsymbol{\xi}_\alpha^{(k)}, \boldsymbol{h}) \approx 0$, $k = 1, 2, 3$, α =1, ..., N. We minimize the *algebraic distance*

$$J=\frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}(\boldsymbol{\xi}_\alpha^{(k)}, \boldsymbol{h})^2 = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}\boldsymbol{h}^\top\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{\xi}_\alpha^{(k)\top}\boldsymbol{h} =(\boldsymbol{h}, \boldsymbol{M}\boldsymbol{h}), \quad (10)$$

where we define the 9×9 matrix $\boldsymbol{M}$ by

$$\boldsymbol{M} = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{\xi}_\alpha^{(k)\top}. \quad (11)$$

Evidently, we need scale normalization on $\boldsymbol{h}$; otherwise, Eq. (10) is minimized by $\boldsymbol{h} = \boldsymbol{0}$. A frequently used convention is $h_{33} = 1$, and $\sum_{i,j=1}^{3} h_{ij}^2 = 1$ is also used, but many other normalizations are conceivable. The important fact is that *the solution depends on the normalization*. Al-Sharadqah and Chernov [1] and Rangarajan and Kanatani [13] exploited this freedom for circle fitting and "optimized" the normalization so that the solution has the highest accuracy. In this paper, we do this for homography estimation.

As Al-Sharadqah and Chernov [1] and Rangarajan and Kanatani [13], we consider the class of normalizations in the form

$$(\boldsymbol{h}, \boldsymbol{N}\boldsymbol{h}) = \text{constant}, \quad (12)$$

for some 9×9 symmetric matrix $\boldsymbol{N}$. If we let $\boldsymbol{N}$ = $\boldsymbol{I}$ (unit matrix), we require that $\|\boldsymbol{h}\|$ = constant. In the following, we call this *standard least-squares*, or simply *LS*. If $\boldsymbol{N}$ is positive definite, Eq. (12) is positive, so no generality is lost if we set it to 1. As Al-Sharadqah and Chernov [1] and Rangarajan and Kanatani [13], however, we do not restrict $\boldsymbol{N}$ to be positive definite. As is well known, the solution $\boldsymbol{h}$ that minimizes Eq. (10) subject to Eq. (12) is obtained by solving the generalized eigenvalue problem

$$\boldsymbol{M}\boldsymbol{h} = \lambda\boldsymbol{N}\boldsymbol{h}. \quad (13)$$

The solution $\boldsymbol{h}$ has scale indeterminacy, so we normalize it to $\|\boldsymbol{h}\| = 1$ rather than Eq. (12). Our task is to select an appropriate $\boldsymbol{N}$ that gives rise to the best solution $\boldsymbol{h}$, applying the perturbation theory of Kanatani [6] to Eq. (13).

4. ERROR ANALYSIS

We assume that the observed positions (x_α, y_α) and (x'_α, y'_α) are perturbations of their true values $(\bar{x}_\alpha, \bar{y}_\alpha)$ and $(\bar{x}'_\alpha, \bar{y}'_\alpha)$ by independent Gaussian noise Δx_α, Δy_α, $\Delta x'_\alpha$, and $\Delta y'_\alpha$ of expectation 0 and standard deviation σ (pixels). The error terms $\Delta\boldsymbol{\xi}_\alpha^{(k)}$ of $\boldsymbol{\xi}_\alpha^{(k)}$ are

$$\Delta\boldsymbol{\xi}_\alpha^{(k)} = \Delta_1\boldsymbol{\xi}_\alpha^{(k)} + \Delta_2\boldsymbol{\xi}_\alpha^{(k)}, \quad (14)$$

where Δ_1 and Δ_2 denote, respectively, terms of orders 1 and 2 in Δx_α, Δy_α, $\Delta x'_\alpha$, and $\Delta y'_\alpha$. The first order terms $\Delta_1\boldsymbol{\xi}_\alpha^{(k)}$ are written as

$$\Delta_1\boldsymbol{\xi}_\alpha^{(k)} = \boldsymbol{T}_\alpha^{(k)}\begin{pmatrix}\Delta x_\alpha \\ \Delta y_\alpha \\ \Delta x'_\alpha \\ \Delta y'_\alpha\end{pmatrix}, \quad (15)$$

where the 9×4 Jacobi matrices $\boldsymbol{T}_\alpha^{(k)}$ of $\boldsymbol{\xi}_\alpha^{(k)}$ are

$$\boldsymbol{T}_\alpha^{(1)}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -f_0 & 0 & 0 & 0 \\ 0 & -f_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{y}'_\alpha & 0 & 0 & \bar{x}_\alpha \\ 0 & \bar{y}'_\alpha & 0 & \bar{y}_\alpha \\ 0 & 0 & 0 & f_0 \end{pmatrix}, \quad \boldsymbol{T}_\alpha^{(2)}=\begin{pmatrix} f_0 & 0 & 0 & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\bar{x}'_\alpha & 0 & -\bar{x}_\alpha & 0 \\ 0 & -\bar{x}'_\alpha & -\bar{y}_\alpha & 0 \\ 0 & 0 & -f_0 & 0 \end{pmatrix},$$

$$\boldsymbol{T}_\alpha^{(3)}=\begin{pmatrix} -\bar{y}'_\alpha & 0 & 0 & -\bar{x}_\alpha \\ 0 & -\bar{y}'_\alpha & 0 & -\bar{y}_\alpha \\ 0 & 0 & 0 & -f_0 \\ \bar{x}'_\alpha & 0 & \bar{x}_\alpha & 0 \\ 0 & \bar{x}'_\alpha & \bar{y}_\alpha & 0 \\ 0 & 0 & f_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (16)$$

We define the covariance matrices of $\boldsymbol{\xi}_\alpha^{(k)}$ by

$$\begin{aligned} &E[\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(l)}] \\ &=\boldsymbol{T}_\alpha^{(k)}E\left[\begin{pmatrix} \Delta x_\alpha^2 & \Delta x_\alpha\Delta y_\alpha & \Delta x_\alpha\Delta x'_\alpha & \Delta x_\alpha\Delta y'_\alpha \\ \Delta y_\alpha\Delta x_\alpha & \Delta y_\alpha^2 & \Delta y_\alpha\Delta x'_\alpha & \Delta y_\alpha\Delta y'_\alpha \\ \Delta x'_\alpha\Delta x_\alpha & \Delta x'_\alpha\Delta y_\alpha & \Delta x'^2_\alpha & \Delta x'_\alpha\Delta y'_\alpha \\ \Delta y'_\alpha\Delta x_\alpha & \Delta y'_\alpha\Delta y_\alpha & \Delta y'_\alpha\Delta x'_\alpha & \Delta y'^2_\alpha \end{pmatrix}\right]\boldsymbol{T}_\alpha^{(l)\top} \\ &=\boldsymbol{T}_\alpha^{(k)}(\sigma^2\boldsymbol{I})\boldsymbol{T}_\alpha^{(l)\top} = \sigma^2\boldsymbol{T}_\alpha^{(k)}\boldsymbol{T}_\alpha^{(l)\top} = \sigma^2 V_0^{(kl)}[\boldsymbol{\xi}_\alpha], \end{aligned} \quad (17)$$

where $E[\,\cdot\,]$ denotes expectation and we put

$$V_0^{(kl)}[\boldsymbol{\xi}_\alpha] \equiv \boldsymbol{T}_\alpha^{(k)}\boldsymbol{T}_\alpha^{(l)\top} \quad (18)$$

The second order error terms $\Delta_2\boldsymbol{\xi}_\alpha^{(k)}$ are given by

$$\begin{aligned} \Delta_2\boldsymbol{\xi}_\alpha^{(1)} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \Delta x_\alpha\Delta y'_\alpha & \Delta y_\alpha\Delta y'_\alpha & 0 \end{pmatrix}^\top, \\ \Delta_2\boldsymbol{\xi}_\alpha^{(2)} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\Delta x'_\alpha\Delta x_\alpha & -\Delta x'_\alpha\Delta y_\alpha & 0 \end{pmatrix}^\top, \\ \Delta_2\boldsymbol{\xi}_\alpha^{(3)} &= \begin{pmatrix} -\Delta y'_\alpha\Delta x_\alpha & -\Delta y'_\alpha\Delta y_\alpha & 0 & \Delta x'_\alpha\Delta x_\alpha & \Delta x'_\alpha\Delta y_\alpha & 0 & 0 & 0 & 0 \end{pmatrix}^\top. \end{aligned} \quad (19)$$

5. PERTURBATION ANALYSIS

Substituting Eq. (14) into Eq. (11), we obtain

$$\boldsymbol{M}=\frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}(\bar{\boldsymbol{\xi}}_\alpha^{(k)}+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}+\Delta_2\boldsymbol{\xi}_\alpha^{(k)})(\bar{\boldsymbol{\xi}}_\alpha^{(k)}+\Delta_1\boldsymbol{\xi}_\alpha^{(k)} +\Delta_2\boldsymbol{\xi}_\alpha^{(k)})^\top = \bar{\boldsymbol{M}} + \Delta_1\boldsymbol{M} + \Delta_2\boldsymbol{M} + \cdots, \quad (20)$$

where $\bar{\boldsymbol{M}}$ is the noise-free term, and $\cdots$ denotes terms of order 3 or higher in noise. The first and second order terms $\Delta_1\boldsymbol{M}$ and $\Delta_2\boldsymbol{M}$ are

$$\Delta_1\boldsymbol{M} = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}(\bar{\boldsymbol{\xi}}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top} + \Delta_1\boldsymbol{\xi}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(k)\top}), \tag{21}$$

$$\Delta_2\boldsymbol{M}=\frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}(\bar{\boldsymbol{\xi}}_\alpha^{(k)}\Delta_2\boldsymbol{\xi}_\alpha^{(k)\top} + \Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top} + \Delta_2\boldsymbol{\xi}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(k)\top}). \tag{22}$$

Accordingly, we expand $\boldsymbol{h}$ and λ in Eq. (13) in the form

$$\boldsymbol{h}=\bar{\boldsymbol{h}}+\Delta_1\boldsymbol{h}+\Delta_2\boldsymbol{h}+\cdots,\ \lambda=\bar{\lambda}+\Delta_1\lambda+\Delta_2\lambda+\cdots. \tag{23}$$

Substituting Eqs. (20) and (23) into Eq. (13), we have

$$(\bar{\boldsymbol{M}}+\Delta_1\boldsymbol{M}+\Delta_2\boldsymbol{M}+\cdots)(\bar{\boldsymbol{h}}+\Delta_1\boldsymbol{h}+\Delta_2\boldsymbol{h}+\cdots) =(\bar{\lambda}+\Delta_1\lambda+\Delta_2\lambda+\cdots)\boldsymbol{N}(\bar{\boldsymbol{h}}+\Delta_1\boldsymbol{h}+\Delta_2\boldsymbol{h}+\cdots). \tag{24}$$

Expanding both sides and equating terms of equal degrees in noise, we obtain

$$\bar{\boldsymbol{M}}\bar{\boldsymbol{h}} = \bar{\lambda}\boldsymbol{N}\bar{\boldsymbol{h}}, \tag{25}$$

$$\bar{\boldsymbol{M}}\Delta_1\boldsymbol{h} + \Delta_1\boldsymbol{M}\bar{\boldsymbol{h}} = \bar{\lambda}\boldsymbol{N}\Delta_1\boldsymbol{h} + \Delta_1\lambda\boldsymbol{N}\bar{\boldsymbol{h}}, \tag{26}$$

$$\bar{\boldsymbol{M}}\Delta_2\boldsymbol{h} + \Delta_1\boldsymbol{M}\Delta_1\boldsymbol{h} + \Delta_2\boldsymbol{M}\bar{\boldsymbol{h}} =\bar{\lambda}\boldsymbol{N}\Delta_2\boldsymbol{h} + \Delta_1\lambda\boldsymbol{N}\Delta_1\boldsymbol{h} + \Delta_2\lambda\boldsymbol{N}\bar{\boldsymbol{h}}. \tag{27}$$

Since $(\bar{\boldsymbol{\xi}}_\alpha^{(k)}, \bar{\boldsymbol{h}}) = 0$ for noise-free data, we have $\bar{\boldsymbol{M}}\bar{\boldsymbol{h}} = \boldsymbol{0}$ and hence $\bar{\lambda} = 0$ from Eq. (25). We see from Eq. (21) that $(\bar{\boldsymbol{h}}, \Delta_1\boldsymbol{M}\bar{\boldsymbol{h}}) = 0$. Computing the inner product of $\bar{\boldsymbol{h}}$ and Eq. (26), we see that $\Delta_1\lambda = 0$. Multiplying Eq. (26) by the pseudoinverse $\bar{\boldsymbol{M}}^-$ from left, we obtain

$$\Delta_1\boldsymbol{h} = -\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}\bar{\boldsymbol{h}}, \tag{28}$$

where we have noted that $\bar{\boldsymbol{h}}$ is a null vector of $\bar{\boldsymbol{M}}$ and hence $\boldsymbol{P}_{\bar{\boldsymbol{h}}} \equiv \bar{\boldsymbol{M}}^-\bar{\boldsymbol{M}}$ is the projection matrix in the direction of $\bar{\boldsymbol{h}}$. We have also noted that $\Delta_1\boldsymbol{h}$ is orthogonal to $\bar{\boldsymbol{h}}$ and hence $\boldsymbol{P}_{\bar{\boldsymbol{h}}}\Delta_1\boldsymbol{h} = \Delta_1\boldsymbol{h}$; this is easily seen by picking out first order terms from $\|\bar{\boldsymbol{h}} + \Delta_1\boldsymbol{h} + \Delta_2\boldsymbol{h} + \cdots\|^2 = 1$.

Substituting Eq. (28) into Eq. (27), we see that $\Delta_2\lambda$ is

$$\Delta_2\lambda=\frac{(\bar{\boldsymbol{h}}, \Delta_2\boldsymbol{M}\bar{\boldsymbol{h}})-(\bar{\boldsymbol{h}}, \Delta_1\boldsymbol{M}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}\bar{\boldsymbol{h}})}{(\bar{\boldsymbol{h}}, \boldsymbol{N}\bar{\boldsymbol{h}})}=\frac{(\bar{\boldsymbol{h}}, \boldsymbol{T}\bar{\boldsymbol{h}})}{(\bar{\boldsymbol{h}}, \boldsymbol{N}\bar{\boldsymbol{h}})}, \tag{29}$$

where we define

$$\boldsymbol{T} \equiv \Delta_2\boldsymbol{M} - \Delta_1\boldsymbol{M}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M} \tag{30}$$

Next, we consider the second order error $\Delta_2\boldsymbol{h}$. Since the magnitude of $\boldsymbol{h}$ is fixed to 1, we are only interested in the component orthogonal to $\bar{\boldsymbol{h}}$, which we denote by

$$\Delta_2\boldsymbol{h}^\perp = \boldsymbol{P}_{\bar{\boldsymbol{h}}}\Delta_2\boldsymbol{h}\ (= \bar{\boldsymbol{M}}^-\bar{\boldsymbol{M}}\Delta_2\boldsymbol{h}). \tag{31}$$

Multiplying Eq. (27) by $\bar{\boldsymbol{M}}^-$ from left and substituting Eq. (28), we obtain

$$\Delta_2\boldsymbol{h}^\perp=\Delta_2\lambda\bar{\boldsymbol{M}}^-\boldsymbol{N}\bar{\boldsymbol{h}}+\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}\bar{\boldsymbol{h}} -\bar{\boldsymbol{M}}^-\Delta_2\boldsymbol{M}\bar{\boldsymbol{h}} =\frac{(\bar{\boldsymbol{h}}, \boldsymbol{T}\bar{\boldsymbol{h}})}{(\bar{\boldsymbol{h}}, \boldsymbol{N}\bar{\boldsymbol{h}})}\bar{\boldsymbol{M}}^-\boldsymbol{N}\bar{\boldsymbol{h}} - \bar{\boldsymbol{M}}^-\boldsymbol{T}\bar{\boldsymbol{h}}. \tag{32}$$

6. COVARIANCE AND BIAS

From Eq. (28), the leading term of the covariance matrix of the solution $\boldsymbol{h}$ is given by

$$\begin{aligned} V[\boldsymbol{h}] &= E[\Delta_1\boldsymbol{h}\Delta_1\boldsymbol{h}^\top] \\ &=\frac{1}{N^2}\bar{\boldsymbol{M}}^- E[(\Delta_1\boldsymbol{M}\boldsymbol{h})(\Delta_1\boldsymbol{M}\boldsymbol{h})^\top]\bar{\boldsymbol{M}}^- \\ &=\frac{1}{N^2}\bar{\boldsymbol{M}}^- E\Big[\sum_{\alpha=1}^{N}\sum_{k=1}^{3}(\Delta\boldsymbol{\xi}_\alpha^{(k)}, \boldsymbol{h})\bar{\boldsymbol{\xi}}_\alpha^{(k)} \sum_{\beta=1}^{N}\sum_{l=1}^{3}(\Delta\boldsymbol{\xi}_\beta^{(l)}, \boldsymbol{h})\bar{\boldsymbol{\xi}}_\beta^{(l)\top}\Big]\bar{\boldsymbol{M}}^- \\ &=\frac{1}{N^2}\bar{\boldsymbol{M}}^-\sum_{\alpha,\beta=1}^{N}\sum_{k,l=1}^{3}(\boldsymbol{h}, E[\Delta\boldsymbol{\xi}_\alpha^{(k)}\Delta\boldsymbol{\xi}_\beta^{(l)\top}]\boldsymbol{h})\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top}\bar{\boldsymbol{M}}^- \\ &=\frac{\sigma^2}{N^2}\bar{\boldsymbol{M}}^-\Big(\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}(\boldsymbol{h}, V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\boldsymbol{h})\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}\Big)\bar{\boldsymbol{M}}^- \\ &=\frac{\sigma^2}{N}\bar{\boldsymbol{M}}^-\bar{\boldsymbol{M}}'\bar{\boldsymbol{M}}^-, \end{aligned} \tag{33}$$

where we define

$$\bar{\boldsymbol{M}}' = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}(\bar{\boldsymbol{h}}, V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\boldsymbol{h})\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}. \tag{34}$$

In the above derivation, we have used our assumption that the noise in $\boldsymbol{\xi}_\alpha$ is independent for each α and that $E[\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}] = \delta_{\alpha\beta}\sigma^2V_0^{(kl)}[\boldsymbol{\xi}_\alpha]$, where $\delta_{\alpha\beta}$ is the Kronecker delta. The important observation is that $V[\boldsymbol{h}]$ does not depend on the normalization weight $\boldsymbol{N}$. Thus, *all algebraic methods have the same covariance matrix in the leading order*, so we are unable to reduce the covariance of $\boldsymbol{h}$ by adjusting $\boldsymbol{N}$. This leads us to focus on the bias.

Since $E[\Delta_1\boldsymbol{h}] = \boldsymbol{0}$, the leading bias is $E[\Delta_2\boldsymbol{h}^\perp]$. To evaluate this, we first compute the expectation $E[\boldsymbol{T}]$ of $\boldsymbol{T}$ in Eq. (30). From Eq. (22), $E[\Delta_2\boldsymbol{M}]$ becomes

$$E[\Delta_2\boldsymbol{M}]=\frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}\Big(\bar{\boldsymbol{\xi}}_\alpha^{(k)}E[\Delta_2\boldsymbol{\xi}_\alpha^{(k)}]^\top$$

$$+E[\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top}] + E[\Delta_2\boldsymbol{\xi}_\alpha^{(k)}]\bar{\boldsymbol{\xi}}_\alpha^{(k)\top}\Big)$$

$$=\frac{\sigma^2}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}V_0^{(kk)}[\boldsymbol{\xi}_\alpha] = \sigma^2\boldsymbol{N}_{\rm T}, \qquad (35)$$

where we put

$$\boldsymbol{N}_{\rm T} = \frac{1}{N}\sum_{k=1}^{3}V_0^{(kk)}[\boldsymbol{\xi}_\alpha]. \qquad (36)$$

The term $E[\Delta_1\boldsymbol{M}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}]$ is evaluated as follows (see Appendix A for the derivation):

$$E[\Delta_1\boldsymbol{M}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}]$$
$$=\frac{\sigma^2}{N^2}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}\Big({\rm tr}[\bar{\boldsymbol{M}}^-V_0^{(kl)}[\boldsymbol{\xi}_\alpha]]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}$$
$$+(\bar{\boldsymbol{\xi}}_\alpha^{(k)}, \bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(l)})V_0^{(kl)}[\boldsymbol{\xi}_\alpha]$$
$$+2\mathcal{S}[V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}]\Big). \qquad (37)$$

Here, ${\rm tr}[\,\cdot\,]$ denotes the trace, and $\mathcal{S}[\,\cdot\,]$ means symmetrization ($\mathcal{S}[\boldsymbol{A}] = (\boldsymbol{A} + \boldsymbol{A}^\top)/2$). From Eqs. (35) and (37), the expectation of $\boldsymbol{T}$ in Eq. (30) is written as

$$E[\boldsymbol{T}]=\sigma^2\Big(\boldsymbol{N}_{\rm T}-\frac{1}{N^2}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}\Big({\rm tr}[\bar{\boldsymbol{M}}^-V_0^{(kl)}[\boldsymbol{\xi}_\alpha]]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}$$
$$+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(l)})V_0^{(kl)}[\boldsymbol{\xi}_\alpha]+2\mathcal{S}[V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}]\Big). \qquad (38)$$

Hence, the expectation of $\Delta_2\boldsymbol{h}^\perp$ in Eq. (32) is

$$E[\Delta_2\boldsymbol{h}^\perp] = \bar{\boldsymbol{M}}^-\Big(\frac{(\bar{\boldsymbol{h}}, E[\boldsymbol{T}]\bar{\boldsymbol{h}})}{(\bar{\boldsymbol{h}}, \boldsymbol{N}\bar{\boldsymbol{h}})}\boldsymbol{N}\bar{\boldsymbol{h}} - E[\boldsymbol{T}]\bar{\boldsymbol{h}}\Big). \qquad (39)$$

7. HIGH ACCURACY ALGEBRAIC METHOD

Careful observation of Eqs. (38) and (39) reveals that if we choose $\boldsymbol{N}$ to be

$$\boldsymbol{N} = \boldsymbol{N}_{\rm T} - \frac{1}{N^2}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}\Big({\rm tr}[\bar{\boldsymbol{M}}^-V_0^{(kl)}[\boldsymbol{\xi}_\alpha]]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(k)\top}$$
$$+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(l)})V_0^{(kl)}[\boldsymbol{\xi}_\alpha]+2\mathcal{S}[V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}], \qquad (40)$$

then $E[\boldsymbol{T}] = \sigma^2\boldsymbol{N}$ from Eq. (38), and hence Eq. (39) becomes

$$E[\Delta_2\boldsymbol{h}^\perp] = \sigma^2\bar{\boldsymbol{M}}^-\Big(\frac{(\bar{\boldsymbol{h}}, \boldsymbol{N}\bar{\boldsymbol{h}})}{(\bar{\boldsymbol{h}}, \boldsymbol{N}\bar{\boldsymbol{h}})}\boldsymbol{N} - \boldsymbol{N}\Big)\bar{\boldsymbol{h}} = \boldsymbol{0}. \qquad (41)$$

Since Eq. (40) contains the true values $\bar{\boldsymbol{\xi}}_\alpha^{(k)}$ and $\bar{\boldsymbol{M}}$, we evaluate them by replacing the true values $(\bar{x}_\alpha, \bar{y}_\alpha)$ and $(\bar{x}'_\alpha, \bar{y}'_\alpha)$ in their definitions by the observations (x_α, y_α) and (x'_α, y'_α), respectively. This does not affect the result, because expectations of odd-order error terms vanish and hence the error in Eq. (41) is at most $O(\sigma^4)$. Thus, the second order bias is *exactly* 0. After Al-Sharadqah and Chernov [1] and Rangarajan and Kanatani [13], we call this solution "hyperaccurate".

Standard linear algebra routines for solving generalized eigenvalue problems in the form of Eq. (13) assume that $\boldsymbol{N}$ is positive definite, but the matrix $\boldsymbol{N}$ in Eq. (40) is not guaranteed to be positive definite. However, this poses no problem, as Eq. (13) can be rewritten as

$$\boldsymbol{N}\boldsymbol{h} = (1/\lambda)\boldsymbol{M}\boldsymbol{h}. \qquad (42)$$

Since the matrix $\boldsymbol{M}$ in Eq. (11) is positive definite for noisy data, we can solve Eq. (42) instead of Eq. (13). If the smallest eigenvalue of $\boldsymbol{M}$ happens to be 0, it indicates that the data are all exact; any method, e.g., LS, gives an exact solution. The perturbation analysis of Kanatani [6] is based on the assumption that $\lambda \approx 0$, so we compute the unit generalized eigenvector for λ with the smallest absolute value.

The second term on the right-hand side of Eq. (40) is $O(1/N)$ and hence is expected to be small when N is large. We call the omission of this term *Taubin approximation*. Letting $\boldsymbol{N} = \boldsymbol{N}_{\rm T}$ and putting $\mu = 1/\lambda$, we obtain Eq. (3).

8. EXPERIMENTS

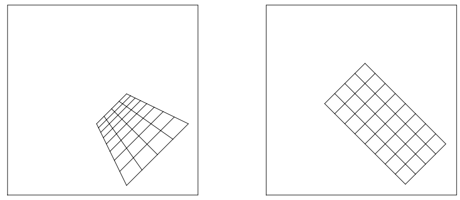


Figure 2: Simulated images of a planar surface.

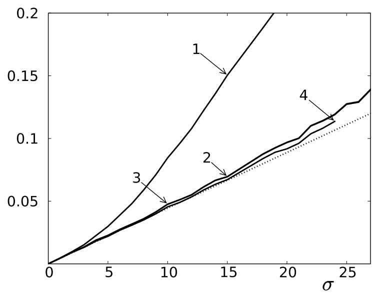


Figure 3: RMS error of the computed homography vs. the standard deviation σ of the added noise. 1. LS. 2. Hyperaccuracy method. 3. Taubin approximation. 4. ML. The halfway termination of the ML plot means that it did not converge beyond that noise level. The dotted line indicates the KCR lower bound.

Fig. 2 shows simulated images of a planar surface viewed from different directions. The image size is assumed to be 800 × 800 pixels with focal length f

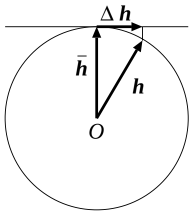


Figure 4: The error component $\Delta \boldsymbol{h}$ of the computed value $\hat{\boldsymbol{h}}$ orthogonal to the true value $\bar{\boldsymbol{h}}$.

= 600 pixels. We added independent Gaussian noise of mean 0 and standard deviation σ (pixels) to the x and y coordinates of the grid points and computed the homography $\boldsymbol{h}$ from them. We measured the error of the computation by

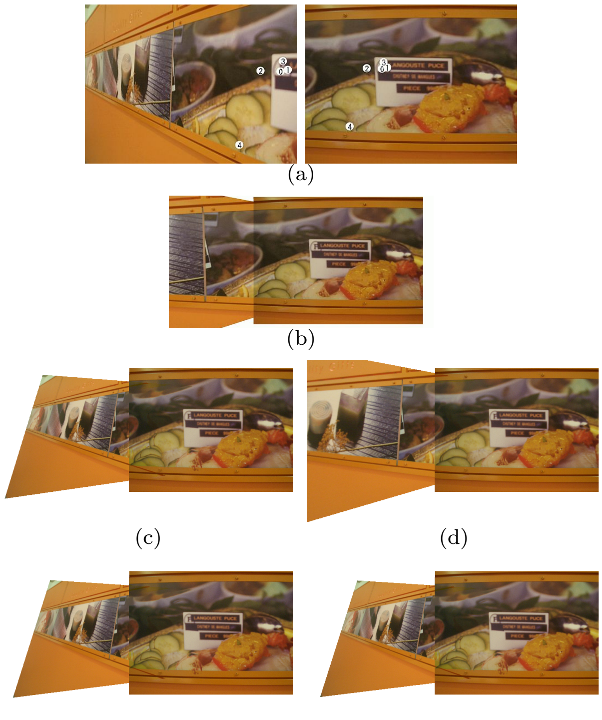
$$\Delta \boldsymbol{h} = \boldsymbol{P}_{\bar{\boldsymbol{h}}} \hat{\boldsymbol{h}}, \qquad \boldsymbol{P}_{\bar{\boldsymbol{h}}} \equiv \boldsymbol{I} - \bar{\boldsymbol{h}} \bar{\boldsymbol{h}}^\top, \tag{43}$$

where $\hat{\boldsymbol{h}}$ and $\bar{\boldsymbol{h}}$ are the computed and the true values, respectively, and $\boldsymbol{P}_{\bar{\boldsymbol{h}}}$ is the projection matrix onto the direction orthogonal to $\bar{\boldsymbol{h}}$; we are only interested in the error of $\hat{\boldsymbol{h}}$, which is a unit vector, orthogonal to $\bar{\boldsymbol{h}}$ (Fig. 4). For each σ, we evaluated the root-mean-square (RMS) error E of $\Delta \boldsymbol{h}$ over 1000 independent trials,

$$E = \sqrt{\frac{1}{1000} \sum_{a=1}^{1000} \|\Delta \boldsymbol{h}^{(a)}\|^2} \tag{44}$$

where the superscript (a) indicates the ath value. Figure 3 plots, for σ on the horizontal axis, the RMS error E of different methods: 1. least squares (LS), 2. our hyperaccuracy method using Eq. (40), 3. Taubin approximation, and 4. ML, for which we derived a new method by extending the FNS principle of Chojnacki [2] (see Appendix B). The dotted line shows the KCR lower bound [4, 5, 6]. The interrupted plot of ML means that the iterations failed to converge for σ larger than that. We can see from Fig. 3 that LS performs very poorly. In contrast, our hyperaccuracy method and its Taubin approximation almost compare with ML. Being algebraic, they do not fail for whatever noise. Our methods and ML (if it converges) almost achieve the theoretical accuracy bound.

Figure 5(a) shows images of a planar scene taken from different directions. From these, we computed the homography from the left image to the right. Figure 5(b) is the generated panoramic image by ML, manually choosing 21 corresponding points. We used `cvWarpPerspective`[1] of OpenCV for image generation. The five marks in Figure 5(a) are corresponding points extracted by `autopano-sift`[2]. Figure 5(c) is the panoramic image computed from them by LS, and Fig. 5(d),(e), and (f) are the corresponding results by ML, our hyperaccuracy method, and its Taubin approximation, respectively. Regarding the homography $\bar{\boldsymbol{h}}$ for Figure 5(b) by ML as the ground truth, we computed the error $\Delta \boldsymbol{h}$ in Eq. (43) and listed at the bottom of Fig. 5 the values of $\|\Delta \boldsymbol{h}\|$ for the four methods. In this example, LS is relatively good and ML is the best, while our method and its Taubin approximation exhibit similar accuracy.



(c)	(d)	(e)	(f)
0.399	0.089	0.433	0.439

Figure 5: Panoramic image generation. (a) Input images of a planar surface and the corresponding points extracted by SIFT. (b) Ground truth. (c) LS. (f) ML. (d) Hyper accuracy method. (e) Taubin approximation. The table lists numerical errors of the computed homography.

Fig. 6 shows another example. The number of corresponding points is seven, and Fig. 6(b) is created by manually choosing 26 corresponding points. For this example, that LS is the worst, while ML, our method, and its Taubin approximation exhibit similar accuracy.

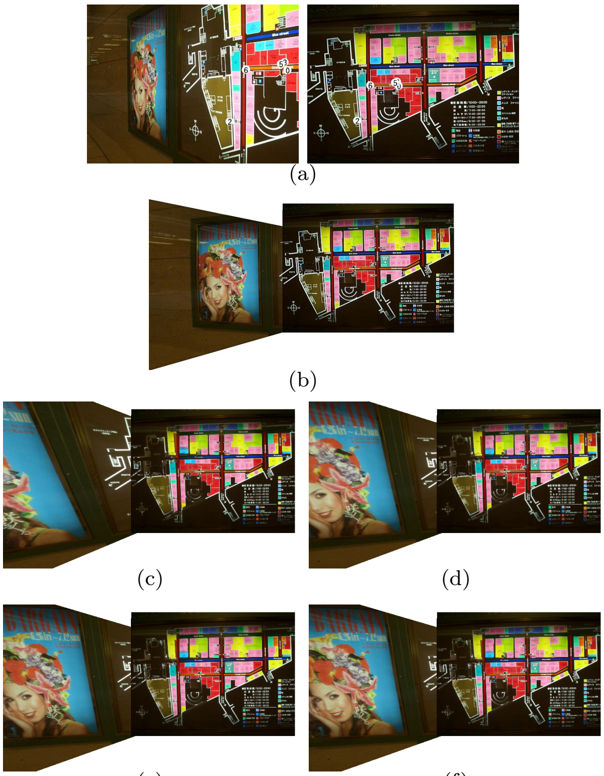
9. CONCLUSIONS

We presented highly accurate LS alternatives to the theoretically optimal ML estimator for homographies[3]. Unlike ML, our hyperaccurate estimator and its Taubin approximation are non-iterative and yield solutions even in the presence of large noise. This is made possible by adjusting the normalization weight matrix $\boldsymbol{N}$ so as to eliminate the bias of the solution up to second order noise terms. By simulation, we

[1] `http://opencv.jp/opencv-1.0.0/document/opencvref_cv_sampling.html#decl_cvWarpPerspective`

[2] `http://user.cs.tu-berlin.de/ nowozin/autopano-sift/`

[3] The code is available at: `http://www....` (our site).



(b)	(c)	(d)	(e)
0.378	0.246	0.210	0.204

Figure 6: Panoramic image generation. (a) Input images of a planar surface and the corresponding points extracted by SIFT. (b) Ground truth. (c) LS. (d) ML. (e) Hyper-accuracy method. (f) Taubin approximation. The table lists numerical errors of the computed homography.

demonstrated that our estimators outperform the traditional LS and has accuracy comparable to ML. We demonstrated that our estimators could be used to create better panoramic images.

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APPENDIX

A. Derivation of Eq. (37)

The term $E[\Delta_1 \boldsymbol{M}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}]$ is computed as follows:

$$
\begin{aligned}
&E[\Delta_1 \boldsymbol{M}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{M}] \\
=&E[\frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{3}\Big(\bar{\boldsymbol{\xi}}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top}+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(k)\top}\Big)\bar{\boldsymbol{M}}^- \\
&\frac{1}{N}\sum_{\beta=1}^{N}\sum_{l=1}^{3}\Big(\bar{\boldsymbol{\xi}}_\beta^{(l)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}+\Delta_1\boldsymbol{\xi}_\beta^{(l)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top}\Big)] \\
=&\frac{1}{N^2}\sum_{\alpha,\beta=1}^{N}\sum_{k,l=1}^{3}E[(\bar{\boldsymbol{\xi}}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top}+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(k)\top})\bar{\boldsymbol{M}}^- \\
&(\bar{\boldsymbol{\xi}}_\beta^{(l)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}+\Delta_1\boldsymbol{\xi}_\beta^{(l)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top})] \\
=&\frac{1}{N^2}\sum_{\alpha,\beta=1}^{N}\sum_{k,l=1}^{3}E[\bar{\boldsymbol{\xi}}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top}\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top} \\
&+\bar{\boldsymbol{\xi}}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{\xi}_\beta^{(l)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top} \\
&+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(k)\top}\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(k)\top}\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{\xi}_\beta^{(l)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top}] \\
=&\frac{1}{N^2}\sum_{\alpha,\beta=1}^N\sum_{k,l=1}^3 E[\bar{\boldsymbol{\xi}}_\alpha^{(k)}(\Delta_1\boldsymbol{\xi}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})\Delta_1\boldsymbol{\xi}_\beta^{(l)\top} \\
&+\bar{\boldsymbol{\xi}}_\alpha^{(k)}(\Delta_1\boldsymbol{\xi}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{\xi}_\beta^{(l)})\bar{\boldsymbol{\xi}}_\beta^{(l)\top} \\
&+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})\Delta_1\boldsymbol{\xi}_\beta^{(l)\top} \\
&+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{\xi}_\beta^{(l)})\bar{\boldsymbol{\xi}}_\beta^{(l)\top}] \\
=&\frac{1}{N^2}\sum_{\alpha,\beta=1}^N\sum_{k,l=1}^3 E[(\Delta_1\boldsymbol{\xi}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})\bar{\boldsymbol{\xi}}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top} \\
&+(\Delta_1\boldsymbol{\xi}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{\xi}_\beta^{(l)})\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top} \\
&+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top} \\
&+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}(\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{\xi}_\beta^{(l)},\bar{\boldsymbol{\xi}}_\alpha^{(k)})\bar{\boldsymbol{\xi}}_\beta^{(l)\top}] \\
=&\frac{1}{N^2}\sum_{\alpha,\beta=1}^N\sum_{k,l=1}^3 E[\bar{\boldsymbol{\xi}}_\alpha^{(k)}((\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})^\top\Delta_1\boldsymbol{\xi}_\alpha^{(k)})\Delta_1\boldsymbol{\xi}_\beta^{(l)\top} \\
&+\mathrm{tr}[\bar{\boldsymbol{M}}^-\Delta_1\boldsymbol{\xi}_\beta^{(l)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top}]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top} \\
&+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top} \\
&+\Delta_1\boldsymbol{\xi}_\alpha^{(k)}(\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)})\bar{\boldsymbol{\xi}}_\beta^{(l)\top}] \\
=&\frac{1}{N^2}\sum_{\alpha,\beta=1}^N\sum_{k,l=1}^3 \Big(\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top}\bar{\boldsymbol{M}}^-E[\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}] \\
&+\mathrm{tr}[\bar{\boldsymbol{M}}^-E[\Delta_1\boldsymbol{\xi}_\beta^{(l)}\Delta_1\boldsymbol{\xi}_\alpha^{(k)\top}]]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top} \\
&+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})E[\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}] \\
&+E[\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\beta^{(l)\top}]\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top}\Big) \\
=&\frac{\sigma^2}{N^2}\sum_{\alpha,\beta=1}^N\sum_{k,l=1}^3 \Big(\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top}\bar{\boldsymbol{M}}^-\delta_{\alpha\beta}V_0^{(kl)}[\boldsymbol{\xi}_\alpha] \\
&+\mathrm{tr}[\bar{\boldsymbol{M}}^-\delta_{\alpha\beta}V_0^{(kl)}[\boldsymbol{\xi}_\alpha]]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top} \\
&+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\beta^{(l)})\delta_{\alpha\beta}V_0^{(kl)}[\boldsymbol{\xi}_\alpha] \\
&+\delta_{\alpha\beta}V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\beta^{(l)\top}\Big) \\
=&\frac{\sigma^2}{N^2}\sum_{\alpha=1}^N\sum_{k,l=1}^3 \Big(\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}\bar{\boldsymbol{M}}^-V_0^{(kl)}[\boldsymbol{\xi}_\alpha] \\
&+\mathrm{tr}[\bar{\boldsymbol{M}}^-V_0^{(kl)}[\boldsymbol{\xi}_\alpha]]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top} \\
&+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(l)})V_0^{(kl)}[\boldsymbol{\xi}_\alpha] \\
&+V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}\Big) \\
=&\frac{\sigma^2}{N^2}\sum_{\alpha=1}^N\sum_{k,l=1}^3 \Big(\mathrm{tr}[\bar{\boldsymbol{M}}^-V_0^{(kl)}[\boldsymbol{\xi}_\alpha]]\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top} \\
&+(\bar{\boldsymbol{\xi}}_\alpha^{(k)},\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(l)})V_0^{(kl)}[\boldsymbol{\xi}_\alpha] \\
&+2\mathcal{S}[V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\bar{\boldsymbol{M}}^-\bar{\boldsymbol{\xi}}_\alpha^{(k)}\bar{\boldsymbol{\xi}}_\alpha^{(l)\top}]\Big).
\end{aligned}
\tag{45}
$$

Thus, Eq. (37) is obtained.

B. ML Homography Estimation

B.1 Formulation

If we assume that noise in $\boldsymbol{\xi}_\alpha^{(k)}$, $k = 1, 2, 3$, $\alpha = 1, ..., N$, is independent, isotropic, and Gaussian, *maximum likelihood* (*ML*) of homography estimation reduces to minimizing the following *Mahalanobis distance*, which equals the negative logarithm of the likelihood function up to a positive multiplicative constant and an additive constant:

$$
J_{\mathrm{ML}} = \frac{1}{N}\sum_{\alpha=1}^N\Big(\begin{pmatrix}\boldsymbol{\xi}_\alpha^{(1)}-\bar{\boldsymbol{\xi}}_\alpha^{(1)}\\ \boldsymbol{\xi}_\alpha^{(2)}-\bar{\boldsymbol{\xi}}_\alpha^{(2)}\\ \boldsymbol{\xi}_\alpha^{(3)}-\bar{\boldsymbol{\xi}}_\alpha^{(3)}\end{pmatrix},
\begin{pmatrix}V_0^{(11)}[\boldsymbol{\xi}_\alpha]V_0^{(12)}[\boldsymbol{\xi}_\alpha]V_0^{(13)}[\boldsymbol{\xi}_\alpha]\\ V_0^{(21)}[\boldsymbol{\xi}_\alpha]V_0^{(22)}[\boldsymbol{\xi}_\alpha]V_0^{(23)}[\boldsymbol{\xi}_\alpha]\\ V_0^{(31)}[\boldsymbol{\xi}_\alpha]V_0^{(32)}[\boldsymbol{\xi}_\alpha]V_0^{(33)}[\boldsymbol{\xi}_\alpha]\end{pmatrix}_4^-
\begin{pmatrix}\boldsymbol{\xi}_\alpha^{(1)}-\bar{\boldsymbol{\xi}}_\alpha^{(1)}\\ \boldsymbol{\xi}_\alpha^{(2)}-\bar{\boldsymbol{\xi}}_\alpha^{(2)}\\ \boldsymbol{\xi}_\alpha^{(3)}-\bar{\boldsymbol{\xi}}_\alpha^{(3)}\end{pmatrix}\Big). \tag{46}
$$

Here, $V_0^{(kl)}[\boldsymbol{\xi}_\alpha]$ are defined from the covariance matrices of $\Delta\boldsymbol{\xi}_\alpha^{(k)}$ by Eq. (18). The notation $(\,\cdot\,)_4^-$ denotes pseudoinverse of rank 4 with eigenvalues except the largest 4 ones being 0: The 27×27 matrix in Eq. (46) has rank 4 because the independent variables in $\Delta\boldsymbol{\xi}_\alpha^{(k)}$, $k = 1, 2, 3$, are only x_α, y_α, x'_α, and y'_α. We minimize Eq. (46) for $\boldsymbol{\xi}_\alpha^{(k)}$, $k = 1, 2, 3$, $\alpha = 1, ..., N$, and $\boldsymbol{h}$ subject to the constraint

$$
(\bar{\boldsymbol{\xi}}_\alpha^{(k)}, \boldsymbol{h}) = 0. \tag{47}
$$

The computational procedure for this was prescribed by Scoleri et al. [15], but their description is rather abstract, using Kronecker products and symbolic differentiations. Here, we evaluate all derivatives directly and write down all equations explicitly, using only standard arithmetics. This will more clearly reveal the underlying mathematical structure of the problem.

If we define 27-D vectors $\boldsymbol{\xi}_\alpha$ and 9×27 matrices $\boldsymbol{I}^{(1)}$, $\boldsymbol{I}^{(2)}$, and $\boldsymbol{I}^{(3)}$ by

$$
\boldsymbol{\xi}_\alpha = \begin{pmatrix}\boldsymbol{\xi}_\alpha^{(1)}\\ \boldsymbol{\xi}_\alpha^{(2)}\\ \boldsymbol{\xi}_\alpha^{(3)}\end{pmatrix}, \tag{48}
$$

$$
\boldsymbol{I}^{(1)} = \begin{pmatrix}\boldsymbol{I}\\ \boldsymbol{O}\\ \boldsymbol{O}\end{pmatrix},\quad
\boldsymbol{I}^{(2)} = \begin{pmatrix}\boldsymbol{O}\\ \boldsymbol{I}\\ \boldsymbol{O}\end{pmatrix},\quad
\boldsymbol{I}^{(3)} = \begin{pmatrix}\boldsymbol{O}\\ \boldsymbol{O}\\ \boldsymbol{I}\end{pmatrix}, \tag{49}
$$

where $\boldsymbol{I}$ is the 9×9 unit matrix, Eq. (47) is rewritten as

$$
(\bar{\boldsymbol{\xi}}_\alpha, \boldsymbol{I}^{(1)}\boldsymbol{h}) = 0,\quad (\bar{\boldsymbol{\xi}}_\alpha, \boldsymbol{I}^{(2)}\boldsymbol{h}) = 0,\quad (\bar{\boldsymbol{\xi}}_\alpha, \boldsymbol{I}^{(3)}\boldsymbol{h}) = 0, \tag{50}
$$

where $\bar{\boldsymbol{\xi}}_\alpha$ is the true value of $\boldsymbol{\xi}_\alpha$. Equation (46) is now rewritten as

$$
J_{\mathrm{ML}} = \frac{1}{N}\sum_{\alpha=1}^N(\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha, V_0[\boldsymbol{\xi}_\alpha]_4^-(\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha)), \tag{51}
$$

where $V_0[\boldsymbol{\xi}_\alpha]_4^-$ is the 27×27 matrix in Eq. (46). Introducing Lagrange multipliers $\lambda_\alpha^{(k)}$ to Eq. (50), we differentiate

$$\frac{1}{2}NJ_{\rm ML} - \sum_{k=1}^{3} \lambda_\alpha^{(k)}(\bar{\boldsymbol{\xi}}_\alpha, \boldsymbol{I}^{(k)}\boldsymbol{h}), \tag{52}$$

with respect to $\bar{\boldsymbol{\xi}}_\alpha$ and set the result to 0. We obtain

$$-V_0[\boldsymbol{\xi}_\alpha]_4^-(\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha) - \sum_{k=1}^{3} \lambda_\alpha^{(k)}\boldsymbol{I}^{(k)}\boldsymbol{h} = \boldsymbol{0}. \tag{53}$$

Multiply this with $V_0[\boldsymbol{\xi}_\alpha]$ from left, we obtain

$$-(\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha) - \sum_{k=1}^{3} \lambda_\alpha^{(k)}V_0[\boldsymbol{\xi}_\alpha]\boldsymbol{I}^{(k)}\boldsymbol{h} = \boldsymbol{0}, \tag{54}$$

where we have noted that the variation $\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha$ due to noise is in the domain of the covariance matrix $V_0[\boldsymbol{\xi}_\alpha]$ and hence is invariant to the projection $V_0[\boldsymbol{\xi}_\alpha]V_0[\boldsymbol{\xi}_\alpha]_4^-$ onto the domain of $V_0[\boldsymbol{\xi}_\alpha]$. Substituting the expression of $V_0[\boldsymbol{\xi}_\alpha]_4^-(\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha)$ obtained from Eq. (53) and the expression of $\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha$ obtained from Eq. (54) into Eq. (51), we can write $J_{\rm ML}$ in the form

$$\begin{aligned}
J_{\rm ML} &= \frac{1}{N}\sum_{\alpha=1}^{N}(\sum_{k=1}^{3}\lambda_\alpha^{(k)}V_0[\boldsymbol{\xi}_\alpha]\boldsymbol{I}^{(k)}\boldsymbol{h}, \sum_{l=1}^{3}\lambda_\alpha^{(l)}\boldsymbol{I}^{(l)}\boldsymbol{h}) \\
&= \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}\lambda_\alpha^{(k)}\lambda_\alpha^{(l)}(\boldsymbol{h}, \boldsymbol{I}^{(kl)\top}V_0[\boldsymbol{\xi}_\alpha]\boldsymbol{I}^{(k)}\boldsymbol{h}) \\
&= \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}\lambda_\alpha^{(k)}\lambda_\alpha^{(l)}(\boldsymbol{h}, V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\boldsymbol{h}) \\
&= \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}\lambda_\alpha^{(k)}\lambda_\alpha^{(l)}V_\alpha^{(kl)},
\end{aligned} \tag{55}$$

where we put

$$V_\alpha^{(kl)} = (\boldsymbol{h}, V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\boldsymbol{h}). \tag{56}$$

If we substitute the expression of $\bar{\boldsymbol{\xi}}_\alpha$ obtained from Eq. (54) into Eq. (50), we have

$$\sum_{l=1}^{3}V_\alpha^{(kl)}\lambda_\alpha^{(l)} = -(\boldsymbol{\xi}_\alpha, \boldsymbol{I}^{(k)}\boldsymbol{h}) = -(\boldsymbol{\xi}_\alpha^{(k)}, \boldsymbol{h}), \quad k = 1, 2, 3, \tag{57}$$

which provides simultaneous linear equations for $\lambda_\alpha^{(k)}$. However, the rank of the coefficient matrix $\boldsymbol{V}_\alpha = (V_\alpha^{(kl)})$ drops to 2 if there is no noise (as described shortly). So, we solve Eq. (57) by least squares, which is equivalent to using the pseudoinverse $\boldsymbol{W}_\alpha = (\boldsymbol{V}_\alpha)_2^-$ of rank 2. We obtain

$$\lambda_\alpha^{(k)} = -\sum_{l=1}^{3}W_\alpha^{(kl)}(\boldsymbol{\xi}_\alpha^{(l)}, \boldsymbol{h}). \tag{58}$$

Substituting this into Eq. (55), we obtain

$$\begin{aligned}
J_{\rm ML} &= \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3}\Big(\sum_{m=1}^{3}W_\alpha^{(km)}(\boldsymbol{\xi}_\alpha^{(m)}, \boldsymbol{h})\Big) \\
&\quad \Big(\sum_{n=1}^{3}W_\alpha^{(ln)}(\boldsymbol{\xi}_\alpha^{(n)}, \boldsymbol{h})\Big)V_\alpha^{(kl)} \\
&= \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{m,n=1}^{3}\Big(\sum_{k,l=1}^{3}W_\alpha^{(km)}V_\alpha^{(kl)}W_\alpha^{(ln)}\Big) \\
&\quad (\boldsymbol{\xi}_\alpha^{(m)}, \boldsymbol{h})(\boldsymbol{\xi}_\alpha^{(n)}, \boldsymbol{h}) \\
&= \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{m,n=1}^{3}W_\alpha^{(mn)}(\boldsymbol{\xi}_\alpha^{(m)}, \boldsymbol{h})(\boldsymbol{\xi}_\alpha^{(n)}, \boldsymbol{h}),
\end{aligned} \tag{59}$$

where we have used the identity for pseudo inverse: $\boldsymbol{W}_\alpha\boldsymbol{V}_\alpha\boldsymbol{W}_\alpha = \boldsymbol{W}_\alpha(\boldsymbol{W}_\alpha)_2^-\boldsymbol{W}_\alpha = \boldsymbol{W}_\alpha$.

The expression of this type is called the *Sampson error*. Note that *no approximation has been introduced* to derive Eq. (59). However, we assumed in the beginning that noise in $\boldsymbol{\xi}_\alpha^{(k)}$ are Gaussian. This is not strictly true if Δx_α, Δy_α, $\Delta x'_\alpha$, and $\Delta y'_\alpha$ is Gaussian. It has been confirmed in many problems that the Gaussian approximation of noise in $\boldsymbol{\xi}_\alpha^{(k)}$, or the *Sampson approximation*, does practically not affect the solution of the strict ML solution [7].

B.2 Minimizing Eq. (59)

It is easily seen from the definition of $\boldsymbol{\xi}_\alpha^{(k)}$ that

$$x'_\alpha\boldsymbol{\xi}_\alpha^{(1)} + y'_\alpha\boldsymbol{\xi}_\alpha^{(2)} + f_0\boldsymbol{\xi}_\alpha^{(3)} = \boldsymbol{0} \tag{60}$$

holds identically. Computing the inner product with $\boldsymbol{h}$ on both sides, we obtain

$$(x'_\alpha\boldsymbol{\xi}_\alpha^{(1)} + y'_\alpha\boldsymbol{\xi}_\alpha^{(2)} + f_0\boldsymbol{\xi}_\alpha^{(3)}, \boldsymbol{h}) = 0. \tag{61}$$

This is an identity in x_α, y_α, x'_α, and y'_α, so its derivatives with respect to these is also identities. Hence, the following identically holds if there is no noise:

$$\begin{aligned}
&(x'_\alpha[\boldsymbol{T}_\alpha^{(1)}]_1 + y'_\alpha[\boldsymbol{T}_\alpha^{(2)}]_1 + f_0[\boldsymbol{T}_\alpha^{(3)}]_1, \boldsymbol{h}) = 0, \\
&(x'_\alpha[\boldsymbol{T}_\alpha^{(1)}]_2 + y'_\alpha[\boldsymbol{T}_\alpha^{(2)}]_2 + f_0[\boldsymbol{T}_\alpha^{(3)}]_2, \boldsymbol{h}) = 0, \\
&(x'_\alpha[\boldsymbol{T}_\alpha^{(1)}]_3 + y'_\alpha[\boldsymbol{T}_\alpha^{(2)}]_3 + f_0[\boldsymbol{T}_\alpha^{(3)}]_3, \boldsymbol{h}) = 0, \\
&(x'_\alpha[\boldsymbol{T}_\alpha^{(1)}]_4 + y'_\alpha[\boldsymbol{T}_\alpha^{(2)}]_4 + f_0[\boldsymbol{T}_\alpha^{(3)}]_4, \boldsymbol{h}) = 0.
\end{aligned} \tag{62}$$

Here, $[\boldsymbol{T}_\alpha^{(k)}]_i$ is the ith column of $\boldsymbol{T}_\alpha^{(k)}$ (= the Jacobi matrix of $\boldsymbol{\xi}_\alpha^{(k)}$), and we have noted that $(\boldsymbol{\xi}_\alpha^{(k)}, \boldsymbol{h}) = 0$ in the absence of noise. From these four equations, we conclude that

$$(x'_\alpha\boldsymbol{T}_\alpha^{(1)} + y'_\alpha\boldsymbol{T}_\alpha^{(2)} + f_0\boldsymbol{T}_\alpha^{(3)})^\top\boldsymbol{h} = \boldsymbol{0}. \tag{63}$$

If we multiply $\boldsymbol{T}_\alpha^{(k)}$ with this and note the definition $V_0^{(kl)}[\boldsymbol{\xi}_\alpha] \equiv \boldsymbol{T}_\alpha^{(k)}\boldsymbol{T}_\alpha^{(l)\top}$, we obtain

$$(x'_\alpha V_0^{(k1)}[\boldsymbol{\xi}_\alpha] + y'_\alpha V_0^{(k2)}[\boldsymbol{\xi}_\alpha] + f_0V_0^{(k3)}[\boldsymbol{\xi}_\alpha])\boldsymbol{h} = \boldsymbol{0}. \tag{64}$$

We write the 3×3 matrix having $(\boldsymbol{h}, V_0^{(kl)}[\boldsymbol{\xi}_\alpha]\boldsymbol{h})$ as its (kl) element as $\boldsymbol{V}_\alpha$. Computing the inner product of $\boldsymbol{h}$ and Eq. (64), we obtain

$$\boldsymbol{V}_\alpha \begin{pmatrix} x'_\alpha \\ y'_\alpha \\ f_0 \end{pmatrix} = \boldsymbol{0}. \tag{65}$$

Thus, $\boldsymbol{x}'_\alpha = \begin{pmatrix} x'_\alpha & y'_\alpha & f_0 \end{pmatrix}^\top$ is a null vector of $\boldsymbol{V}_\alpha$. From the definition of pseudoinverse, it is also a null vector of $\boldsymbol{W}_\alpha = (\boldsymbol{V}_\alpha)_2^-$. It follows that $\boldsymbol{W}_\alpha\boldsymbol{V}_\alpha$ and $\boldsymbol{V}_\alpha\boldsymbol{W}_\alpha$ are both projection matrices onto the subspace orthogonal to $\boldsymbol{x}'_\alpha$. Hence, we can write

$$\boldsymbol{W}_\alpha\boldsymbol{V}_\alpha = \boldsymbol{V}_\alpha\boldsymbol{W}_\alpha = \boldsymbol{I} - \mathcal{N}[\boldsymbol{x}'_\alpha]\mathcal{N}[\boldsymbol{x}'_\alpha]^\top, \tag{66}$$

where $\mathcal{N}[\,\cdot\,]$ denotes normalization into unit norm. Differentiating Eq. (66) with respect to h_i, we obtain

$$\frac{\partial \boldsymbol{V}_\alpha}{\partial h_i}\boldsymbol{W}_\alpha + \boldsymbol{V}_\alpha\frac{\partial \boldsymbol{W}_\alpha}{\partial h_i} = \boldsymbol{O}. \tag{67}$$

Multiplying this by $\boldsymbol{W}_\alpha$ from left and noting that $\partial\boldsymbol{W}_\alpha/\partial h_i$ also has $\boldsymbol{x}'_\alpha$ as its null vector and hence is invariant to the projection $\boldsymbol{W}_\alpha\boldsymbol{V}_\alpha$, we obtain the following identity:

$$\frac{\partial \boldsymbol{W}_\alpha}{\partial h_i} = -\boldsymbol{W}_\alpha\frac{\partial \boldsymbol{V}_\alpha}{\partial h_i}\boldsymbol{W}_\alpha. \tag{68}$$

Now, if we define the 9×3 matrix

$$\boldsymbol{\Xi}_\alpha = \begin{pmatrix} \boldsymbol{\xi}_\alpha^{(1)}\boldsymbol{\xi}_\alpha^{(2)}\boldsymbol{\xi}_\alpha^{(3)} \end{pmatrix}, \tag{69}$$

Eq. (59) can be rewritten as follows:

$$J_{\mathrm{ML}} = \frac{1}{N}\sum_{\alpha=1}^{N}(\boldsymbol{h}, \boldsymbol{\Xi}_\alpha\boldsymbol{W}_\alpha\boldsymbol{\Xi}_\alpha^\top\boldsymbol{h}). \tag{70}$$

Differentiating this with respect to h_i and using Eq. (68), we obtain

$$\frac{\partial J_{\mathrm{ML}}}{\partial h_i} = \frac{2}{N}\sum_{\alpha=1}^{N}(\boldsymbol{\Xi}_\alpha\boldsymbol{W}_\alpha\boldsymbol{\Xi}_\alpha^\top\boldsymbol{h})_i - \frac{2}{N}\sum_{\alpha=1}^{N}(\boldsymbol{h}, \boldsymbol{\Xi}_\alpha\boldsymbol{W}_\alpha\frac{\partial \boldsymbol{V}_\alpha}{\partial h_i}\boldsymbol{W}_\alpha\boldsymbol{\Xi}_\alpha^\top\boldsymbol{h}), \tag{71}$$

where $(\,\cdot\,)_i$ denotes the ith component. If we put

$$v_\alpha^{(k)} = \sum_{l=1}^{3} W_\alpha^{(kl)}(\boldsymbol{\xi}_\alpha^{(l)}, \boldsymbol{h}), \tag{72}$$

and define $\boldsymbol{v}_\alpha$ to be the 3-D vector with components $v_\alpha^{(k)}$, $k = 1, 2, 3$, Eq. (72) is written as

$$\boldsymbol{v}_\alpha = \boldsymbol{W}_\alpha\boldsymbol{\Xi}_\alpha^\top\boldsymbol{h} \tag{73}$$

From the definition of the matrix $\boldsymbol{V}_\alpha$, we see that $\partial\boldsymbol{V}_\alpha/\partial h_i$ is a 3×3 matrix whose (kl) element is $2\sum_{j=1}^{9} V_0^{(kl)}[\boldsymbol{\xi}_\alpha]_{ij}h_j$. Hence, the last term of the right-hand side of Eq. (71) is

$$\frac{2}{N}\sum_{\alpha=1}^{N}(\boldsymbol{h}, \boldsymbol{\Xi}_\alpha\boldsymbol{W}_\alpha\frac{\partial \boldsymbol{V}_\alpha}{\partial h_i}\boldsymbol{W}_\alpha\boldsymbol{\Xi}_\alpha^\top\boldsymbol{h}) = \frac{2}{N}\sum_{\alpha=1}^{N}(\boldsymbol{v}_\alpha, \frac{\partial \boldsymbol{V}_\alpha}{\partial h_i}\boldsymbol{v}_\alpha) = \sum_{j=1}^{9}\Big(\frac{2}{N}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{3} V_0^{(kl)}[\boldsymbol{\xi}_\alpha]_{ij}v_\alpha^{(k)}v_\alpha^{(l)}\Big)h_j. \tag{74}$$

If we define 9×9 matrices $\boldsymbol{M}_{\mathrm{ML}}$ and $\boldsymbol{L}_{\mathrm{ML}}$ by

$$\boldsymbol{M}^{\mathrm{ML}} = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha^{(kl)}\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{\xi}_\alpha^{(l)\top}, \tag{75}$$

$$\boldsymbol{L}^{\mathrm{ML}} = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{N} v_\alpha^{(k)}v_\alpha^{(l)}V_0^{(kl)}[\boldsymbol{\xi}_\alpha], \tag{76}$$

the first term on the right-hand side of Eq. (71) is simply $2\boldsymbol{M}^{\mathrm{ML}}$. Equation (74) is written as $2\boldsymbol{L}^{\mathrm{ML}}\boldsymbol{h}$. Thus, we obtain the following expression of the derivative of J_{ML} in Eq. (70):

$$\nabla_{\boldsymbol{h}}J_{\mathrm{ML}} = 2(\boldsymbol{M}^{\mathrm{ML}} - \boldsymbol{L}^{\mathrm{ML}})\boldsymbol{h} \tag{77}$$

It follows that to minimize J_{ML} we need to solve

$$(\boldsymbol{M}^{\mathrm{ML}} - \boldsymbol{L}^{\mathrm{ML}})\boldsymbol{h} = \boldsymbol{0}. \tag{78}$$

In the above derivation, we have assumed that there is no noise. In the presence of noise, the only difference is that Eq. (65) does not exactly hold, and $\boldsymbol{V}_\alpha$ is nonsingular with the smallest eigenvalue close to 0. So, we regard the definition of $\boldsymbol{W}_\alpha = (\boldsymbol{V}_\alpha)_2^-$ as obtained by curtailing the smallest eigenvalue of $\boldsymbol{V}_\alpha$ to 0.

B.3 Solving Eq. (78)

In order to solve Eq. (78), we use the FNS principle of Chojnacki et al. [2], though we may as well use the HEIV principle of Leedan and Meer [11] and Matei and Meer [12]. The FNS procedure goes as follows:

1. Provide an initial value $\boldsymbol{h}_0$ for $\boldsymbol{h}$ (e.g., by LS).
2. Compute the matrices $\boldsymbol{M}^{\mathrm{ML}}$ and $\boldsymbol{L}^{\mathrm{ML}}$ in Eqs. (75) and (76).
3. Solve the eigenvalue problem

$$(\boldsymbol{M}^{\mathrm{ML}} - \boldsymbol{L}^{\mathrm{ML}})\boldsymbol{h} = \lambda\boldsymbol{h}, \tag{79}$$

   and compute the unit eigenvector $\boldsymbol{h}$ for the smallest eigenvalue λ.
4. If $\boldsymbol{h} \approx \boldsymbol{h}_0$, return $\boldsymbol{h}$ and stop. Else, let $\boldsymbol{h}_0 \leftarrow \mathcal{N}[\boldsymbol{h}_0 + \boldsymbol{h}]$, and go back to Step 2.

The term $\mathcal{N}[\boldsymbol{h}_0 + \boldsymbol{h}]$ means $\mathcal{N}[(\boldsymbol{h}_0 + \boldsymbol{h})/2]$. This average taking, not originally shown by Chojnacki et al. [2], was shown to stabilize the convergence in many problems [10].