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Variational Formulations for Electromagnetic Field and Charged-Particle Stream Configurations and Their Linearization

Yoshitaka MORIKAWA* and Hiroshi HAMADA*

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Synopsis

Two variational formulations for electromagnetic field and charged-particle stream configurations, in which both field and particles are described by the field-like variables suited for the problems of electrodynamics, are presented. One of them is directly obtained through slight modifications of Sturrock's original procedure but has a complicated form. The other is obtained through linearization of the preceding one and has a compact form. Both formulations lend themselves to straightforward derivation of the well-known energy-momentum tensor and/or its conservation law. Specifically the latter one is of academic interest because of its compact form. Moreover, as a proof of its practical usefulness the variational principle under the small-amplitude approximation is derived from it, which is known to provide a basis for the study of certain types of instability in plasmas. It is, however, hoped that it will find main applications in the electrodynamics problems concerned with large-amplitude behavior.

1. Introduction

The classical description of the electromagnetic field and

* Department of Electronics

charged-particle stream configurations by a variational principle, in which the stream is expressed in terms of "field-like" or Eulerian variables, has been derived by various authors. Dirac has obtained the variational principle in the course of setting up a classical theory of electrons as a basis for a passage to the quantum theory [1,2]. Low has developed that principle with the hope that it might provide a powerful tool for attacking practical plasma problems [3]. Recently certain errors in Low's original formulation have be pointed out and corrected by Galloway and Kim, who have applied the resultant principle to the analysis of the problem of non-linear wave interaction in warm plasmas [4]. Furthermore, Sturrock has set up that principle, who has intended to apply it the studies of amplification of disturbances in electron beams and certain types of instability in plasma [5].

Dirac's variational principle has the benefit that the energy-momentum tensor is directly derivable through application of the well-known technique, but is described in terms of the Clebsch variables, which are not otherwise well suited for the problems of electrodynamics. On the other hand, the remaining three vairational formulations do not lend themselves to the derivation of the energy-momentum tensor, although they are described in terms of the variables which are suited for the problems of electrodynamics. They are in a form convenient to use under the assumption of small-amplitude disturbances. Under this assumption the small-amplitude power theorem or Manley-Low relation is easily derivable from them.

In the present paper we shall present the variational principle which is expressed only in terms of the variables appropriate for the problems of electrodynamics and yet allows one the direct derivation of the energy-momentum tensor from the corresponding Lagrangian density. The method for our variational formulation is, in essence, similar to Sturrock's one, which is based on an artificial perturbation for charged-particle assembly by introducing the displacement vector, which is chosen as dynamical variables, but there exist the following differences between them; in our formulation the location of the event in the perturbed system is chosen as independent variables and on the other hand that of the event in the unperturbed system corresponding to the perturbed event as dynamical variables, in contrast to Sturrock's choice. In §3 the formulation will be given, being compared with Sturrock's one. The Lagrangian density directly resulting from such modifications is in a complicated form.

It is, therefore, put in a simpler form by assuming that the perturbation is infinitesimally small and then linearizing it. In §4 we shall represent how this linearization is carried out and verify that the resultant Lagrangian density leads back to the appropriate set of equations and energy-momentum tensor in that physical configurations. It is, for practical purposes, of importance to consider in what form the Lagrangian density is expressed under the small-amplitude disturbances, which was, indeed, the subject of Sturrock's paper. This expression may be derived from the variational principle given in §4 as well as that in §3. In §5 we shall show how the derivation is worked out from the former one as a proof of its practical usefulness and present the two resultant expressions; one is in a simpler form than the other, which is in agreement with Sturrock's one, whereas it is not suited for the derivation of the power theorem.

2. Notation

For the convenience of comparison, we shall use the same notations as those by Sturrock [5]. In this section they will be reviewed briefly.

Our theory will be set up in covariant relativistic form and with Gaussian system of units. We write x^μ, U^μ, and A^μ for the four-dimensional position, velocity, and magnetic potential vectors, respectively, which are related to the original three-dimensional vectors by the relations,

$$x^0 = ct \ , \qquad (x^i) = (x_r) \ , \tag{1}$$

$$(i, r = 1,2,3)$$

$$U^0 = \gamma \ , \qquad (U^i) = (\frac{\gamma}{c} v_r) \ , \tag{2}$$

where (x_r) and (v_r) denote the original three-dimensional position and velocity vectors, respectively, t the time, c the velocity of light, and

$$\gamma = [1 - (\frac{v}{c})^2]^{-\frac{1}{2}} \ , \tag{3}$$

so that

$$U^\mu U_\mu = 1 \ , \tag{4}$$

and

$$A^{o} = \phi , \quad (A^{i}) = (A_{r}) , \quad (i,r = 1,2,3) \tag{5}$$

where ϕ is the electric potential and A_r the original three-dimensional magnetic potential.

We adopt the following form for the metric tensor:

$$g_{oo} = 1, \quad g_{oi} = 0, \quad g_{io} = 0, \quad g_{ij} = -\delta_{ij}, \quad (i,j = 1,2,3). \tag{6}$$

Then the element of world distance ds is defined by

$$ds = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} . \tag{7}$$

With this definition the four-dimensional velocity U^{μ} is expressed as

$$U^{\mu} = \frac{dx^{\mu}}{ds} . \tag{8}$$

The antisymmetric field tensor is defined by

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \equiv \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} . \tag{9}$$

The semicolon notation for differentiation will be used, as seen in (9).

We denote by $N(x)$ the proper number density of charged-particles which is related to ρ, the number density measured with respect to the given coordinate system, by

$$N = \frac{1}{\gamma}\rho . \tag{10}$$

Then the vector defined by

$$N^{\mu} \equiv NU^{\mu} , \tag{11}$$

which has components

$$N^{o} = \rho , \quad (N^{i}) = (\rho \frac{v_r}{c}) , \quad (i,r = 1,2,3) \tag{12}$$

satisfies the equation of conservation of particles,

$$N^{\mu}_{\ ;\mu} = 0 . \tag{13}$$

If we denote by q the charge of the particles under consideration, Maxwell's equations and Newton-Lorentz equations take the following forms, respectively:

$$F^{\mu\nu}_{\ ;\nu} = -4\pi q N^{\mu} \ , \tag{14}$$

and

$$U^{\nu}U_{\mu;\nu} = \frac{q}{mc^2}\, U^{\nu}F_{\mu\nu} \ . \tag{15}$$

3. Variational Formulation

The action function for an assembly of charged-particles interacting with an electromagnetic field may be written as the sum of three terms:

$$S = S_f + S_p + S_i \tag{16}$$

where S_f, S_p, and S_i are the field, particle, and interaction contributions, respectively. Under the approximation of the assembly of discrete particles by a fluid model, the above functions take the following forms:

$$S_f = -\frac{1}{16\pi c}\int d^4x (A_{\nu;\mu} - A_{\mu;\nu})(A^{\nu;\mu} - A^{\mu;\nu}) \ , \tag{17}$$

$$S_p = -mc\int d^4x\, N \ , \tag{18}$$

and

$$S_i = -\frac{q}{c}\int d^4x\, NU^{\mu}A_{\mu} \ . \tag{19}$$

The above formulas do not yet represent a variational principle for the particle trajectories, since the representation of the trajectories does not explicitly appears in them.

In order to describe an assembly of particles by field-like coordinates and yet set up a variational principle for the trajectories of particles, Sturrock has introduced a "displacement vector", as defined as follows [5]. At first he has adopted as the base of his system of variables an assembly of world lines characterized by scalar and vector functions $N(x)$, $U^{\mu}(x)$ and considered a perturbed set of world lines, which is traced out by the same set of particles as the unperturbed assembly of world lines. Then he has introduced the displacement vector $\xi^{\mu}(x)$, which is the function characterizing the perturbed set of world lines such that the particle which, in the unperturbed system, passed through the event with coordinates x^μ is, in the perturbed system, found to pass through the event with coordinates $x^\mu+\xi^\mu(x)$ (see Fig.1 (a)). He has shown that when the action function is expressed in terms of the displacement vector, the resultant variational principle yields the appropriate field equations and equations of motion. It might also be shown that we obtain the same result by the adoption of $\tilde{x}^\mu(x)\equiv x^\mu+\xi^\mu(x)$ in place of $\xi^\mu(x)$ as dynamical variables. His Lagrangian density involves some quantities not to be varied and therefore is not in the form of a closed system. Accordingly one fails to derive the energy-momentum tensor and/or its conservation law from it. Furthermore, the set of equations resulting from his variational principle is that in the perturbed system, expressed as the function of the unperturbed position. This is another reason for the above failure.

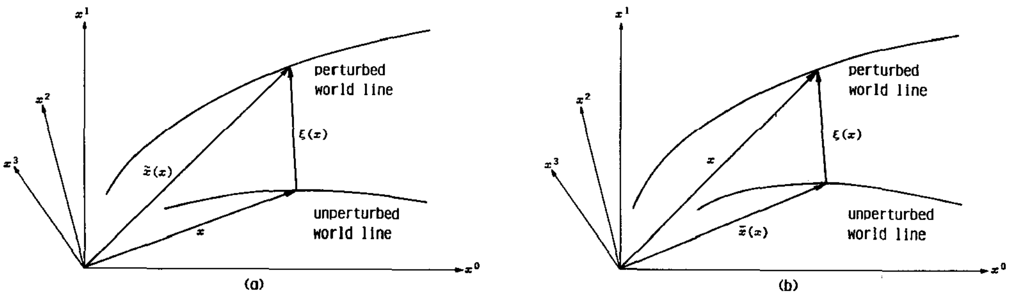

Fig.1 Definitions of dynamical variables; (a) dynamical variables by Sturrock, $\xi(x)$, and (b) those by the authors, $\bar{x}(x)$.

These circumstances may be improved if we consider that the particle which, in the perturbed system, passes through the event with coordinate x^μ is, in the unperturbed system, found to have passed through the event with coordinates $\bar{x}^\mu(x)$ and then express the action functions in terms of $\bar{x}^\mu(x)$, which are taken as dynamical variables, in contrast to Sturrock's original procedure (see Fig.1 (b)). We shall show bellow how this improvement is worked out.

It should first be noted that U^μ and N in the formulas (18) and (19) are those in the perturbed system. Then we want to express $U^\mu(x)$ and $N(x)$ in terms of $\bar{U}^\mu(\bar{x})$, $\bar{N}(\bar{x})$, and $\bar{x}^\mu$, where $\bar{U}^\mu(\bar{x})$ and $\bar{N}(\bar{x})$ are the velocity vector and proper number density of particles in the unperturbed system. First of all, differentiate x^μ, which may be regarded as the functions of $\bar{x}^{\mu}$, with respect to the world distance $\bar{s}$ in the unperturbed system and we obtain

$$\frac{dx^{\mu}}{d\bar{s}} = \frac{ds}{d\bar{s}} U^{\mu}(x) = \bar{U}^{\nu}(\bar{x})\, x^{\mu}_{;\bar{\nu}} \quad , \tag{20}$$

where $;\bar{\nu}$ denotes the differentiation with respect to $\bar{x}^{\nu}$. In the above equation the tensor $(x^{\mu}_{;\bar{\nu}})$ is the inverse of the tensor $(\bar{x}^{\mu}_{;\nu})$ and its elements are expressed in terms of $\bar{x}^{\mu}_{;\nu}$ as

$$x^{\mu}_{;\bar{\nu}} = X^{\mu}{}_{\nu}\left(\frac{\partial^4\bar{x}}{\partial^4 x}\right)^{-1} \quad , \tag{21}$$

where $\partial^4\bar{x}/\partial^4 x$ denotes the Jaccobian and $X^{\mu}{}_{\nu}$ the cofactor of the element $\bar{x}^{\nu}_{;\mu}$ in the tensor $(\bar{x}^{\mu}_{;\nu})$, which is expressible in terms of $\bar{x}^{\mu}_{;\nu}$ as follows:

$$X^{\mu}{}_{\nu} = \frac{\partial}{\partial\bar{x}^{\nu}_{;\mu}}\left(\frac{\partial^4\bar{x}}{\partial^4 x}\right) \quad . \tag{22}$$

From (4), (20), and (21) we obtain

$$\frac{ds}{d\bar{s}} = [\bar{U}^{\nu}(\bar{x})\,\bar{U}^{\sigma}(\bar{x})\,X^{\mu}{}_{\nu}\,X_{\mu\sigma}]^{\frac{1}{2}} \,/\, \left(\frac{\partial^4\bar{x}}{\partial^4 x}\right) \quad , \tag{23}$$

so that again from (20) and (21)

$$U^{\mu}(x) = \bar{U}^{\nu}(\bar{x})\,X^{\mu}{}_{\nu}[\bar{U}^{\tau}(\bar{x})\,\bar{U}^{\sigma}(\bar{x})\,X^{\rho}{}_{\tau}\,X_{\rho\sigma}]^{-\frac{1}{2}} \quad . \tag{24}$$

From the preceding assumption of conservation of particles between the perturbed and unperturbed systems, it follows that

$$\rho(x)\, d^3x = \bar{\rho}(\bar{x})\, d^3\bar{x} \quad , \tag{25}$$

where $\rho(x)$ and $\bar{\rho}(\bar{x})$ denote the number density measured with respect to the perturbed and unperturbed coordinate systems, respectively, and d^3x and $d^3\bar{x}$ the three-dimensional volume elements of the perturbed and unperturbed systems, respectively. Equation (25) may be rewritten as

$$N(x)\, d^4x = \bar{N}(\bar{x})\, d^4\bar{x}\, \frac{ds}{d\bar{s}} = \bar{N}(\bar{x})\, d^4x \left(\frac{\partial^4\bar{x}}{\partial^4 x}\right) \frac{ds}{d\bar{s}} \quad , \tag{26}$$

where $d^4\bar{x}$ denotes the four-dimensional volume element in the unperturbed system. Equations (23) and (26) lead to

$$N(x) = \bar{N}(\bar{x})[\bar{U}^{\nu}(\bar{x})\,\bar{U}^{\sigma}(\bar{x})\,X^{\mu}{}_{\nu}\,X_{\mu\sigma}]^{\frac{1}{2}} \quad . \tag{27}$$

Substituting (24) and (27) into (18) and (19), we obtain

$$S_p = -mc \int d^4x\, \bar{N}(\bar{x}) [\bar{U}^\nu(\bar{x})\, \bar{U}^\sigma(\bar{x})\, X^\mu{}_\nu X_{\mu\sigma}]^{\frac{1}{2}} \quad , \tag{28}$$

and

$$S_i = -\frac{q}{c} \int d^4x\, \bar{N}^\nu(\bar{x})\, A_\mu X^\mu{}_\nu \quad , \tag{29}$$

where $\bar{N}^\nu(\bar{x})$ denotes the current density of particles in the unperturbed system. It is important to note that the contributions to the integrals are characterized by, and integrated over, the variables x^μ which denote just the location of the event giving rise to the contribution.

The variational principle may now be expressed in terms of a Lagrangian density,

$$\delta S = \delta \int d^4x\, L = 0 \quad , \tag{30}$$

which is expressible as follows:

$$L = L_f + L_p + L_i \quad , \tag{31}$$

where

$$L_f = -\frac{1}{16\pi c}(A^{\nu;\mu} - A^{\mu;\nu})(A_{\nu;\mu} - A_{\mu;\nu}) \quad , \tag{32}$$

$$L_p = -mc\bar{N}(\bar{x})[\bar{U}^\nu(\bar{x})\, \bar{U}^\sigma(\bar{x})\, X^\mu{}_\nu X_{\mu\sigma}]^{\frac{1}{2}} \quad , \tag{33}$$

and

$$L_i = -\frac{q}{c}\, \bar{N}^\nu(\bar{x})\, A_\mu X^\mu{}_\nu \quad . \tag{34}$$

It should be noted that the above Lagrangian density is expressed only in terms of the variables to be varied, A^μ and $\bar{x}^\mu$, which is in a form appropriate to obtain the energy-momentum tensor and/or its conservation law. It might be shown that the Lagrangian density leads back to Maxwell's equations and Newton-Lorentz equations in the perturbed system, expressed as the function of the perturbed position and, indeed, yields the appropriate energy-momentum tensor [6].

4. Another Expression of Variational Principle

The variational principle obtained in the preceding section has a complicated form when the cofactor $X^{\mu}{}_{\nu}$ is expressed in terms of the dynamical variables $\bar{x}^{\mu}$. In this section it will be cast in a simpler form, which may be convenient to use in some cases. It should be noted, for this purpose, that the amount of the perturbation given the distribution of world lines is arbitrary. Accordingly we may assume that the amount is infinitesimally small. Then, we may identify $\bar{x}$ with x and the proper number density and velocity vector of particles in the unperturbed system, $\overline{N}(\bar{x})$ and $\overline{U}^{\mu}(\bar{x})$, with those in the perturbed system, $N(\bar{x})$ and $U^{\mu}(\bar{x})$, respectively. It is noted here that the arguments of $N(\bar{x})$ and $U^{\mu}(\bar{x})$ must not be changed to x, since they remain to be dynamical variables. Furthermore, under the above assumption the cofactor $X^{\mu}{}_{\nu}$ goes to

$$- 2\delta^{\mu}{}_{\nu} + \delta^{\mu}{}_{\nu}\bar{x}^{\sigma}{}_{;\sigma} - \bar{x}^{\mu}{}_{;\nu} \quad ,$$

and $[\overline{U}^{\nu}(\bar{x})\, \overline{U}^{\sigma}(\bar{x})\, X^{\mu}{}_{\nu}\, X_{\mu\sigma}]^{\frac{1}{2}}$ in (33) to

$$U^{\nu}(\bar{x})\, U^{\sigma}(\bar{x})(- 2\delta_{\nu\sigma} + \delta_{\nu\sigma}\bar{x}^{\tau}{}_{;\tau} - \bar{x}_{\nu;\sigma}) \quad ,$$

as shown in Appendix A. It follows from the substitution of the above transitions into (33) and (34) that the field, particle, and interaction contributions take the following forms, respectively:

$$L_f = - \frac{1}{16\pi c}(A^{\nu;\mu} - A^{\mu;\nu})(A_{\nu;\mu} - A_{\mu;\nu}) \quad , \tag{35}$$

$$L_p = - mc\, N^{\nu}(\bar{x})\, U_{\mu}(\bar{x})(- 2\delta^{\mu}{}_{\nu} + \delta^{\mu}{}_{\nu}\bar{x}^{\sigma}{}_{;\sigma} - \bar{x}^{\mu}{}_{;\nu}) \quad , \tag{36}$$

and

$$L_i = - \frac{q}{c}\, N^{\nu}(\bar{x})\, A_{\mu}(-2\delta^{\mu}{}_{\nu} + \delta^{\mu}{}_{\nu}\bar{x}^{\sigma}{}_{;\sigma} - \bar{x}^{\mu}{}_{;\nu}) \quad , \tag{37}$$

with $x \equiv \bar{x}$.

We shall show below that the above Lagrangian density leads to the appropriate set of equations and energy-momentum tensor in the physical system under consideration.

The variables conjugate to A^{μ} and $\bar{x}^{\mu}$, which are defined by

$$\Pi^{\mu\nu} = \frac{\partial L}{\partial A_{\mu;\nu}} \quad , \tag{38}$$

and

$$\Omega^{\mu\nu} = \frac{\partial L}{\partial \bar{x}_{\mu;\nu}} \quad , \tag{39}$$

take from the preceding formulas the following expressions:

$$\Pi^{\mu\nu} = \frac{1}{4\pi c} F^{\mu\nu} \quad , \tag{40}$$

and

$$\Omega^{\mu\nu} = - mc(N^\sigma U_\sigma \delta^{\mu\nu} - N^\nu U^\mu) - \frac{q}{c}(N^\sigma A_\sigma \delta^{\mu\nu} - N^\nu A^\mu) \tag{41}$$

The Euler-Lagrange equations derivable from (30) are

$$\Pi^{\mu\nu}_{;\nu} = \frac{\partial L}{\partial A_\mu} \quad , \tag{42}$$

and

$$\Omega^{\mu\nu}_{;\nu} = \frac{\partial L}{\partial \bar{x}_\mu} \quad . \tag{43}$$

If we evaluate the right hand sides of (42) and (43), considering that $\bar{x}=x$, $\bar{x}^\sigma_{;\sigma}=4$, and $\bar{x}^\mu_{;\nu}=\delta^\mu{}_\nu$, we obtain the following equations:

$$F^{\mu\nu}_{;\nu} = - 4\pi q N^\mu \quad , \tag{44}$$

and

$$U^\nu U^\mu_{;\nu} = \frac{q}{mc^2} U^\nu F^\mu{}_\nu \quad , \tag{45}$$

where we have used the equation of conservation of particles (13) and the relation (11). Equations (44) and (45) are Maxwell's equations and Newton-Lorentz equations, respectively.

The canonical energy-momentum tensor for our problem is defined as follows [7]:

$$T_\mu{}^\nu = A_{\sigma;\mu}\Pi^{\sigma\nu} + \bar{x}_{\sigma;\mu}\Omega^{\sigma\nu} - L\delta_\mu{}^\nu \quad . \tag{46}$$

It follows immediately from the definitions (38) and (39), and from the Euler-Lagrange equations (42) and (43), that

$$T^{\mu\nu}_{;\nu} = 0 \quad . \tag{47}$$

This is the equation of conservation of energy-momentum tensor. Furthermore, the physical energy-momentum tensor, which is symmetrical, is given as follows [7]:

$$\Theta_\mu{}^\nu = T_\mu{}^\nu + (A_\mu \Pi^{\nu\sigma})_{;\sigma} \quad , \tag{48}$$

which also satisfies the law of conservation,

$$\Theta^{\mu\nu}_{;\nu} = 0 \quad . \tag{49}$$

If we evaluate the right hand sides of (46) and (48), using the the preceding expressions relevant to each term appearing in them and the relation $\bar{x}_{\sigma;\mu}=\delta_{\sigma\mu}$, we obtain for the canonical energy-momentum tensor,

$$T_{\mu}{}^{\nu} = \frac{1}{4\pi c} A_{\sigma;\mu} F^{\sigma\nu} + mcU_{\mu}N^{\nu} + \frac{q}{c} A_{\mu}N^{\nu} + \frac{1}{16\pi c} F^{\sigma\tau}F_{\sigma\tau}\delta_{\mu}{}^{\nu} \quad , \tag{50}$$

and for the physical energy-momentum tensor,

$$\Theta_{\mu}{}^{\nu} = \frac{1}{4\pi c} F_{\mu\sigma} F^{\sigma\nu} + mcU_{\mu}N^{\nu} + \frac{1}{16\pi c} F^{\sigma\tau} F_{\sigma\tau}\delta_{\mu}{}^{\nu} \quad , \tag{51}$$

where we have used the Newton-Lorentz equations (45). These expressions are well-known as the energy-momentum tensor for the electromagnetic field and charged-particle stream configurations.

5. The Variational Principle under the Assumption of Small Disturbances

In the case that small-amplitude disturbaces are impressed in the physical system under consideration, we may linearize the set of equations governing the disturbances. The variational principle for leading to the linearized set of equations is obtained as follows; express all quantities as the sum of the undisturbed and disturbed parts and expand the action function in terms of the disturbed quantities to retain only the second-order terms of them. The Lagrangian density for the small-amplitude approximation might be obtained from the action function given in §3, resulting in the same expression as Sturrock's one. In this section we shall verify that the same result may also be obtained from the action function given in §4, although the mathematical manipulations used are rather complicated. This verification may give a proof of its practical usefulness.

In order to characterize the perturbations of all quantities involved in the action function, we introduce the following transitions in notation:

$$\bar{x}^{\mu} \rightarrow \bar{X}^{\mu} + \bar{x}^{\mu} , \qquad A^{\mu} \rightarrow A^{\mu} + a^{\mu} ,$$

$$N^{\mu} \rightarrow N^{\mu} + n^{\mu} , \qquad U^{\mu} \rightarrow U^{\mu} + u^{\mu} ,$$

where, on the right-hand side, capital letters denote the corresponding quantities of the unperturbed system and small letters the contributions to the corresponding quantities due to the perturbation. It should be noted that the first transition is accompanied by the transition $x^{\mu} \rightarrow X^{\mu} + x^{\mu}$, where $\bar{X}^{\mu} = X^{\mu}$ and $\bar{x}^{\mu} = x^{\mu}$, since the original $\bar{x}^{\mu}$ should be indentified with the original x^{μ}, as pointed out in §4. It follows, therefore, that the independent variables change from the original x^{μ} to X^{μ}. This change of independent variables requires that the original four-dimensional volume element d^4x in the action function is replaced by $\{\partial^4(X+x)/\partial^4(X)\}d^4X$ and the differentiations with respect to the original x^{μ} by those with respect to X^{μ}. Under these circumstances the small-amplitude approximation requires for the original quantities to be expressed as follows:

$$N^{\mu}(\bar{x}) \rightarrow N^{\mu} + N^{\mu}_{;\nu}\bar{x}^{\nu} + \frac{1}{2}N^{\mu}_{;\nu;\sigma}\bar{x}^{\nu}\bar{x}^{\sigma} + n^{\mu} + n^{\mu}_{;\nu}\bar{x}^{\nu} ,$$

$$U^{\mu}(\bar{x}) \rightarrow U^{\mu} + U^{\mu}_{;\nu}\bar{x}^{\nu} + \frac{1}{2}U^{\mu}_{;\nu;\sigma}\bar{x}^{\nu}\bar{x}^{\sigma} + u^{\mu} + u^{\mu}_{;\nu}\bar{x}^{\nu} ,$$

$$A^{\mu}(x) \rightarrow A^{\mu} + A^{\mu}_{;\nu}x^{\nu} + \frac{1}{2}A^{\mu}_{;\nu;\sigma}x^{\nu}x^{\sigma} + a^{\mu} + a^{\mu}_{;\nu}x^{\nu} ,$$

$$A^{\mu}_{;\nu}(x) \rightarrow [A^{\mu}_{;\sigma} + (A^{\mu}_{;\tau}x^{\tau})_{;\sigma} + \frac{1}{2}(A^{\mu}_{;\tau;\gamma}x^{\tau}x^{\gamma})_{;\sigma}$$

$$+ a^{\mu}_{;\sigma} + (a^{\mu}_{;\tau}x^{\tau})_{;\sigma}](\delta^{\sigma}{}_{\nu} - x^{\sigma}_{;\nu}) ,$$

$$\bar{x}^{\mu}_{;\nu}(x) \rightarrow \delta^{\mu}{}_{\nu} + \bar{x}^{\mu}_{;\nu} - x^{\mu}_{;\nu} - \bar{x}^{\mu}_{;\sigma}x^{\sigma}_{;\nu} ,$$

and

$$d^4x \rightarrow (1 + x^{\mu}_{;\mu})d^4X ,$$

where on the right hand side of each transition all quantities, whether perturbed or unperturbed, now are the functions of the new independent variables X^{μ} and therefore the notation $;\mu$ denotes the differentiation with respect to X^{μ}. The derivation of the last three transitions will be given in Appendix B.

We substitute the above transitions into the action functions corresponding to the field, particle, and interaction contributions to the Lagrangian density, (35), (36), and (37), and extract only the terms of the second order with respect to the perturbed quantities, and then ingnore the terms including neither the new dynamical variables $\bar{x}^\mu$ nor a^μ. It should also be noted that we may remove the terms resulting in the equations in the unperturbed system but at the perturbed position from the action functions. In fact, this assertion may be justified, since they are expressible as the action integral of a total divergence by using the unperturbed set of equations. After taking these procedures we obtain the following action functions under the small-amplitude approximation:

$$S_f^{(2)} = -\frac{1}{16\pi c}\int d^4X(a^{\nu;\mu} - a^{\mu;\nu})(a_{\nu;\mu} - a_{\mu;\nu}) \quad , \tag{52}$$

$$S_p^{(2)} = -mc\int d^4X\{[(n^\mu U_\mu + N^\mu u_\mu)\bar{x}^\nu]_{;\nu} - (n^\nu U_\mu + N^\nu u_\mu)(\bar{x}^\mu_{;\nu} - x^\mu_{;\nu})\} \quad , \tag{53}$$

and

$$S_i^{(2)} = -\frac{q}{c}\int d^4X[(n^\mu_{;\nu}A_\mu + N^\mu_{;\nu}a_\mu)\bar{x}^\nu + (n^\mu A_\mu + N^\mu a_\mu)\bar{x}^\nu_{;\nu} + (n^\mu A_{\mu;\nu} + N^\mu a_{\mu;\nu})x^\nu - (n^\nu A_\mu + N^\nu a_\mu)(\bar{x}^\mu_{;\nu} - x^\mu_{;\nu}) + n^\mu a_\mu] \quad , \tag{54}$$

where the supperscript (2) denotes the second-order contribution. In (53) the action integral of a total divergence may be omitted. By adding to (53) and (54) the action integrals of a total divergence,

$$-mc\int d^4X[(n^\nu U_\mu + N^\nu u_\mu)(\bar{x}^\mu - x^\mu)]_{;\nu} \quad ,$$

and

$$\frac{q}{c}\int d^4X\{[(n^\mu A_\mu + N^\mu a_\mu)\bar{x}^\nu]_{;\nu} - [(n^\nu A_\mu + N^\nu a_\mu)(\bar{x}^\mu - x^\mu)]_{;\nu}\} \quad ,$$

respectively, and using the equations of conservation of particles in the unperturbed and perturbed systems,

$$N^\mu_{;\mu} = 0 \quad , \tag{55a}$$

and

$$n^{\mu}_{;\mu} = 0 \tag{55b}$$

and then ignoring the terms not including dynamical variables, we obtain the more convenient expressions for S_p and S_i. Thus the expression of the second-order contribution for the Lagrangian density becomes as follows:

$$L^{(2)} = L_f^{(2)} + L_p^{(2)} + L_i^{(2)} \quad , \tag{56}$$

where

$$L_f^{(2)} = -\frac{1}{16\pi c}(a^{\nu;\mu} - a^{\mu;\nu})(a_{\nu;\mu} - a_{\mu;\nu}) \quad , \tag{57}$$

$$L_p^{(2)} = -mc(n^{\nu}U_{\mu;\nu} + N^{\nu}u_{\mu;\nu})\bar{x}^{\mu} \quad , \tag{58}$$

and

$$L_i^{(2)} = -\frac{q}{c}[n^{\mu}a_{\mu} - n^{\nu}F_{\mu\nu}\bar{x}^{\mu} - N^{\nu}(a_{\nu;\mu} - a_{\mu;\nu})(\bar{x}^{\mu} - x^{\mu})] \quad , \tag{59}$$

with $\bar{x}^{\mu} \equiv x^{\mu}$.

We shall now show that this Lagrangian density leads to the linearized field equations and equations of motion.

The Euler-Lagrange equations under the small-amplitude approximation now take the following form:

$$\pi^{\mu\nu}_{;\nu} = \frac{\partial L^{(2)}}{\partial a_{\mu}} \quad , \tag{60}$$

and

$$\omega^{\mu\nu}_{;\nu} = \frac{\partial L^{(2)}}{\partial \bar{x}_{\mu}} \quad , \tag{61}$$

where we have introduced the definitions for the canonical momenta,

$$\pi^{\mu\nu} = \frac{\partial L^{(2)}}{\partial a_{\mu;\nu}} \quad , \tag{62}$$

and

$$\omega^{\mu\nu} = \frac{\partial L^{(2)}}{\partial \bar{x}_{\mu;\nu}} \quad . \tag{63}$$

If we evaluate both sides of (60) and (61), using (56)-(59), we obtain the following equations:

$$f^{\mu\nu}_{;\nu} = -4\pi q n^{\mu} \quad , \tag{64}$$

and

$$n^{\nu}U^{\mu}_{;\nu} + N^{\nu}u^{\mu}_{;\nu} = \frac{q}{mc^2}(n^{\nu}F^{\mu}{}_{\nu} + N^{\nu}f^{\mu}{}_{\nu}) \quad , \tag{65}$$

where we have introduced the notation

$$f_{\mu\nu} \equiv a_{\nu;\mu} - a_{\mu;\nu} \quad . \tag{66}$$

By using the relations (11) and

$$n^{\mu} = nU^{\mu} + Nu^{\mu} \quad , \tag{67}$$

and the unperturbed equations of motion, (65) reduces to

$$u^{\nu}U^{\mu}_{;\nu} + U^{\nu}u^{\mu}_{;\nu} = \frac{q}{mc^2}(u_{\nu}F^{\mu}{}_{\nu} + U^{\nu}f^{\mu}{}_{\nu}) \quad . \tag{68}$$

Equations (64) and (68) are just the linearized forms for Maxwell's equations (14) and Newton-Lorentz equations (15), respectively. Although the Lagrangian density as given in equations (56)-(59) leads to the appropriate linearized set of equations, it is not of the form suited for evaluating the energy-momentum tensor as defined by

$$t_{\mu}{}^{\nu} = a_{\sigma;\mu}\pi^{\sigma\nu} + \bar{x}_{\sigma;\mu}\omega^{\sigma\nu} - L^{(2)}\delta_{\mu}{}^{\nu} \quad . \tag{69}$$

The reason is as follows; it follows from the definitions (62) and (63), and from the Euler-Lagrange equations (60) and (61), that

$$t^{\mu\nu}_{;\nu} = -\frac{\partial L^{(2)}}{\partial X_{\mu}} \quad , \tag{70}$$

and the existence of the perturbed quatities not to be varied in (56)-(59), n^{μ}, u^{μ}, and x^{μ}, violates the conservation of energy, which would hold without them, when the unperturbed system is static. The method for resolving this problem is that one expresses the quadratic Lagrangian density in terms of the unperturbed quantities N and U^{μ}, and the dynamical variables a^{μ} and $\bar{x}^{\mu}$. We shall now show how this is worked out.

The small-amplitude contribution to the current density of particles, n^{μ}, is expressed as

$$n^{\mu} = (N^{\nu}x^{\mu} - N^{\mu}x^{\nu})_{;\nu} \quad , \tag{71}$$

as verified in Appendix C. Accordingly, $n^{\mu}a_{\mu}$ in (59) may be written as follows:

$$\begin{aligned} n^{\mu}a_{\mu} &= (N^{\nu}x^{\mu} - N^{\mu}x^{\nu})_{;\nu}a_{\mu} \\ &= [(N^{\nu}x^{\mu} - N^{\mu}x^{\nu})a_{\mu}]_{;\nu} - N^{\nu}f_{\nu\mu}x^{\mu} \quad . \end{aligned} \tag{72}$$

Substituting (72) into (59) and using the relation (67) and the equations of motion in the unperturbed system, we obtain

$$\begin{aligned} L^{(2)} = &- \frac{1}{16\pi c} f^{\mu\nu} f_{\mu\nu} - mc(Nu^{\nu}U_{\mu;\nu} + N^{\nu}u_{\mu;\nu})\bar{x}^{\mu} \\ &+ \frac{q}{c}[Nu^{\nu}F_{\mu\nu}\bar{x}^{\mu} + N^{\nu}f_{\mu\nu}(\bar{x}^{\mu} - 2x^{\mu})] \quad , \end{aligned} \tag{73}$$

where we have ingnored the term expressed as a total divergence. Furthermore, (73) may be written as follows:

$$\begin{aligned} L^{(2)} = &- \frac{1}{16\pi c} f^{\mu\nu} f_{\mu\nu} + mc(Nu^{\nu}U_{\mu;\nu} + N^{\nu}u_{\mu;\nu})\bar{x}^{\mu} \\ &- \frac{q}{c}(Nu^{\nu}F_{\mu\nu}\bar{x}^{\mu} + N^{\nu}f_{\mu\nu}\bar{x}^{\mu}) \quad . \end{aligned} \tag{74}$$

The reason is that, since the term in the right hand side of (73), $N^{\nu}f_{\mu\nu}(\bar{x}^{\mu}-2x^{\mu})$, may be put as $-N^{\nu}f_{\mu\nu}\bar{x}^{\mu}$ by identifying x^{μ} with $\bar{x}^{\mu}$ when one wants to vary a^{μ}, and as $N^{\nu}f_{\mu\nu}\bar{x}^{\mu}$ by neglecting x^{μ} when to do $\bar{x}^{\mu}$, we may replace it by $-N^{\nu}f_{\mu\nu}\bar{x}^{\mu}$ if one inverts the sign of other terms involving $\bar{x}^{\mu}$ in the right hand side of (73). Adding to the right hand side of (74)

$$+ mc[N(U^{\nu}U_{\mu;\nu})_{;\sigma}x^{\sigma}\bar{x}^{\mu}] - \frac{q}{c}[N(U^{\nu}F_{\mu\nu})_{;\sigma}x^{\sigma}\bar{x}^{\mu}] \quad ,$$

which gives the unperturbed equations of motion at the position as displaced by x^{μ} from X^{μ} when varied with respect to $\bar{x}^{\mu}$, and using the expression for the small-amplitude contribution to the velocity, u^{μ}, in terms of U^{μ} and x^{μ}, as verified in Appendix C,

$$u^{\mu} = U^{\nu}x^{\mu}_{;\nu} - U^{\mu}_{;\nu}x^{\nu} - U^{\mu}U^{\nu}U^{\sigma}x_{\nu;\sigma} \quad , \tag{75}$$

we obtain

$$L^{(2)} = -\frac{1}{16\pi c} f^{\mu\nu} f_{\mu\nu} + mc(-N^{\nu}U^{\sigma}x_{\mu;\sigma}\bar{x}^{\mu}_{;\nu} + N^{\nu}U^{\mu}U^{\sigma}U^{\tau}x_{\sigma;\tau}\bar{x}_{\mu;\nu})$$
$$-\frac{q}{c}(N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + N^{\sigma}F_{\mu\nu}x^{\nu}_{;\sigma}\bar{x}^{\mu} + N^{\nu}f_{\mu\nu}\bar{x}^{\mu}) \quad , \tag{76}$$

where we have used the relation (11) and the equation of conservation of particles and equations of motion in the unperturbed system and further neglected the term expressible as a total divergence. In (76) the following replacements are possible:

$$N^{\nu}U^{\sigma}x_{\mu;\sigma}\bar{x}^{\mu}_{;\nu} \rightarrow \frac{1}{2}N^{\nu}U^{\sigma}\bar{x}_{\mu;\sigma}\bar{x}^{\mu}_{;\nu} \quad ,$$

and

$$N^{\nu}U^{\mu}U^{\sigma}U^{\tau}x_{\sigma;\tau}\bar{x}_{\mu;\nu} \rightarrow \frac{1}{2}N^{\nu}U^{\mu}U^{\sigma}U^{\tau}\bar{x}_{\sigma;\tau}\bar{x}_{\mu;\nu} \quad .$$

Furthermore, in (76) it is possible to substitute

$$\frac{1}{2}N^{\nu}F_{\mu\nu;\sigma}(x^{\sigma}\bar{x}^{\mu} + x^{\mu}\bar{x}^{\sigma}) - \frac{1}{2}N^{\nu}F_{\mu\sigma}(x^{\sigma}\bar{x}^{\mu}_{;\nu} + x^{\mu}_{;\nu}\bar{x}^{\sigma})$$

for

$$N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + N^{\sigma}F_{\mu\nu}x^{\nu}_{;\sigma}\bar{x}^{\mu} \quad ,$$

as shown in Appendix D. Here again the following replacements are possible:

$$N^{\nu}F_{\mu\nu;\sigma}(x^{\sigma}\bar{x}^{\mu} + x^{\mu}\bar{x}^{\sigma}) \rightarrow N^{\nu}F_{\mu\nu;\sigma}\bar{x}^{\sigma}\bar{x}^{\mu} \quad ,$$

and

$$N^{\nu}F_{\mu\sigma}(x^{\sigma}\bar{x}^{\mu}_{;\nu} + x^{\mu}_{;\nu}\bar{x}^{\sigma}) \rightarrow N^{\nu}F_{\mu\sigma}\bar{x}^{\sigma}\bar{x}^{\mu}_{;\nu} \quad .$$

Thus we obtain the following expression for the Lagrangian density under the small-amplitude approximation:

$$L^{(2)} = -\frac{1}{16\pi c}(a^{\nu;\mu} - a^{\mu;\nu})(a_{\nu;\mu} - a_{\mu;\nu})$$
$$- mc(\frac{1}{2}N^{\nu}U^{\sigma}\bar{x}_{\mu;\sigma}\bar{x}^{\mu}_{;\nu} - \frac{1}{2}N^{\nu}U^{\mu}U^{\sigma}U^{\tau}\bar{x}_{\sigma;\tau}\bar{x}_{\mu;\nu})$$
$$- \frac{q}{c}[\frac{1}{2}N^{\nu}F_{\mu\nu;\sigma}\bar{x}^{\sigma}\bar{x}^{\mu} - \frac{1}{2}N^{\nu}F_{\mu\sigma}\bar{x}^{\sigma}\bar{x}^{\mu}_{;\nu} + N^{\nu}\bar{x}^{\mu}(a_{\nu;\mu} - a_{\mu;\nu})] \quad . \tag{77}$$

This is expressed only in terms of the unperturbed quantities and the dynamical variables. The expression (77) is in agreement with that derived by Sturrock, if we indentify $\bar{x}^{\mu}$ with his displacement vector ξ^μ [5]. The energy term of the energy-momentum tensor, as defined by (69), which is evaluated from this Lagrangian density, is subject to the conservation of energy when the unperturbed system is static.

6. Conclusion

Although two variational formulations for electromagnetic field and charged-particle stream configurations have been presented, we have a special interest in the variational principle derived in §4 because of its compact form. Academically, we are interested in the formal similarity between the particle and interaction contributions to the Lagrangean density. Practically, it is hoped that it will find main applications in the electrodynamics problems concerned with large-amplitude behavior, although the variational principle under the small-amplitude approximation has been derived from it as an immediate proof of its practical usefulness.

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Appendix A

If we put as

$$\bar{x}^\mu = x^\mu + \xi^\mu \quad , \tag{A1}$$

the Jaccobian $\partial^4\bar{x}/\partial^4 x$ is approximately expressed as

$$\frac{\partial^4\bar{x}}{\partial^4 x} \simeq 1 + \xi^\mu_{;\mu} + \frac{1}{2}(\xi^\mu_{;\mu}\xi^\nu_{;\nu} - \xi^\mu_{;\nu}\xi^\nu_{;\mu}) \quad , \tag{A2}$$

retaining the terms up to the second-order with respect to ξ^μ. Writing down (A2) with $\bar{x}^\mu$, we obtain

$$\frac{\partial^4 \bar{x}}{\partial^4 x} \simeq \bar{x}^\mu_{;\mu} - 3 + \frac{1}{2}(\bar{x}^\mu_{;\mu} - 4)(\bar{x}^\nu_{;\nu} - 4) - \frac{1}{2}(\bar{x}^\mu_{;\nu} - \delta^\mu{}_\nu)(\bar{x}^\nu_{;\mu} - \delta^\nu{}_\mu) \quad . \qquad \text{(A3)}$$

From (22) and (A3), the cofactor $X^\mu{}_\nu$ is approximated as

$$X^\mu{}_\nu \approx -2\delta^\mu{}_\nu + \delta^\mu{}_\nu \bar{x}^\sigma_{;\sigma} - \bar{x}^\mu_{;\nu} \quad , \qquad \text{(A4)}$$

or

$$= \delta^\mu{}_\nu + \delta^\mu{}_\nu \xi^\sigma_{;\sigma} - \xi^\mu_{;\nu} \quad . \qquad \text{(A4')}$$

This leads to the approximate expression,

$$[\bar{U}^\nu(\bar{x})\bar{U}^\sigma(\bar{x})X^\mu{}_\nu X_{\mu\sigma}]^{\frac{1}{2}}$$

$$\simeq [U^\nu(\bar{x})U^\sigma(\bar{x})(\delta^\mu{}_\nu + \delta^\mu{}_\nu\xi^\tau_{;\tau} - \xi^\mu_{;\nu})(\delta_{\mu\sigma} + \delta_{\mu\sigma}\xi^\gamma_{;\gamma} - \xi_{\mu;\sigma})]^{\frac{1}{2}} \quad .$$

Retaining the terms up to the first-order with respect to ξ^μ, we obtain

$$[\bar{U}^\nu(\bar{x})\bar{U}^\sigma(\bar{x})X^\mu{}_\nu X_{\mu\sigma}]^{\frac{1}{2}} \simeq 1 + \xi^\tau_{;\tau} - U^\nu(\bar{x})U^\sigma(\bar{x})\xi_{\nu;\sigma}$$

$$= U^\nu(\bar{x})U^\sigma(\bar{x})(\delta_{\nu\sigma} + \delta_{\nu\sigma}\xi^\tau_{;\tau} - \xi_{\nu;\sigma}) \quad , \qquad \text{(A5)}$$

or

$$= U^\nu(\bar{x})U^\sigma(\bar{x})(-2\delta_{\nu\sigma} + \delta_{\nu\sigma}\bar{x}^\tau_{;\tau} - \bar{x}_{\nu;\sigma}) \quad , \qquad \text{(A5')}$$

where we have used the relation (4).

Appendix B

Under the transitions,

$$\bar{x}^\mu \rightarrow \bar{X}^\mu + \bar{x}^\mu \quad ,$$

and

$$x^\mu \rightarrow X^\mu + x^\mu \quad ,$$

it follows that

$$\bar{x}^{\mu}_{;\nu} \quad \rightarrow \quad \frac{\partial(\bar{X}^{\mu} + \bar{x}^{\mu})}{\partial X^{\sigma}} \frac{\partial X^{\sigma}}{\partial(X^{\nu} + x^{\nu})} \quad .$$

By making use of (21) and (22), we obtain

$$\frac{\partial X^{\sigma}}{\partial(X^{\nu} + x^{\nu})} = \{\frac{\partial}{\partial(X^{\nu} + x^{\nu})_{;\sigma}}[\frac{\partial^4(X + x)}{\partial^4 X}]\}[\frac{\partial^4(X + x)}{\partial^4 X}]^{-1} \quad . \tag{B1}$$

By identifying x^{μ} and ξ^{μ} with X^{μ} and x^{μ}, respectively, in (A2) and (A4´), (B1) may be approximated as

$$\frac{\partial X^{\sigma}}{\partial(X^{\nu} + x^{\nu})} \simeq (\delta^{\sigma}{}_{\nu} + \delta^{\sigma}{}_{\nu} x^{\gamma}_{;\gamma} - x^{\sigma}_{;\nu})(1 + x^{\tau}_{;\tau})^{-1}$$

$$\simeq (\delta^{\sigma}{}_{\nu} - x^{\sigma}_{;\nu}) \quad , \tag{B2}$$

where we have ingnored the terms of the second-order with respect to x^{μ}, since they does not take part in the variation. It follows from (B2) that

$$\bar{x}^{\mu}_{;\nu} \rightarrow (\delta^{\mu}{}_{\sigma} + \bar{x}^{\mu}_{;\sigma})(\delta^{\sigma}{}_{\nu} - x^{\sigma}_{;\nu})$$

$$= \delta^{\mu}{}_{\nu} + \bar{x}^{\mu}_{;\nu} - x^{\mu}_{;\nu} - \bar{x}^{\mu}_{;\sigma} x^{\sigma}_{;\nu} \quad .$$

By reference to the above results, it is straightforward to show that the transitions for $A^{\mu}_{;\nu}$ and $d^4\boldsymbol{x}$ as given in §5 are followed.

Appendix C

Let us consider the particle velocity of the perturbed system at the position $X^{\mu}+x^{\mu}$. This is approximated as follows:

$$U^{\mu} + U^{\mu}_{;\nu} x^{\nu} + \boldsymbol{u}^{\mu} \simeq U^{\nu}(\delta^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu} x^{\sigma}_{;\sigma} + x^{\mu}_{;\nu})(1 + x^{\tau}_{;\tau} - U^{\sigma} U^{\gamma} x_{\sigma;\gamma})$$

$$\simeq U^{\mu} - U^{\mu} U^{\sigma} U^{\gamma} x_{\sigma;\gamma} + U^{\nu} x^{\mu}_{;\nu} \quad , \tag{C1}$$

by making use of (24), where we identify $\bar{x}^{\mu}$, $\boldsymbol{x}^{\mu}$ and $\bar{U}^{\mu}(\bar{\boldsymbol{x}})$ with X^{μ}, $X^{\mu}+x^{\mu}$ and U^{μ}, and of (A4´) and (A5), where ξ^{μ} and $U^{\mu}(\bar{x})$ with $-x^{\mu}$ and U^{μ}. Accordingly we obtain

$$\boldsymbol{u}^{\mu} = U^{\sigma} x^{\mu}_{;\sigma} - U^{\mu}_{;\sigma} x^{\sigma} - U^{\mu} U^{\sigma} U^{\tau} x_{\sigma;\tau} \quad . \tag{C2}$$

Next, let us consider the current density of particles of the perturbed system at the position $X^{\mu}+x^{\mu}$. It follows from (24) and (27) that

$$N^{\mu} = \bar{N}^{\nu}(\bar{x})X^{\mu}{}_{\nu} \ . \tag{C3}$$

By using the procedure similar to that which led to (C2), we obtain

$$n^{\mu} = N^{\nu}x^{\mu}_{;\nu} - N^{\mu}x^{\nu}_{;\nu} - N^{\mu}_{;\nu}x^{\nu} \ . \tag{C4}$$

The use of the law of the conservation of particles in the unperturbed system leads to

$$n^{\mu} = (N^{\nu}x^{\mu} - N^{\mu}x^{\nu})_{;\nu} \ , \tag{C5}$$

Appendix D

Since we may write as

$$\begin{aligned} F_{\mu\nu;\sigma} &= A_{\nu;\mu;\sigma} - A_{\mu;\nu;\sigma} \\ &= A_{\sigma;\mu;\nu} - A_{\mu;\sigma;\nu} + A_{\nu;\sigma;\mu} - A_{\sigma;\nu;\mu} \\ &= F_{\sigma\mu;\nu} + F_{\sigma\nu;\mu} \ , \end{aligned} \tag{D1}$$

we obtain the following expressions:

$$\begin{aligned} &N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + N^{\sigma}F_{\mu\nu}x^{\nu}_{;\sigma}\bar{x}^{\mu} \\ &= \frac{1}{2}N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + \frac{1}{2}N^{\nu}(F_{\mu\sigma;\nu} + F_{\sigma\nu;\mu})x^{\sigma}\bar{x}^{\mu} + N^{\sigma}F_{\mu\nu}x^{\nu}_{;\sigma}\bar{x}^{\mu} \\ &= \frac{1}{2}N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + \frac{1}{2}(N^{\nu}F_{\mu\sigma}x^{\sigma}\bar{x}^{\mu})_{;\nu} - \frac{1}{2}N^{\nu}F_{\mu\sigma}x^{\sigma}_{;\nu}\bar{x}^{\mu} \\ &\quad - \frac{1}{2}N^{\nu}F_{\mu\sigma}x^{\sigma}\bar{x}^{\mu}_{;\nu} + \frac{1}{2}N^{\nu}F_{\sigma\nu;\mu}x^{\sigma}\bar{x}^{\mu} + N^{\sigma}F_{\mu\nu}x^{\nu}_{;\sigma}\bar{x}^{\mu} \ , \end{aligned} \tag{D2}$$

where we have used the law of the conservation of particles. Neglecting the second term, which is a total divergence, and exchanging the suffixes μ and σ in the fifth term and ν and σ in the sixth term each other in the last expression of (D2), the following replacement is possible:

$$N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + N^{\sigma}F_{\mu\nu}x^{\nu}_{;\sigma}\bar{x}^{\mu}$$

$$\rightarrow \frac{1}{2}N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + \frac{1}{2}N^{\nu}F_{\mu\nu;\sigma}x^{\mu}\bar{x}^{\sigma} - \frac{1}{2}N^{\nu}F_{\mu\sigma}x^{\sigma}\bar{x}^{\mu}_{;\nu} + \frac{1}{2}N^{\nu}F_{\mu\sigma}x^{\sigma}_{;\nu}\bar{x}^{\mu} \quad .$$

Exchanging the suffixes μ and σ each other in the fourth term of the right hand side of the above replacement and using the identity

$$F_{\mu\sigma} = -F_{\sigma\mu}$$

lead to the further replacement,

$$N^{\nu}F_{\mu\nu;\sigma}x^{\sigma}\bar{x}^{\mu} + N^{\sigma}F_{\mu\nu}x^{\nu}_{;\sigma}\bar{x}^{\mu}$$

$$\rightarrow \frac{1}{2}N^{\nu}F_{\mu\nu;\sigma}(x^{\sigma}\bar{x}^{\mu} + x^{\mu}\bar{x}^{\sigma}) - \frac{1}{2}N^{\nu}F_{\mu\sigma}(x^{\sigma}\bar{x}^{\mu}_{;\nu} + x^{\mu}_{;\nu}\bar{x}^{\sigma}) \quad .$$