Memoirs of the School of Engineering, Okayama University Vol. 17, No. 1, February 1983

Optimal Toll Rate and Expansion of Urban Expressway

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(Received November 15, 1982)

SYNOPSIS

Some extension is made of the previous papers of the same title in order to investigate (1) what aspect consumers' surplus has according to the characteristics of inverse demand curve and (2) where the maximum consumers' surplus is reached in an equilibrium of toll revenues and cost.

As for (1) three kinds of inverse demand curves are assumed in general form. The marginal consumers' surplus to expansion of expressway network is proved to be of definite sign (positive or negative) or equal to zero according to each curve assumed.

For each of curves, the region where consumers' surplus finds its maximum is also shown on the expressway users ~ network expansion plane.

1. INTRODUCTION

The paper is some extension of the previous papers on optimal toll-rating and network expanding of urban expressway in an equilibrium of toll revenues and cost of expressway service supplied [1], [2]. One of the previous conclusions is that, in case of a certain kind of demand curve, the extremum condition to consumers' (expressway users') surplus be equivalent to that to the number of users (called

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diverted traffic hereafter in this paper). It was shown that the marginal consumers' surplus to network expansion was always equal to zero in case of such demand curve, which was found to give reasons for the equivalence of both extremum conditions.

In the present paper, some general aspects of optimal solution are shown assuming three kinds of inverse demand curves, one of which is such as mentioned above.

2. MODEL AND THE PRESENT PROBLEM

2.1 Model [3]

The Model is formulated as follows;

Maximize total surplus (consumers' surplus)

$$S = \int_0^q f(\xi, s)d\xi - C(s) \tag{1}$$

subject to

$$f(q, s)q = C(s), \tag{2}$$

where

$$f(q, s) = p \text{ ; inverse demand curve,} \tag{3}$$
$$C(s) = C \text{ ; total service cost,} \tag{4}$$

q ; diverted traffic,

s ; network expansion and

p ; toll rate.

Eq.(2) means an equilibrium of toll revenues and service cost. With the further premise that a flat toll rate is imposed upon diverted traffic, following assumptions are made

(1) the whole expressway users are homogeneous,

(2) static formulation is admitted of and

(3) congestion cost is disregarded.

2.2 The Present Problem

The previous papers showed that the following relationship is necessary in optimality, if any,

$$\frac{dq}{ds} \begin{cases} < 0, \text{ if } \partial S/\partial s > 0 \\ > 0, \text{ if } \partial S/\partial s < 0 \\ = 0, \text{ if } \partial S/\partial s = 0. \end{cases} \tag{5}$$

where $\partial S/\partial s$ is the marginal consumers' surplus to network expansion.

The present problem is to show

(1) that it is uniquely determined whether the marginal consumers' surplus to network expansion is positive, negative or equal to zero for arbitrary s and q and

(2) the region of optimum solution on $q \sim s$ plane, for each of three inverse demand curves assumed.

By the way, the relationship (5) is derived as follows. By substituting equilibrium condition (2) into (1) and differentiating by s we have

$$\frac{dS}{ds} = \frac{\partial S}{\partial s} - \frac{\partial f}{\partial q} q \frac{dq}{ds} . \tag{6}$$

Since $\partial f/\partial q < 0$, $q \geqq 0$ in general, we have the relationship (5) by putting $dS/ds = 0$. We have to pay attention to that q in eq.(6), accordingly in (5), must satisfy eq.(2).

3. INVERS DEMAND CURVE AND CONSUMERS' SURPLUS

3.1 Assumptions

Two assumptions are made to inverse demand curve as follows.

(1) The curve shifts toward northeast on $p \sim q$ plane.

(2) The curve is characterized by any one of $\partial^2 f/\partial q\partial s < 0$, >0 and $=0$ for arbitrary s and q.

Some complementary explanation is made in the following. Assumption (1) is expressed by $\partial f/\partial s > 0$, which is based on the following way of thinking. The whole population [A1] of diverted traffic increases with network expansion. Diverted traffic increases with network expansion for a specified toll rate, inversely, toll rate has to be raised with network expansion for a specified diverted traffic.

The following is a brief illustration of the three kinds of inverse demand curves characterized by assumption (2).

For arbitrary value of q, we have $\Delta p_s = (\partial f/\partial s)\Delta s$ (>0) by approximation of the first order, where Δp_s is an increment of toll rate caused by network expansion from s to s + Δs. Partial differentiation of Δp_s by q gives $\partial(\Delta p_s)/\partial q = (\partial^2 f/\partial q\partial s)\Delta s$. Accordingly, Δp_s decreases as q increases if $\partial^2 f/\partial q\partial s < 0$ and increases if $\partial^2 f/\partial q\partial s > 0$. Especially, if $\partial^2 f/\partial q\partial s = 0$ for arbitrary s and q, Δp_s is independent of q. Another explanation is as follows; inverse demand curve increases (decreases) with network expansion in absolute value of the gradient as to q if $\partial^2 f/\partial q\partial s < 0$ (>0) and is independent of s if $\partial^2 f/\partial q\partial s = 0$.

Fig. 1 is illustrative examples of inverse demand curves assumed above.

3.2 Behavior of Consumers' Surplus

The first term $\partial S/\partial s$ on the righthand side of eq.(6) is expressed as

$$\frac{\partial S}{\partial s} = \int_0^q \frac{\partial f(\xi,s)}{\partial s} d\xi - \frac{\partial f(q,s)}{\partial s} q \quad (7)$$

where q satisfies eq.(2).

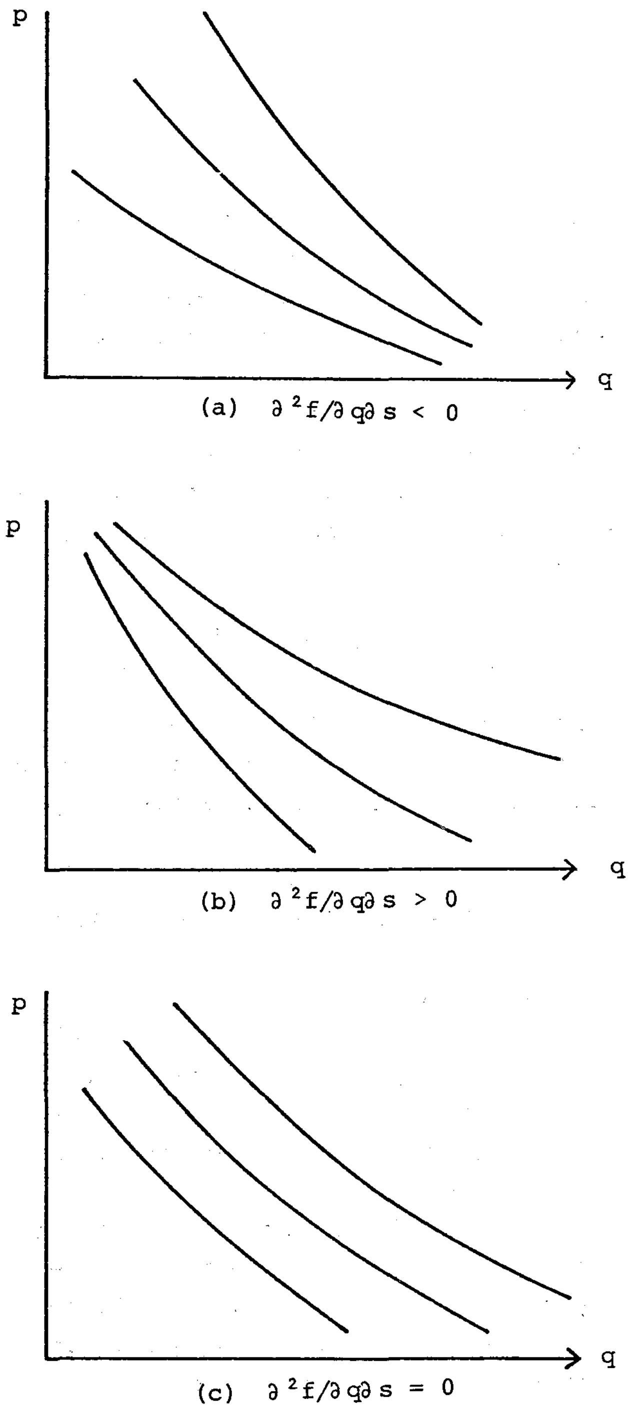

Fig. 1 Demand curve

Partial assumptive differentiation of eq. (7) by q gives $\partial(\partial S/\partial s)/\partial q = -(\partial^2 f/\partial q \partial s)q$. We also have $\partial S/\partial s = 0$ by putting temporarily $q = 0$ in eq. (7).

Afterall the following relationship is obtained.

$$\frac{\partial S}{\partial s} \begin{cases} > 0 & \text{for } \partial^2 f/\partial q \partial s < 0 \\ < 0 & \text{for } \partial^2 f/\partial q \partial s > 0 \\ = 0 & \text{for } \partial^2 f/\partial q \partial s = 0 . \end{cases} \tag{8}$$

4. OPTIMUM SOLUTION ON q ~ s PLANE

The solution of eq. (2) as to q is expressed as $q = q(s)$, that is called in this section equilibrium curve on q ~ s plane. The region of optimum solution on equilibrium curve is shown in the following.

Eqs. (5) and (8) give the necessary conditions that

$$\frac{dq}{ds} \begin{cases} < 0 & \text{if } \partial^2 f/\partial q \partial s < 0 \\ > 0 & \text{if } \partial^2 f/\partial q \partial s > 0 \\ = 0 & \text{if } \partial^2 f/\partial q \partial s = 0 . \end{cases} \tag{9}$$

Optimum solution lies on the part of equilibrium curve where diverted traffic decreases (increases) for such demand curve as shown in Fig. 1(a) (Fig. 1(b)). For Fig. 1(c), optimum solution lies at maximal diverted traffic [A2]. The foregoing is illustrated in Fig. 2, in which equilibrium curve is shown by $q = q(s)$ and maximum toll revenue curve by $\partial R/\partial q = 0$ ($R = fq$). In Fig. 2 convexity is assumed of the set of points within and on equilibrium curve [A3].

Since, for arbitrary value of s, consumers' surplus is larger on the upper half ABC of equilibrium curve than on the lower ADC, optimum solution lies on the parts BC and AC for demand curves shown in Fig. 1(a) and (b), respectively. Point B is optimal for curve in Fig. 1(c).

The following is an additional investigation in the behavior of producers' surplus in the region of optimum solution on q ~ s plane.

Differentiation of both sides of eq. (2) by s gives $\partial(R - C)/\partial s = -(\partial R/\partial q)(dq/ds)$. Since $\partial R/\partial q < 0$ in the optimum region, the marginal producers' surplus to network expansion is negative on the part BC where $dq/ds < 0$, positive on the part AB and equal to zero at point B.

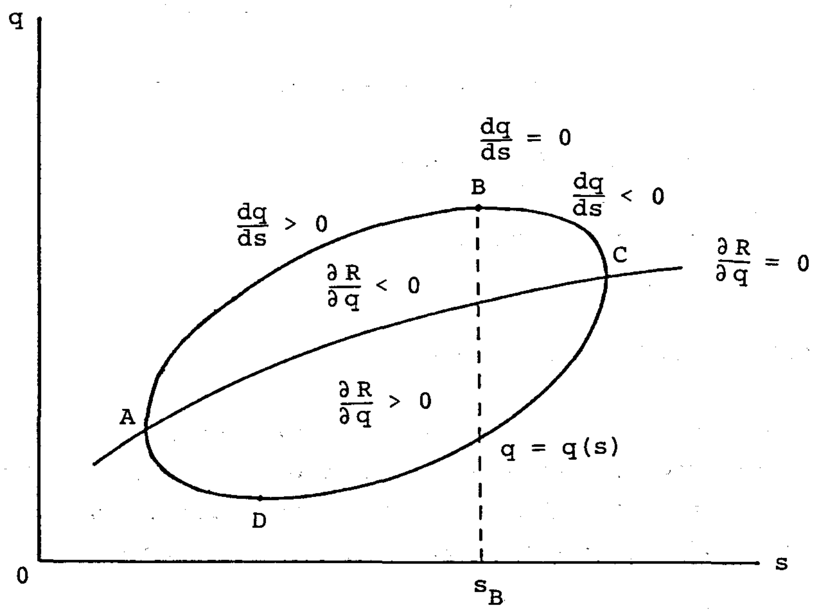

Fig. 2 Optimum solution on equilibrium curve

5. CONCLUSIONS

The region where consumers' surplus finds its maximum was shown for three typical kinds of inverse demand curves through making clear the corresponding behavior of the marginal consumers' surplus to network expansion. Optimum solution, if any, lies in such a region that diverted traffic decreases (increases) if $\partial^2 f/\partial q \partial s < 0$ (> 0) and at any one of the points of the maximal diverted traffic if $\partial^2 f/\partial q \partial s = 0$.

NOTE

[A1] The number of persons expected to use expressway if it is of free use. The number is supposed to be a monotonic increasing function of network expansion.

[A2] Since in eq.(6) $\partial S/\partial s = 0$ for arbitrary s and q, we have $dS/ds = -(\partial f/\partial q)q(dq/ds)$. Note that $\partial f/\partial q < 0$, then we see that extremum condition to S is equivalent to that to q.

[A3] The condition is expressed by

$$\left(\frac{\partial R}{\partial q}\right)^2 \frac{\partial^2 (C-R)}{\partial s^2} - \left\{\frac{\partial (C-R)}{\partial s}\right\}^2 \frac{\partial^2 R}{\partial q^2} - 2\frac{\partial R}{\partial q} \frac{\partial (C-R)}{\partial s} \frac{\partial^2 R}{\partial q \partial s} > 0$$

where R = f(q, s)q and C = C(s).

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[3] H. Yamada : do., 11(1968), 9, 17.