

Optimal Toll Rate and Expansion of Urban Expressway

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Synopsis

A welfare economic approach is tried to an optimal decision of toll rate and expansion of urban expressway network in an equilibrium of toll revenues and cost of service supplied. The model, originated with Yamada, is such that the decision comes into optimality when the maximum consumers' surplus is reached in the equilibrium condition.

The paper is concerned with some general aspects of the optimal solution and reexamination of the solution obtained in the past when used a specific demand curve.

General aspects obtained are as follows; The extremum condition to consumers' surplus is equivalent to that to diverted traffic (the realized number of expressway users) only when demand curve has such a property that the marginal consumers' surplus to network expansion vanishes. In case that the marginal consumers' surplus does not vanish, the extrema of consumers' surplus is found in the regions of negative marginal diverted traffic if demand curve yields positive marginal surplus, and in the regions of the positive if it gives negative marginal surplus. The contact points of demand and average cost curves give extrema of neither consumers' surplus nor diverted traffic.

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An implicative finding, made out by Yamada by using a specific demand curve, that optimal network expansion is reached when the marginal service cost to expansion averaged to the marginal diverted traffic to expansion is equal to the value of time saved by using expressway just by mean trip length holds good at the points of the maximum diverted traffic, but not at the contacts of demand and average cost curves. In case of the demand curve, the condition for an equilibrium of revenues and cost to come into existence is that the minimum of the ratio of service cost averaged to the whole population of expressway users to the value of time mentioned above is less than or equal to e^{-1} .

1. Introduction

Urban expressway service is supplied under self-financing system in this country. Under the system service cost, consisting of those of construction, maintenance, management, etc., must be just paid for by expressway users. Some theoretical problem results from the system that the toll has to be rated so as to meet toll revenues with cost of service supplied. Accordingly, the system requests one to solve a problem of simultaneous decision of expressway network expansion and toll rate, since it requires one to keep an equilibrium of service cost, that is supposed to be some function of network expansion, and toll revenues, that is given by toll rate multiplied by the realized number of expressway users that is dependent upon both toll rate and network expansion.

To the problem were made two ways of Approach by Research Groupe of Toll Rate System to Hanshin Expressway [1]; one by an operations research way of thinking [2] and the other by welfare economics. The latter was developed later by Yamada [3] and was applied by Iida [4] to the planning of Kita-kyushu Urban Expressway Network. On assumptions that there exist homogeneous users alone on whom a flat toll rate is imposed, and congestion cost can be disregarded, Yamada has built a static model in which the maximum consumers' surplus criterion is adopted to

find optimal network expansion. Using specific demand curve, he showed very meaningful implication on a rather intuitive assumption that the optimum solution, in, general, would be found at the contact point of two curves; demand curve and average cost curve. Further assumption was made by Iida, in his application of the model to the planning of Kita-kyushu Urban Expressway Network, that the demand (that is, the number of expressway users realized in an equilibrium of toll revenues and service cost) would reach its maximum as well at the maximum feasible expansion of network, that is, at the contact of two curves.

We are concerned in this paper with

- (1) the assumptions mentioned above,
- (2) the illustrative presentation of several kinds of expansion paths for specific demand curve used by Yamada in his model and
- (3) some further discussions about Yamada's model.

2. Model Description and Some Preliminary Investigations

2.1 Model Description

With the premises that an equilibrium of toll revenues and service cost must be kept and a flat toll rate is imposed upon expressway users, following assumptions are made [3]

- (1) expressway users are homogeneous,
- (2) a static formulation is admitted of
and
- (3) congestion cost may be disregarded.

The first assumption implies that there exists demand curve of a single kind.

Following functions are introduced;

$$X = X(s), \quad X'(s) > 0 \quad (1)$$

$$C = C(s), \quad C'(s) > 0 \quad (2)$$

$$p = f(q,s), \quad (3)$$

where

X = the whole population of expressway users under study, a part of which is realized to be expressway

users (in vehicles)

C = expressway service cost,

p = toll rate,

s = expressway network expansion (that is, an area covered by expressway network)

and

q = the realized number of expressway users that is some part of X (in vehicles).

The relationship between p and q in eq.(3) is so called demand curve. Hereafter in the paper q is called diverted traffic. The reason why network expansion s is included in eq.(3) is that, for an arbitrary toll rate, diverted traffic may as well be regarded as some function of s . Further definition of X is that X is diverted traffic only when expressway is of free use, that is, $q = X$ only when $p = 0$.

Average service cost is given by

$$\pi = \frac{C(s)}{q} \quad (4)$$

which is called average cost curve.

The problem is to find an optimal network expansion and toll rate where consumers' surplus reaches its maximum in an equilibrium of toll revenues and service cost. The problem is now stated as follows;

Maximize

$$S = \int_0^q f(\xi, s) d\xi - C(s) \quad (5)$$

subject to

$$f(q, s)q = C(s). \quad (6)$$

Eq.(6) shows an equilibrium condition to toll revenues and service cost for an arbitrary expansion s .

An illustrative explanation is given in Fig.1, in which E_1

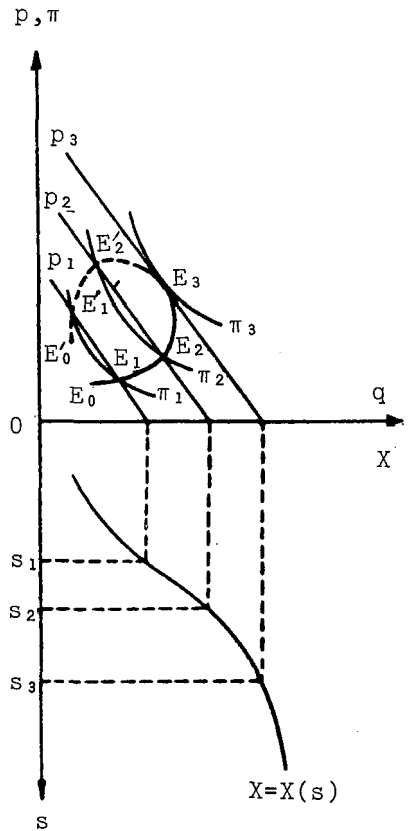


Fig.1 Expansion Path [3]

and E_i^1 , $i = 0, 1, 2, 3$, are intersections or contact points of demand curves and average cost ones. Needless to say, eq.(6) is satisfied on the loci $E_0E_1E_2E_3$ and $E_0^1E_1^1E_2^1E_3$. It is apparent that consumers' surplus is larger on locus $E_0E_1E_2E_3$ than on $E_0^1E_1^1E_2^1E_3$. That is the reason why only the former is named expansion path in [3]. Both loci, however, are called expansion path in this paper, because we have not solved the problem yet. Point E_3 is a contact of demand curve and average cost curve, where the maximum consumers' surplus was supposed to be reached [3] and the maximum diverted traffic to be obtainable as well [4]. Point E_3 is named the terminus of expansion path [3].

The problem may be solved by the Lagrangian method of undetermined multiplier. The Lagrangian is introduced to form

$$L = \int_0^q f(\xi, s)d\xi - C(s) + \mu \{C(s) - f(q, s)q\}, \quad (7)$$

where μ is an undetermined multiplier. The solution is obtained by solving $\partial L/\partial q = 0$, $\partial L/\partial s = 0$ and $\partial L/\partial \mu = 0$. Optimal toll rate is calculated by putting the solution into eq.(3).

2.2 Some Preliminary Investigations

From eq.(7) we have

$$\frac{\partial L}{\partial q} = f(q, s) - \mu \{f(q, s) + \frac{\partial f(q, s)}{\partial q}q\} \quad (8)$$

$$\frac{\partial L}{\partial s} = \int_0^q \frac{\partial f(\xi, s)}{\partial s} d\xi - C'(s) - \mu \left\{ \frac{\partial f(q, s)}{\partial s}q - C'(s) \right\}. \quad (9)$$

Let's suppose the following type of demand curve

$$\begin{aligned} p &= f(q, s) \\ &= f_1(q) + f_2(s). \end{aligned} \quad (10)$$

Though the type is not so general, but it is a generalization of the one used by Yamada. By general property of demand curve, df_1/dq is negative. The function f_2 is proved to be an increasing function of s . By the way, we have $q = X$ when $p = 0$, that is,

$$f_1(x) + f_2(s) = 0.$$

Ordinary differentiation of the above by s yields

$$\frac{df_1}{dX} \frac{dX}{ds} + \frac{df_2}{ds} = 0,$$

So we have

$$\frac{df_2}{ds} = -\frac{df_1}{dX} \frac{dX}{ds}$$

in which the right hand side is always positive since we have $df_1/dX < 0$ and $dX/ds > 0$. It is clear that demand curves given by eq.(10) are in parallel with each other on p - q plane.

Putting eq.(10) into eqs.(8), (9) and equating each to zero give

$$(f_1 + f_2) - \mu \left\{ (f_1 + f_2) + \frac{df_1}{dq} q \right\} = 0$$

$$(1 - \mu) \left\{ \frac{df_2}{ds} q - C'(s) \right\} = 0.$$

The second equation above leads to

$$\frac{df_2}{ds} q - C'(s) = 0 \quad (11)$$

since we have $\mu \neq 1$ because $df_1/dq \neq 0$ in the first equation.

In general, point E_3 , the contact of general form of demand curve (3) and average cost curve (4) is on the locus that gives the maximum toll revenues. Toll revenues are given by $R = fq$, from which we have

$$\begin{aligned} \frac{\partial R}{\partial q} &= f + \frac{\partial f}{\partial q} q \\ &= f \left(1 + \frac{\partial f}{\partial q} \frac{q}{f} \right). \end{aligned}$$

At the contact, $f = \pi$ and $\partial f / \partial q = \partial \pi / \partial q$. Eq.(4) always gives $(\partial \pi / \partial q)(q/\pi) = -1$. After all the contact satisfies $\partial R / \partial q = 0$.

3. Extremum Condition to Consumers' Surplus

The relationship between extremum condition to consumers' surplus and that to diverted traffic is investigated in this section, when assumed demand curve of the type given by eq.(10).

Under equilibrium condition (6), eq.(5) is rewritten as follows

$$S = \int_0^q f(\xi, s) d\xi - f(q, s)q \quad (12)$$

Note that q in eq.(12) is a certain function of s satisfying equilibrium condition (6). Accordingly, S is, in itself, a function of s alone. Ordinary differentiation of S by s yields

$$\frac{dS}{ds} = \frac{\partial S}{\partial q} \frac{dq}{ds} + \frac{\partial S}{\partial s},$$

where

$$\begin{aligned} \frac{\partial S}{\partial q} &= f - (f + \frac{\partial f}{\partial q}q) \\ &= -\frac{\partial f}{\partial q}q. \end{aligned}$$

We have

$$\frac{dS}{ds} = -\frac{\partial f}{\partial q}q \frac{dq}{ds} + \frac{\partial S}{\partial s}. \quad (13)$$

The second term on the right hand side of eq.(13) is the marginal consumers' surplus to network expansion under equilibrium condition.

Noting that $\partial f/\partial q < 0$ in general, we can say that the extrema of consumers' surplus have, if any, to be found in such regions or at points that

$$\frac{dq}{ds} < 0 \quad \text{when} \quad \frac{\partial S}{\partial s} > 0$$

$$\frac{dq}{ds} > 0 \quad \text{when} \quad \frac{\partial S}{\partial s} < 0$$

$$\frac{dq}{ds} = 0 \quad \text{when} \quad \frac{\partial S}{\partial s} = 0.$$

It will depend upon the property of demand curve whether the marginal consumers' surplus to network expansion is positive, negative or zero.

Let's assume the type given by eq.(10), and we have

$$\frac{\partial f}{\partial s} = \frac{df_2}{ds}, \quad \frac{\partial f}{\partial q} = \frac{df_1}{dq}$$

The marginal consumers' surplus to expansion is expressed by

$$\frac{\partial S}{\partial s} = \int_0^q \frac{\partial f}{\partial s} d\xi - \frac{\partial f}{\partial s} q ,$$

which vanishes when used the first of the two equations obtained above.

Now we have from eq.(13)

$$\frac{dS}{ds} = - \frac{df_1}{dq} q \frac{dq}{ds} , \quad (14)$$

which implies that the extremum condition to consumers' surplus is equivalent to that to diverted traffic in equilibrium condition, since df_1/dq is negative. In another word, eq.(14) means that the optimal expansion of network is found at any one of the points that give the maximal diverted traffic in the same condition.

Reminding that q in eq.(14) satisfies eq.(6), we have by differentiating both sides of eq.(6) by s

$$(f + \frac{\partial f}{\partial q} q) \frac{dq}{ds} + \frac{\partial f}{\partial s} q = C'(s) .$$

So we have

$$\frac{dq}{ds} = \frac{C'(s) - (\partial f/\partial s)q}{f + (\partial f/\partial q)q} \quad (15)$$

where

$$f + \frac{\partial f}{\partial q} q \neq 0 . \quad (16)$$

Eq.(15) is a general requisite to diverted traffic in an equilibrium of toll revenues and service cost. The extremum condition to diverted traffic is given by

$$C'(s) - \frac{\partial f}{\partial s} q = 0 . \quad (17)$$

The above equation shows the natural fact that the marginal toll revenues to network expansion should be equal to the marginal service cost to network expansion at the extrema of diverted traffic in an equilibrium. The conclusion is that, when demand curve is assumed to be given by eq.(10), the extremum

condition to consumers' surplus is given by eq.(17) that is the same as eq.(11). Attention should be paid to the fact that the conclusion, by eq.(16), is valid somewhere else except for the contact points of demand and average cost curves.

The following is a specified demand curve adopted by Yamada.

$$p = \frac{1}{\alpha} \ln \frac{AX(s)}{q} \quad (18)$$

where α and A are constants. Eq.(18) is a special case of eq.(10) since it is obtained by putting in eq.(10)

$$f_1 = -\frac{1}{\alpha} \ln q, \quad f_2 = \frac{1}{\alpha} \ln AX(s) .$$

In this special case we have, correspondingly to eq.(11) or (17),

$$\frac{C'(s)}{X'(s)(q/X(s))} = \frac{1}{\alpha} . \quad (19)$$

This is a very implicative finding made by Yamada, since eq.(19) implies "that network expansion is optimized just when the marginal service cost to expansion, averaged to the marginal diverted traffic to expansion, be equal to the value of time saved by using expressway just by mean trip length." The same attention as mentioned before is to be paid as well in this case.

4. Special Case

In this section, assuming eq.(18) as demand curve, we are concerned with the condition under which an equilibrium can exist of toll revenues and service cost and with some illustrative presentation of various kinds of paths on which an equilibrium is kept. The illustrative work will give a visible explanation of what was discussed so far in the paper. Diversion ratio is newly introduced for the purpose.

4.1 Equilibrium Condition

Diversion ratio is defined by

$$r = \frac{q}{AX(s)}, \quad (20)$$

which is the ratio of diverted traffic to the whole population of expressway users. Though real diversion ratio is to be defined by $q/X(s)$, there is no loss of generality in defining the ratio by eq.(20) since A is constant.

Putting r defined above into eqs.(18) and (4), we have, respectively,

$$p = -\frac{1}{\alpha} \ln r, \quad (21)$$

$$\pi = \frac{D(s)}{r}, \quad (22)$$

where

$$D(s) = \frac{C(s)}{AX(s)}.$$

$D(s)$ is an average of service cost to the whole population of expressway users.

Equilibrium condition is expressed by

$$-r \ln r = \alpha D(s) \quad (23)$$

Let the solution of eq.(23) be written as $r = r(s)$. Differentiation of the both sides of eq.(23) by s gives

$$\frac{dr(s)}{ds} = -\frac{\alpha D'(s)}{1 + \ln r(s)}, \quad r(s) \neq e^{-1} \quad (24)$$

Some properties of the function $r = r(s)$ are made clear using eqs.(23) and (24). Those will depend on the characteristics of $\alpha D(s)$.

Eq.(23) gives the condition under which an equilibrium exists of toll revenues and service cost, that is, the following inequality is solvable as to s

$$\alpha D(s) \leq e^{-1} \quad (25)$$

since we have $-r \ln r \leq e^{-1}$ for any value of r ($0 \leq r \leq 1$).

The above condition is interpreted as follows; the minimum value of the ratio of an average of service cost to the whole population of expressway users to the value of time saved by using expressway just by mean trip length is less than or equal to e^{-1} . By the way, $1/\alpha$ is a constant defined, by Yamada, as the value of time by using expressway just by mean trip length [2, 3].

4.2 Equilibrium Path to Diversion Ratio

The function $r = r(s)$ that satisfies eq.(23) is called here equilibrium path to diversion ratio. The property of the path depends upon that of $\alpha D(s)$. A typical example is illustrated in the following.

Assuming that $D(s)$ is such a convex function that has a single minimum at $s = s_0$ with $\alpha D(s_0) \leq e^{-1}$ and $\alpha D(s) > e^{-1}$ as $s \rightarrow 0$ and ∞ . One of such cases are shown on $y \sim s$ plane in Fig.2. The case like this may be possible when service cost $C(s)$ is supposed to be a linear function of s with fixed cost while the whole population $X(s)$, with smaller value of $X(0)$, to increase increasingly as s increases to a certain value and decreasingly thereafter.

The curve $y = -r \ln r$ is shown on $r \sim y$ plane in Fig.2. Using two curves on $y \sim s$ and $r \sim y$ planes, we have equilibrium path to diversion ratio as shown on $r \sim s$ plane in Fig.2. In the following, some detailed investigation is made in the properties of the path $r = r(s)$ shown in Fig.2.

- (1) The two points $s = s_{\min}$ and s_{\max} are minimum and maximum feasible expansions of network, where diversion ratio is just equal to e^{-1} . Toll rate is given by $1/\alpha$ by eq.(21).
- (2) The points $s = s_{\min}$ and s_{\max} are both the contacts of demand and average cost curves given by eqs.(21) and (22), respectively.
- (3) By assumption that $D'(s) \neq 0$ at $s = s_{\min}$ and s_{\max} , we have

Even though the type of function $D(s)$ assumed above is considerably likely, another types of $D(s)$ are assumable according to the properties of $C(s)$ and $X(s)$. So long as we use demand curve as given by eq.(18), similar analysis is applicable to them.

4.3 Equilibrium Path to Diverted Traffic

Similar analysis is made of diverted traffic by use of $r = r(s)$ obtained in the preceding.

Diverted traffic is given by

$$q = Ar(s)X(s),$$

which, differentiated by s , yields

$$\frac{dq}{ds} = AX(s) \left\{ \frac{dr(s)}{ds} + r(s) \frac{X'(s)}{X(s)} \right\}.$$

Putting eq.(24) into the above, we have

$$\frac{dq}{ds} = AX'(s) \frac{r(s) - r_0(s)}{1 + \ln r(s)}, \quad r(s) \neq e^{-1} \quad (26)$$

where

$$r_0(s) = \frac{\alpha C'(s)}{A X'(s)}. \quad (27)$$

Note that $r_0(s)$ is the ratio of an average of the marginal cost to the marginal whole population of expressway users, both to network expansion, to the value of time saved by using expressway just by mean trip length.

The behavior of diverted traffic depends on those of $r(s)$ and $r_0(s)$. Note that $AX'(s) > 0$ in eq.(26).

We have the following inequalities

- (a) $r_0(s) < e^{-1}, \quad s = s_{\min}$
- (b) $r_{\min} < r_0(s) < e^{-1}, \quad s = s_0$
- (c) $r_0(s) > e^{-1}, \quad s = s_{\max}$

where r_{\min} is the smaller of the two solutions of eq.(23) at $s = s_0$. These are shown as follows. At the beginning, we have

$$\left. \begin{aligned} r_0(s) &< \alpha D(s), & s < s_0 \\ r_0(s) &= \alpha D(s), & s = s_0 \\ r_0(s) &> \alpha D(s), & s > s_0 \end{aligned} \right\} \quad (28)$$

These are obtained on the foregoing assumption that

$$D'(s) \begin{cases} < 0, & s < s_0 \\ = 0, & s = s_0 \\ > 0, & s > s_0 \end{cases}$$

where, by definition that $D(s) = C(s)/AX(s)$,

$$D'(s) = \frac{1}{\alpha} \frac{X'(s)}{X(s)} \{r_0(s) - \alpha D(s)\}, \quad X'(s) > 0$$

(a) $\alpha D(s) = e^{-1}$ at $s = s_{\min}$, while $r_0(s) < \alpha D(s)$, $s < s_0$ by the first of (28). Noting that $s_{\min} < s_0$, we have $r_0(s) < e^{-1}$, $s = s_{\min}$.

(b) $\alpha D(s) < e^{-1}$, $s = s_0$, while $r_0(s) = \alpha D(s)$, $s = s_0$ by the second of (28). And so we have $r_0(s) < e^{-1}$, $s = s_0$. Note that r_{\min} is smaller than $\alpha D(s_0)$ since, in general, the smaller one of the two solutions of $-r \ln r = B$ (an arbitrary positive constant) is smaller than B . After all we have $r_{\min} < r_0(s) < e^{-1}$, $s = s_0$.

(c) $\alpha D(s) = e^{-1}$, $s = s_{\max}$, while $r_0(s) > \alpha D(s)$, $s > s_0$ by the third of (28). Noting that $s_{\max} > s_0$, we have $r_0(s) > e^{-1}$, $s = s_{\max}$.

Inequalities (a), (b) and (c) suggest us that the curve $r = r_0(s)$ intersects, at least once, each of the upper and the lower halves, abd and acd, in regions $s_0 < s < s_{\max}$ and $s_{\min} < s < s_0$, respectively. Fig.3 shows such cases that there exists a single intersection of the curve $r = r_0(s)$ and each of the upper and the lower halves of equilibrium path.

In case of Fig.3, it follows from eq.(26) that diverted traffic finds its maximum and minimum at intersections on the upper and the lower halves of the path, respectively, since

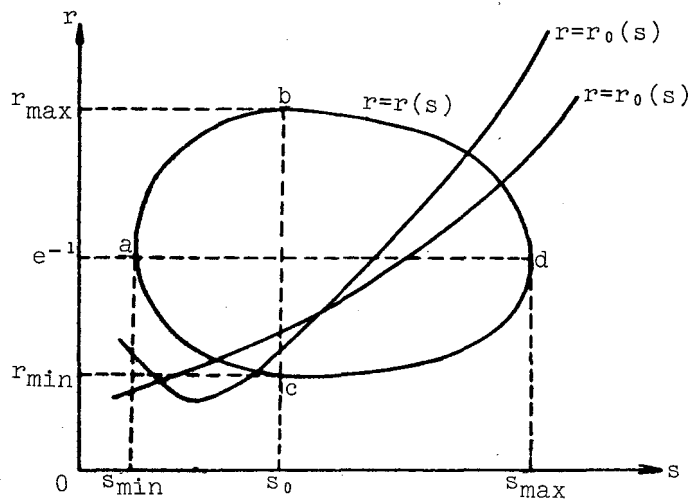


Fig.3 $r_0(s)$

the denominator on the right hand side of eq.(26) is positive on the upper half and negative on the lower.

Indeed there may exist many other cases of the curve $r = r_0(s)$ than the case as shown in Fig.3, but the preceding example is well enough to show equilibrium path to diverted traffic and the points where it reaches the extrema. Equilibrium path to diverted traffic is shown on $q \sim s$ plane in Fig.4, in which diverted traffic reaches its maximum and minimum at $s = s_2$ and s_1 , respectively. Note that the curve $r = r_0(s)$ intersects equilibrium path $r = r(s)$ at $s = s_2$ and s_1 .

4.4 Expansion Path on $p \sim q$ Plane

The locus of (p, q) that keeps an equilibrium of toll revenues and service cost is called here expansion path on $p \sim q$ plane.

From eqs.(21) and (24),

$$\frac{dp}{ds} = \frac{D'(s)}{r(s)(1 + \ln r(s))}, \quad r(s) \neq e^{-1}$$

Using the above and eq.(26), we have

$$\frac{dp}{dq} = \frac{1}{Ar(s)X'(s)} \frac{D'(s)}{r(s) - r_0(s)} \quad (29)$$

where $Ar(s)X'(s)$ is positive. On $r \sim s$ plane in Fig.4, $r(s) - r_0(s)$ is positive on the part g_1abg_2 of equilibrium path to diversion ratio and negative on g_1cdg_2 . By assumption $D'(s)$ is negative for $s < s_0$, positive for $s > s_0$ and equal to zero when $s = s_0$.

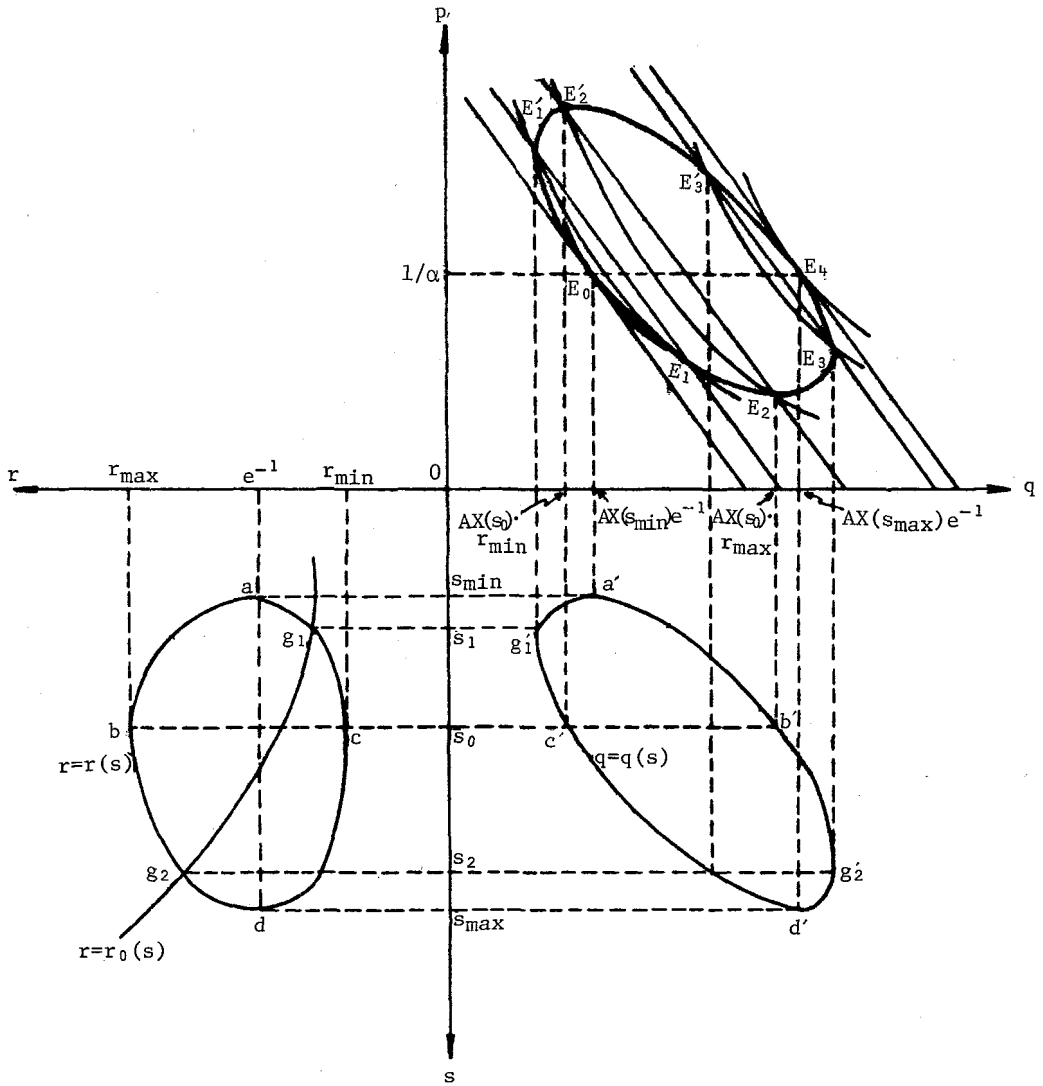


Fig.4 Diverted Traffic and Expansion Path

Expansion path is shown as $E_0E_1E_2E_3E_4E_3'E_2'E_1'$ on $p \sim q$ plane in Fig.4, together with equilibrium path to diverted traffic on $q \sim s$ plane. Some properties are as follows;

- (1) The upper and the lower halves, abd and acd , of equilibrium path on $r \sim s$ plane are corresponding to the paths $E_0E_1E_2E_3E_4$ and $E_0E_1'E_2'E_3'E_4$ on $p \sim q$ plane, respectively.
- (2) Points E_0 and E_4 , corresponding to points a and d on $r \sim s$ plane, respectively, are both contacts of demand and average cost curves. The two points correspond to the minimum and maximum feasible network expansions.
- (3) Points E_3 and E_1' are corresponding to g_2' and g_1' , respectively, on equilibrium path to diverted traffic. The points g_2' and g_1' give the maximum and the minimum diverted traffic, respectively, in an equilibrium of toll revenues and service cost.
- (4) The maximum and minimum toll rates are found at points E_2 and E_2' , respectively, on expansion path. Note that these rates occur at $s = s_0$ where $D'(s) = 0$. It is noticeable that at $s = s_0$ is minimized an average of service cost to the whole population of expressway users.

4.5 Maximum Consumers' Surplus

Consumers' surplus is maximized at point E_3 , which is corresponding to point g_2' where diverted traffic reaches its maximum. The corresponding network expansion is given by the larger one of the two solutions of

$$-r_0(s) \ln r_0(s) = \alpha D(s).$$

The larger one is shown as s_2 on s -axis in Fig.4. The corresponding toll rate is given by

$$p = -\frac{1}{\alpha} \ln r_0(s_2)$$

or equivalently

$$p = \frac{D(s_2)}{r_0(s_2)}$$

Some special case may happen that the maximum consumers'

surplus is reached at the contact point of demand and average cost curves. Fig.2 suggests that such a case occurs only when

$$\alpha D(s_0) = e^{-1},$$

which means that an equilibrium of toll revenues and service cost is kept at point $s = s_0$ alone. Such a case is trivial.

5. Conclusion

A brief summary of the paper is made in the following.

(1) The extremum condition to consumers' surplus is equivalent, in general, to the one to diverted traffic in an equilibrium of toll revenues and service cost, only when demand curve has such a property that the marginal consumers' surplus to network expansion vanishes.

(2) The extrema of diverted traffic in an equilibrium are reached, in general, when the marginal toll revenues to network expansion are equal to the marginal service cost to network expansion. The contacts of demand and average cost curves give, in general, no extrema of diverted traffic. Accordingly, the contacts give no extrema of consumers' surplus even though demand curve is such that the marginal consumers' surplus to expansion vanishes.

(3) In case that demand curve has such a property that the marginal consumers' surplus to network expansion does not vanish, the extrema of consumers' surplus are to be found in the regions of negative marginal diverted traffic to network expansion if demand curve yields positive marginal consumers' surplus, while in the regions of positive marginal diverted traffic if demand curve gives negative marginal consumers' surplus.

The followings are summary for specific demand curve used by Yamada.

(4) The foregoings (1) and (2) are valid for this type of demand curve since the marginal consumers' surplus to network expansion vanishes for the curve. A very implicative finding by Yamada that optimal expansion of network is reached just when the marginal service cost to expansion, averaged to the

marginal diverted traffic to expansion, be equal to the value of time saved by using expressway just by mean trip length holds good at any one of the points of the maximum diverted traffic, but not at the contacts of demand and average cost curves.

(5) The condition of an equilibrium of toll revenues and service cost to come into existence is that the minimum of the ratio of an average of service cost to the whole population of expressway users to the value of time saved by using expressway just by mean trip length is less than or equal to e^{-1} . Especially when the minimum is equal to e^{-1} , equilibrium of toll revenues and service cost is kept only at the contact of demand and average cost curves.

(6) The extrema of toll rate and diversion ratio is reached at the same time when an average of service cost to the whole population of expressway users reaches its extrema.

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