A General Purpose Simulator for Multivariable Control Systems

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Synopsis

A high speed and high accuracy simulator whose data-handling is light and whose weak stability is assured even for stiff systems is discussed.

This work is a part of CAD package : CADPACS-T which has been under development for the purpose of designing control systems utilizing TSS of a large scale digital computer in our laboratory.

This simulator realizes high accuracy, high compu-tational speed and high reliability adopting A-stable extrapolation methods to transform into a difference system, a recursive formula of an output interval to solve the difference equation, automatic choice of an optimal step length such that minimizes an predicted error, and evaluating error bounds closely near the actual errors in the point of numerical calculation.

Moreover, in the point of data management, this realizes separation of program modules from data, common utilization of data and separation of investi-gation from simulation.

1. Introduction

In the design problem of the controllers for multivariable sys-tems like industrial evaporators whose dynamics are perturbed by load changes and disturbances, trial and error-like procedures with com-

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puter simulations are inevitable.

For this purpose, heretofore a lot of CAD systems have been developed [1] and also recently, those on the basis of the state space method become to be developed. [2,3] But, these CAD systems attach importance to development and realization of synthesis theories for control systems.

Whereas, many of physical systems are stiff. Therefore, it is not necessarily easy to simulate these systems with high accuracy and low computational cost. For the purpose, the necessity of a high speed and high accuracy simulator whose data-handling is light and whose weak stability is assured even for stiff systems is being understood gradually.

This paper intends to discuss the structure and the function which a simulator for stiff multivariable systems should have, introducing the general purpose simulator which has been developed as a support ofCAD package : CADPACS-T [4] under development for the purpose of designing control systems utilizing TSS of a large scale digital computer.

Chapter 2 is concerned with the principle. Chapter 3 deals with the structure and the function. Chapter 4 illustrates the performance with some examples. Chapter 5 is concernes with the conclution.

2. Principle [5-7]

2.1 Object and Environment

Consider computational methods for multivariable systems which are described in a linear ordinary differential equation.
   SYSTEM

$$x(t) = A x(t) + B u(t) \qquad (1.a)$$

$$y(t) = C x(t) + D u(t) \qquad (1.b)$$

$$x \in R^n, \quad u \in R^{nu}, \quad y \in R^{ny}$$

Suppose that each component of the external input u(t) is represented in a piece-wise linear equation as eq.(2).
   INPUT

$$u^j(t) = \sum_i \{\alpha_i^j(t-\tau_i^j) + \beta_i^j\}\{1(t-\tau_i^j) - 1(t-\tau_{i+1}^j)\} \qquad (2)$$

$$(j = 1,\ldots,n_u)$$

where i(t) is unit step function.

Suppose that output times to observe are given in eq.(3) deter-
mined by intervals $[\sigma_i, \sigma_{i+1}]$ and output numbers L_i (i=0,1,...).

OUTPUT TIME

$$t \; \epsilon \; \{\sigma_i + l \, w_i \mid l = 0,1,\ldots,L_i \; ; \; w_i \equiv (\sigma_{i+1} - \sigma_i)/L_i\} \qquad (3)$$

for each $[\sigma_i, \sigma_{i+1}]$, L_i (i = 0,1,..,I-1)

2.2 Transform into Difference System

In choosing numerical methods three conditions must be taken into
consideration, namely, i) accuracy, ii) computational time (cost),
iii) weak stability. Especially, for stiff systems consideration for
weak stability is inevitable. Therefore, A-stable numerical methods
are favourable which assure weak stability for choosing any step
length as far as the original differential system is stable [8,9].

Well, when the external input in eq.(2) is represented in a mono-
linear equation as eq.(2)' by selecting an appropriate time interval
$[\tau_i, \tau_{i+1}]$, the difference equation obtained by applying an one-step
numerical method for the system in eq.(1), is represented as eq.(4).

In $[\tau_i, \tau_{i+1}]$,

$$u(t) = \{\alpha(t-\tau_i) + \beta\}\{1(t-\tau_i) - 1(t-\tau_{i+1})\} \qquad (2)'$$

DIFFERENCE SYSTEM

$$x_{k+1} = P\,x_k + Q\,u_k + R\,\alpha \qquad (4.a)$$

$$y_k \;\; = C\,x_k + D\,u_k \qquad (4.b)$$

where

$$P \equiv \exp(A\,h)\,, \qquad Q \equiv A^{-1}(P-I)\,B\,, \qquad (5)$$

$$R \equiv A^{-2}(P-I-A\,h)\,B$$

$$u_k = \alpha\,k\,h + \beta \qquad (6)$$

(h : step length , k = 0,1,2,...)

On the other hand, it has been shown by Axelsson that the differ-
ence system obtained by applying Padé approximation for $\exp(A\,h)$ as
eq.(7) or (8) to the equation $\dot{x} = A\,x$, is stable and is stiffly A-
stable in case of eq.(8) [9].

i) (μ,μ) Pade approximation   (order : 2μ)

$$Q_{\mu}^{-1}(\theta)\ P_{\mu}(\theta) = \exp(\theta) + o(\theta^{2\mu+1}) \tag{7}$$

where $P_{\mu}(\theta)$ and $Q_{\mu}(\theta)$ are μ-th order polynomial of θ, respectively, and θ represents Ah.

ii) $(\mu-1,\mu)$ Pade approximation   (order : $2\mu-1$)

$$Q_{\mu}^{-1}(\theta)\ P_{\mu}(\theta) = \exp(\theta) + o(\theta^{2\mu}) \tag{8}$$

Well, the case $\mu = 1$ in eq.(7) is known as Trapezoidal rule. We adopt the above method and call one of eq.(7) μ-stage trapezoidal method and call one of eq.(8) μ-stage Stiff method. Namely, adopting eq.(7) or eq.(8) as P in eq.(4),

  METHOD

 i) μ-T (A-stable)

$$P \leftarrow Q_{\mu}^{-1}(\theta)\ P_{\mu}(\theta)$$

 ii) μ-S (Stiffly A-stable)

$$P \leftarrow Q_{\mu}^{-1}(\theta)\ P_{\mu-1}(\theta) \qquad .$$

2.3 Numerical Method with Recursive Formula

    Generally, though break points τ_i^j in eq.(2) and output time intervals $[\sigma_i,\sigma_{i+1}]$ are in a complicate relation, selecting an appropriate interval set $\{[t_i,t_{i+1}]\ ;\ i=0,1,\dots\}$, input and output time can be represented in each interval as eqs.(9) and (10), respectively.
    Transforming input-output environment data into the above form, we arrange data to represent as eqs.(9) and (10) every interval in a table what we call SIMTB (canonical table for simulation environment)

  ENVIRON

    In each interval of SIMTB,

(INPUT)

$$u_k = \alpha k h + \beta \tag{9}$$

(OUTPUT TIME)

$$k_0\ \varepsilon\ \{m,2m,\dots,\ell m,\dots,Lm\},\quad m \equiv w/h \tag{10}$$

        (w; output observation interval, m : ineger)

    Now, in the eqs.(4) and (9) system, considering that we have only

to observe the outputs every m-step length represented as eq.(10), we can utilize the recursive formula eq.(11) with respect to the analytical solution on every m-step length.

RECURSIVE FORMULA

$$\xi_{l+1} = P^m \, \xi_l,$$ (11)

$$(l = 0, 1, \ldots, L-1)$$

where

$$\xi_l \equiv x_{m \, l} - (\Psi + l \, \phi)$$ (12.a)

$$\phi \equiv -A^{-1} B \, \alpha \, w$$ (12.b)

$$\Psi \equiv -A^{-1} (B\beta + A^{-1} B \, \alpha)$$ (12.c)

Notice that the computational time of P^m which is primary part of computational operations of eqs.(11) and (12), is approximately proportional to $\log_2(h^{-1})$, because expanding P^m with binary representation of m, P^m can be represented as eq.(13).

$$P^m = \prod_{i=0}^{\bar{m}} P^{b_i 2^i} \, , \quad (b_i = 0 \text{ or } 1)$$ (13)

where

$$\bar{m} \equiv [\log_2 m], \quad [\cdot] \; ; \text{ Gauss bracket.}$$

That is to say, in case of $m = 2^{\bar{m}}$, P^m can be calculated by $\bar{m} + 1$ times multiplication and $m = w/h$.

2.4 Choice of Step Length

Usually, the step length h is chosen such that the local truncation error is acceptably small. But, it is still essential to choose h such that the total error consisting of the truncation error and the round off error on every m-step length is acceptably small or minimum. Generally, it is difficult to choose an adequate h on the above step-control policy and also it is necessary to consider the computational cost.

In our method, as the computational time is approximately proportional to $\log_2 h^{-1}$, a sufficiently small h can be chosen. Therefore, we adopt the optimal step length h_{opt} which minimizes a predicted error bound for each interval in SIMTB as discussed after. Practi-

cally, as m must be equal to $2^{\bar{m}}$ ($\bar{m}$: integer) such that computational time of P^m is to be minimum, adopting h_{opt} as a desired h, we determine h as $w/2^{\bar{m}}$ with $\bar{m}$ satisfying eq.(14). In this time, the relation (15) is valid.

$$\log_2 (w/hd) \leqq \bar{m} \leqq \log (w/hd) + 1 \qquad (14)$$

$$hd/2 < h \leq hd \qquad (15)$$

2.5 Estimation of Error Bound

Paying attention to the truncation error and the round off error in transforming into a difference system with a numerical method, we now consider to estimate the error bound of the above numerical method using the recursive formula.

Now, representing P in eq.(4) with the true value $\bar{P}$, truncation error E_{pt} and round off error E_{pr} as eq.(16), consider the approximate error of P^m.

$$P \equiv \bar{P} + E_{pt} + E_{pr} \qquad (16)$$

The truncation error E_{pt} is represented by the principal local truncation error as eq.(17).

$$E_{pt} \equiv C_{p+1}(A h)^{p+1} \qquad (17)$$

where

    p   ; the order of rational approximation transforming into
         a difference system,
    C_{p+1} ; (p+1)-th error constant,

The round off error E_{pr} is represented as eq.(18) with the relative machine precision.

$$E_{pr} \equiv (\varepsilon_{ij}P_{ij}) \ , \ |\varepsilon_{ij}| \leq \hat{\varepsilon}_m \ , \ P_{ij} = (P)_{ij} \qquad (18)$$

where

$$\hat{\varepsilon}_m \equiv \max_\varepsilon \{\varepsilon | 1+\varepsilon=1 \ ; \ \text{internal expression in the machine}\}$$

Let the truncation error of P^m be E_{pmt}. Considering exchangeability of $\bar{P}$ and E_{pt}, E_{pmt} is easily expressed as eq.(19).

$$E_{pmt} = m \bar{P}^{m-1} E_{pt} \qquad (19)$$

On the other hand, as $\bar{P}$ and E_{pr} are not exchangeable, it is difficult to estimate the effect of round off error of P^m. Then, in

general, let express a round off matrix Eϵ for a matrix E as eq.(20).

$$E\epsilon \equiv (\epsilon_{ij} e_{ij}) \quad ; \quad E \equiv (e_{ij}) \ \epsilon \ R^{n \times m} \tag{20}$$

$$|\epsilon_{ij}| \leq \hat{\epsilon}_m \quad ; \quad i = 1, \ldots, n \quad ; \quad j = 1, \ldots, m$$

And let introduce the idea of the following bound round off matrix.

[Definition]

For a matrix X and a round off matrix Eϵ, we define the round off matrix Fϵ satisfying eq.(21) as the bound round off matrix for X Eϵ. And let express the relation of eq.(21) as $F\epsilon \supseteqq X E\epsilon$.

$$\max_\epsilon |(F\epsilon)_{ij}| \ \overset{\geqq}{=} \ \max_\epsilon |(X E\epsilon)_{ij}| \tag{21}$$

$$\text{for all } i, j$$

When the matrix composed with the absolute value of every component of a matrix E is put a sign ^ as eq.(22), eq.(23) is valid.

$$\hat{E} \equiv (|e_{ij}|) \quad ; \quad \hat{E}\epsilon \equiv (\epsilon_{ij}|e_{ij}|) \tag{22}$$

$$\hat{X} \hat{E}\epsilon \ \supseteqq \ X E\epsilon \quad , \quad \hat{E}\epsilon \hat{X} \ \supseteqq \ E\epsilon X \tag{23}$$

When the operations involving round off matrix are replaced by those of estimating maximum error ranges with bound round off matrix like eq.(23), eq.(24) is valid for the round off matrix E_{pmr} of P^m.

$$E_{pmr} \ \supseteqq \ m \ \hat{P}\hat{\epsilon}^m \tag{24}$$

Well, when it is taken into consideration that ϕ and Ψ are not influenced by the error in P and that practically, h<<1, m>>1 and so P nearly equals to unit matrix I, the truncation error $e_{xt,l}$ and the round off error $(e_{xr,l})\epsilon$ of the state x_{ml} at ml steps are approximately calculated from the following recursive formulas.

$$e_{xt,l+1} \ \simeq \ P^m (e_{xt,l} + m E_{pt} \xi_l) \tag{25}$$

$$\hat{e}_{xr,l+1} \ \simeq \ P^m \hat{e}_{xr,l} + m \hat{P}^m \hat{\xi}_l \tag{26.a}$$

$$(e_{xr,l})_\epsilon \equiv (\epsilon_i \hat{e}_{xr,l}^i), \qquad |\epsilon_i| \ \leq \ \hat{\epsilon}_m \tag{26.b}$$

Therefore, the output bound error e_{yl} of $y_{m,l}$ is obtained as eq.(27) every component.

$$e_{y,l}^j = |c_j e_{xt,l}| + \hat{c}_j \hat{e}_{xr,l} \hat{\epsilon}_m \qquad (27)$$

$$(j = 1,\ldots,n_y)$$

where c_j is the j-th row vector of C in eq.(4).

2.6 Optimal Step Length

To adopt an adequate step length minimizes the output error (in a norm with an appropriate weight) at some point in each interval of SIMTB. We call such a step length an optimal step length h_{opt}.

Well, in the above estimation of the error bound, the step length needs to be already given in the stage when the recursive formulas of the state and the error are carried out. Therefore, from the results we can not determine the value of the step length. However, if the canonical form of the system eq.(1) is given which is obtained by simular transformation with the eigenvectors of A in eq.(1), the calculations of eqs.(25) - (27) are essentially replaced by those of scalar variables and the output bound error $e_{y,w}$ at time w can be represented as eq.(28).

$$e_{y,w}^j = e_{t_0}^j + e_{r_0}^j + w h^p f_t^j + w h^{-1} f_r^j \qquad (28)$$

where

$$e_{t_0}^j \equiv |\textstyle\sum c_{ji} e_{x_i t,0} \exp(\lambda_i w)| \qquad (29.a)$$

$$e_{r_0}^j \equiv \hat{\epsilon}_m \textstyle\sum |c_{ji} \hat{e}_{x_i r,0} \exp(\lambda_i w)| \qquad (29.b)$$

$$f_t^j \equiv |C_{p+1} \textstyle\sum c_{ji} \lambda_i^{p+1} \xi_{i_0} \exp(\lambda_i w)| \qquad (29.c)$$

$$f_r^j \equiv \hat{\epsilon}_m \textstyle\sum |c_{ji} \xi_{i_0} \exp(\lambda_i w)| \qquad (29.d)$$

$$(j = 1,\ldots,n_y)$$

where $\sum$ implys the sum over 1 to n w.r.t. i.

When we express the norm with a weight γ of $e_{y,w}$ as $\|e_{y,w}\|_\gamma$, the optimal step length h_{opt} is obtained from eq.(28) as eq.(30)

$$h_{opt} = \{\|f_r\|_\gamma / p\|f_t\|_\gamma\}^{1/(p+1)} \qquad (30)$$

where f_t and f_r are n_y-th order vector consisted of f_t^j and f_r^j of eq.(29), respectively.

3. Structure and Function

We now present the structure of the simulator which is a realization of the above simulation principle. Fig.1 shows the structure of the simulator module in CADPACS-T and Table 1 explains functions of modules or commands in Fig.1.

The structure shown in Fig.1 is built up on the following four bases.

i) to separate program modules from data,

ii) to make commonly use of data by some modules,

iii) to separate investigation of results from each simulation,

iv) to manage data attaching importance to relation between many kinds of data,

i) is for general purpose, ii) avoids duplication of data and repetition of program execution, iii) feeds more efficient information from comparison of results on some different conditions and iv) releases from troublesome data-handling and enables to make up detail reports with respect to results automatically.
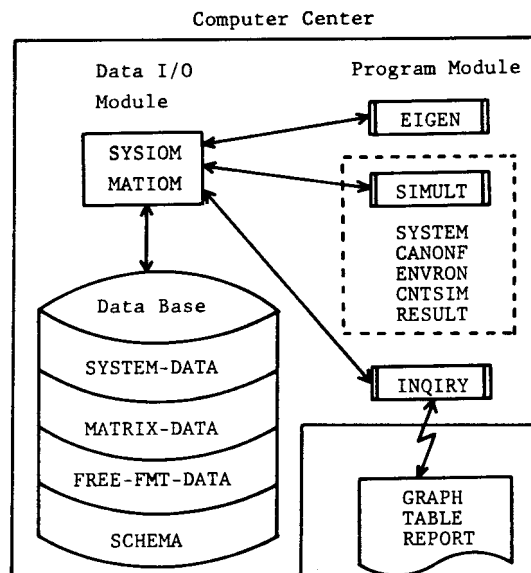

Fig.1 Structure of Simulator Module in CAD Package : CADPACS-T

The structure of CNTSIM shown in Table 1 is composed on the following three bases.

i) to asure high accuracy through all intervals of SIMTB by choosing the optimal step length which minimizes numerical errors consisted of truncation errors (mono-decrease with h) and round off errors (mono-increase with h) each interval of SIMTB.

ii) to realize high speed computation by minimizing the number of computational operations using a m-step wise recursive formula and PRESIM performing common calculations through each interval.

iii) to improve reliability of simulation results by obtaining an actual error on each outputtime with EBOUND (option).

Table 1 Description of Simulation Module
in CADPACS-T

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┌─────────────────────────────────────────────────┐
│   Functions of Program Modules                   │
│ [1] EIGEN  ; to obtain eigenvalues and eigenvectors │
│ [2] SIMULT ; to obtain time response of a system │
│ [3] INQIRY ; to display simulation or analysis   │
│             results and report                   │
│   Functions of Commands in SIMULT                │
│ 1) SYSTEM ; to input system data                 │
│ 2) CANONF ; to make canonical form for error bound │
│             with eigenvalues and eigenvectors    │
│ 3) ENVRON ; to make SIMTB with input-output      │
│             environment data                     │
│ 4) CNTSIM ; to solve linear differential equation │
│ 5) RESULT ; to store simulation results          │
│   Functions of Subroutines in CNTSIM             │
│ 1. PRESIM ; to prepare with calculations         │
│      each interval of SIMTB                       │
│ 2. HOPTIM ; to determine optimal step length     │
│ 3. TRAPEZ ; to transform into difference equation │
│ 4. SIMSUB ; to set up recursive formula and to   │
│             carry out                            │
│ 5. EBOUND ; to estimate error bound (option)     │
└─────────────────────────────────────────────────┘
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4. Performance and Example

To illustrate the performance of SIMLT, some simulation results with respect to the step length vs. actual and estimated bound errors are shown.

1) A 5th order canonical form system (single output)
  Initial Value Problem

Fig.2 shows the simulation results w.r.t. step length vs. actual and bound errors for the system eq.(31).

$$x(t) = A\,x(t) \quad; x(0) = x_0$$

$$y(t) = C\,x(t)$$

(31)

where $A \equiv$ block $\mathrm{diag}\left(-100, \begin{bmatrix} -3, & 3 \\ -3, & -3 \end{bmatrix}, \begin{bmatrix} -0.3, & 0.3 \\ -0.3, & -0.3 \end{bmatrix}\right),$

$c_1 x_{10} = 1, \ c_2 x_{20} = 0, \ c_3 x_{30} = 1, \ c_4 x_{40} = 1/2, \ c_5 x_{50} = \sqrt{3}/2 \ .$

2) A 20th order system (two inputs - one output)

Consider the two inputs - one output system whose transfer function is given by eq.(32).

$$G(s) = [G_1(s)\ G_2(s)] \tag{32.a}$$

$$G_i(s) \equiv \frac{1}{2}\sum_{k=1}^{n_p}(\frac{rik}{s-\lambda_k} + \frac{rik^*}{s-\lambda_k^*}) \tag{32.b}$$

$$(i = 1,\ 2)$$

where the set of eigenvalues $\{\lambda_k,\lambda_k^*\}$, k=1,...,$n_p$ is given with Table 2*[10] (n_p = 15, n = 20), and the set of residues r_{ik}, i=1,2 ; k=1,...,n is given by eq.(33) excepting that $r_{ik}\equiv f_{ik}/n_p$ in the case of the initial value problem.

$$r_{ik} \equiv \frac{1}{n_p}(-\lambda_k)f_{ik}\ ,\quad (i = 1,2\ ;\ k = 1,...,n_p) \tag{33.a}$$

$$f_{ik} \equiv 1.0 + 0.1\ (-1)^{i+k}\ ,\quad (k = 1,...,n_p-1) \tag{33.b}$$

$$f_{i\,n_p} = 1.0\ ,\quad (i = 1,\ 2)$$

Table 2 Pole's set of the 20-th order system

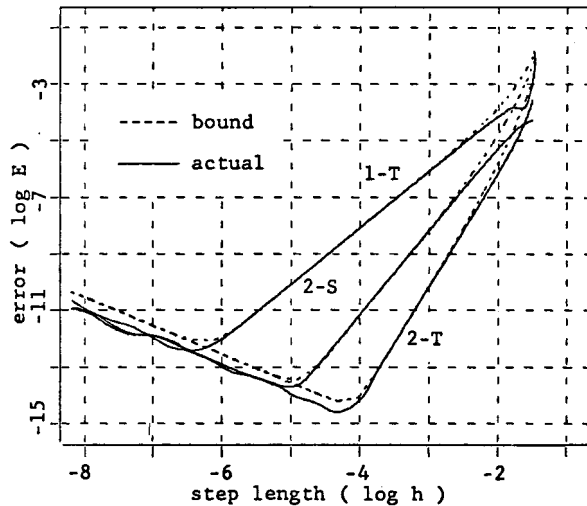| k	real	imag.
1	-0.4371	0.
2	-0.3329	0.
3	-0.8020	±0.1547
4	-1.0925	0.
5	-1.1547	±0.3178
6	-1.8294	0.
7	-1.5163	±1.8327
8	-2.9843	0.
9	-3.5710	0.
10	-4.0642	0.
11	-3.2541	±2.3971
12	-9.0686	±5.2021
13	-48.273	0.
14	-209.29	0.
15	-328.96	0.

Fig.2 Step length vs. actual and bound errors with numerical methods (1-T, 2-S, 2-T) : W = 0.1

Here, eqs.(32) and (33) are for the sake of analytical solution to calculate the actual error, and for the simulation, the system is used which is minimum realized with the observable canonical form.

* This table refers from Table 2 in P.223 of [10].

Initial Value Problem

Fig.3 shows the simulation results w.r.t. step length vs. actual and bound errors for the system eqs. (32) and (33) with the initial value which is the map in the state space from $u(t) = (1/2, 1/2)^T \delta(t)$.

Response with External Input

Fig.4(b) shows the simulation results for the system eqs.(32) and (33) under the environment which is given with the input shown in Fig.4(a) and the output time that the end points σ_i, i=0, 1,...,5 of the output request intervals are 0, 0.2, 2.5, 5.0, 7.5 and 10.0, respectively and the number L_i of each interval is 20 in eq.(3).

By the way, h_{opt} determined with eq.(29) is adopted as the desired step length, and the first division time w of each interval is adopted as the time which the estimated error ought to be minimized at. And also, the computational cpu time required for this simulation is approximately 5.6 seconds with double precision on the NEAC Acos-700S.

From these results, it is shown that SIMULT in CADPACS-T has excellent performance at accuracy,
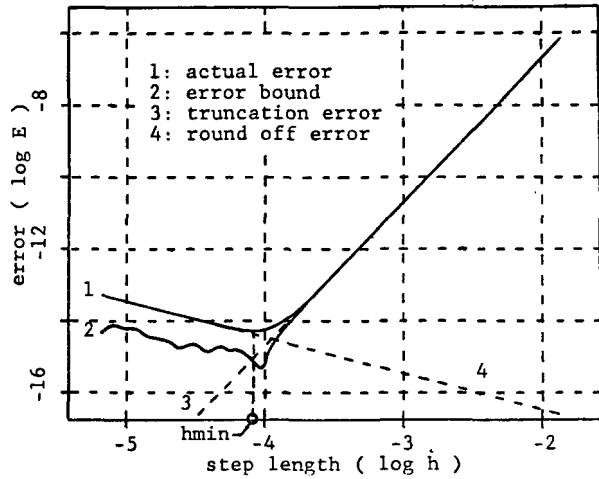

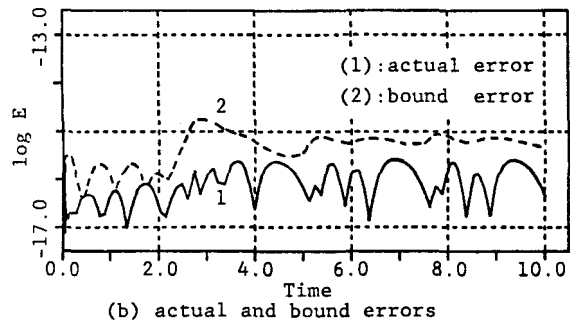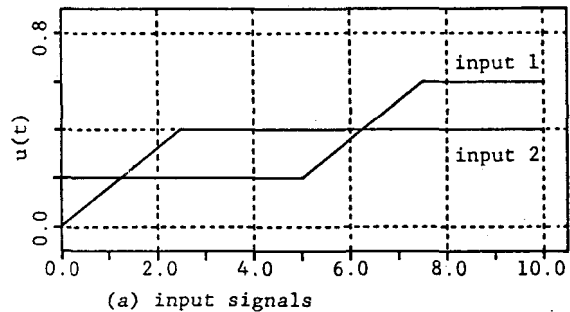Fig.3 Step length vs. actual and bound errors with method (2-T) : W = 0.1

(a) input signals

(b) actual and bound errors

Fig.4 Results of simulation for the 20-th order system (observable canonical form).

hd = min{h_{opt}|error bound at t = w}

computational speed and reliability.

 3) A 23th order actual system

    The object system is a drum type boiler with the steam flow rate
of 200 ton/hour. This state space model is represented as the 23th
order linear system by applying 2nd order Padé approximation to pure
lag elements [10].

    Constructing the closed-loop system with a state feedback optimal
control law for this object, we performed simulations for the optimal
regulator system under a step load change and obtained the transient
responses. Fig.5* shows the graphic outputs of the controlled out-
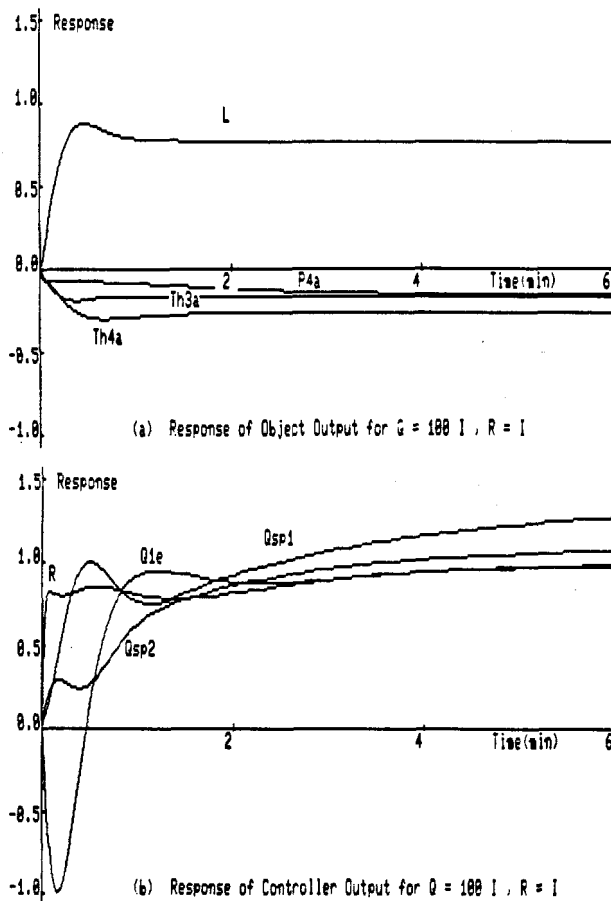puts and the actuating signals of the results.

(a) Response of Object Output for $G = 100\ I$, $R = I$

(b) Response of Controller Output for $Q = 100\ I$, $R = I$

Fig. 5   Response of Optimal Regulator System
         for $Q = 100\ I$, $R = I$

* Fig.5 corresponds to Fig.9 of the reference [10].

5. Conclusion

This paper discussed the principle, the structure and the function of the general purpose simulator which was developed as a support of CAD package : CADPACS-T under development for designing control systems. And some examples illustrated the performance.

The fundamental functions of a simulator seem to be achieved.

In the future, it seems that it is important to improve the structure so as to allow more general external inputs for simulation of a partial control system, and to accumulate know-how like a balance operation [11] in order to reduce the equivalent error level.

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