

Analysis of Superconducting Microstructures: Critical Temperature of Two-Dimensional Structures

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SYNOPSIS

Critical temperatures of two-dimensional microstructures with superconducting proximity effect in the dirty limit are evaluated for various geometrical constructions. As a numerical method, the finite element method is applied. Guidelines in estimating critical temperatures are given for the case where the decay of superconducting order parameter is either sufficiently slow or fast in comparison with the scale length of the structure.

I. INTRODUCTION

The critical parameters of microstructures including superconductors are key quantities in their applications to electronic devices.¹ They are related to the geometry of the structure as well as the bulk properties of constituent materials. Their dependency on the size and other characteristics is also of interest from theoretical point of view.

Among important cases of realistic applications, there may be the one where the behavior of the superconducting order parameter is described by a relatively simple equation in the dirty limit.^{2,3}

The analyses of critical parameters of such microstructures so far have been done mainly in one-dimensional cases.⁴ The purpose of this paper is to extend them to the two-dimensional cases where geometrical effects may play a more important role due to increased freedom of shape of the structure.

This is the third of a series of papers on related subjects: Preceding papers will be cited as I³ and II.⁵ In I, the superconducting proximity effect has been revisited and generalized to inhomogeneous media, and in II, a numerical method has been formulated. We here confine ourselves within the critical temperature without magnetic field. The critical magnetic field will be discussed in a forthcoming paper.

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II. METHOD OF ANALYSIS

We follow the method described in II and will not reproduce the formulation. We also use the same notations unless defined otherwise here.

At the temperature T , the characteristic length ξ of the diffusion process with the diffusion coefficient D_0 is given by

$$\xi = \left(\frac{\hbar D_0}{2\pi k_B T} \right)^{1/2} \quad (2.1)$$

For a structure with the scale of length L_0 , the ratio ξ/L_0 determines the behavior of the order parameter. We use the parameter η defined by

$$\eta = 2\pi(\xi/L_0)^2 \quad (2.2)$$

to express the effect of this ratio.

The diffusion coefficient D_0 is related to the mean free path l as

$$D_0 = lv_F/3, \quad (2.3)$$

where v_F is the Fermi velocity. Thus the parameter η is expressed by the mean free path as

$$\eta = \frac{1}{3} \frac{l}{L_0} \frac{v_F(\hbar/k_B T)}{L_0}. \quad (2.4)$$

Though the mean free path l is much smaller than the system size L_0 , the ratio $v_F(\hbar/k_B T)/L_0$ can be larger than unity and the resultant ratio η may take values up to, say, 0.5: For example, when $L_0 \sim 10^{-5} \text{cm}$, $l \sim 10^{-6} \text{cm}$, $v_F \sim 10^8 \text{cm/s}$, and $T \sim 10 \text{K}$, $v_F(\hbar/k_B T)/L_0 \sim 10^{-3}$ and $\eta \sim 0.3$. Note that the parameter η is smaller, but not always much smaller, than unity.

For a bulk superconductor with the density of states N_0 and the effective interaction (attraction) between electrons V_0 , the critical temperature $T_c^{(bulk)}$ given by the BCS theory is

$$T_c^{(bulk)} = 1.13\Theta_D \exp(-1/V_0 N_0), \quad (2.5)$$

where Θ_D is the Debye temperature. We consider microstructures composed of this superconductor and other materials such as normal metals and superconductors with lower critical temperatures.

When we normalize the temperature T by $T_c^{(bulk)}$ as

$$t = T/T_c^{(bulk)}, \quad (2.6)$$

the equation which determines the critical parameters is written as⁵

$$\det | C_{nn'} | = 0, \quad (2.7)$$

with

$$C_{nn'} = \delta_{nn'} - V_0 N_0 A_n \int d\mathbf{r} \nu(\mathbf{r})^2 \phi_n(\mathbf{r}) \nu(\mathbf{r}) \phi_{n'}(\mathbf{r}) / \left[\int d\mathbf{r} \nu(\mathbf{r}) \phi_n(\mathbf{r})^2 \int d\mathbf{r}' \nu(\mathbf{r}') \phi_{n'}(\mathbf{r}')^2 \right]^{\frac{1}{2}}, \quad (2.8)$$

$$A_n = \frac{1}{V_0 N_0} + \ln\left(\frac{1}{t}\right) + \ln\left(\frac{\varepsilon_n + \theta_D}{\theta_D}\right) + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{\eta \varepsilon_n}{4\pi}\right). \quad (2.9)$$

Here $\phi_n(\mathbf{r})$ is the eigenfunction of an operator \hat{L} with the eigenvalue E_n as

$$[\hat{L} - E_n] \phi_n(\mathbf{r}) = 0, \quad (2.10)$$

$$\hat{L} = -\frac{\hbar}{N(\mathbf{r})} \nabla \cdot D(\mathbf{r}) N(\mathbf{r}) \nabla, \quad (2.11)$$

$D(\mathbf{r})$ being the diffusion tensor, and $N(\mathbf{r})$, the local density of states.

In eq.(2.8) the effective interaction between electrons $V(\mathbf{r})$, the eigenvalue E_n , and $N(\mathbf{r})$ are normalized as

$$v(\mathbf{r}) = V(\mathbf{r})/V_0, \quad (2.12)$$

$$\nu(\mathbf{r}) = N(\mathbf{r})/N_0, \quad (2.13)$$

$$\varepsilon_n = E_n / \left(\frac{\hbar D_0}{L_0^2} \right). \quad (2.14)$$

The parameter θ_D is defined by

$$\theta_D = k_B \Theta_D / (\hbar D_0 / L_0^2) \quad (2.15)$$

and its typical value may be of the order of 10^2 : For example, $\Theta_D \sim 3 \cdot 10^2 K$, $L_0 \sim 10^{-5} cm$, $l \sim 10^{-6} cm$, and $v_F \sim 10^8 cm/s$ give $\theta_D \sim 120$.

From eq.(2.9) we see that the properties of microstructures related to the superconducting proximity effect are determined by the parameter η and the geometrical characteristics of the structure, as well as the obvious parameters such as V_0 , N_0 , and Θ_D .

III. QUASI-ONE-DIMENSIONAL STRUCTURES

For a cylinder composed of layers stacked in the direction perpendicular to the cross section, the critical temperature is exactly the same as the one for the one-dimensional system with corresponding structure. The order parameter given for the one-dimensional system becomes the solution for the cylinder when its value outside the cylinder is defined to be zero. The boundary condition at the surface is that the normal derivative vanishes. It is clear that one-dimensional solutions limited within the cylinder satisfy the condition at the surface along the side, as well as on the top or bottom, of the cylinder. We note that the above statement holds for any shape of the cross section of the cylinder.

There have been considerable amount of investigations on the one-dimensional structures⁴ and the possibility of controlling superconducting channel based on the proximity effect has also been discussed.⁶

In this case, the critical temperature is determined by the ratio of the super part in the system and is decreased with its decrease.

As is expected naturally, the critical temperature is also reduced with the increase of the parameter η : Larger value of η enables the Cooper pairs to penetrate to the normal (or weakly superconducting) part and has the effect of lowering the critical temperature, since the effective interaction between electrons is not (or only weakly) attractive in the latter part.

An example of the dependence of the critical temperature on the ratio of the super part and the parameter η is shown in Fig.1a. In this example, the structure is a bilayer composed of super and normal metals, with effective interactions V_0 and 0, respectively, and the density of states and the diffusion coefficient is assumed to be uniform in the whole system. We see that the critical temperature changes as expected.

In all the results presented in this paper, we specify the value of η by η_0 at the bulk critical temperature $T_c^{(bulk)}$:

$$\eta_0 = \eta(T = T_c^{(bulk)}). \quad (3.1)$$

It is therefore to be noted that the actual value of η at the critical point T_c is larger than η_0 by the factor of $T_c^{(bulk)}/T_c$.

In Fig.1b we plot the critical temperature as functions of the ratio in the case where the density of states of the normal part is changed. We have higher critical temperatures when the density of states is larger in the super part. This tendency may also explained by the localization of the Cooper pair in the super part in these cases.

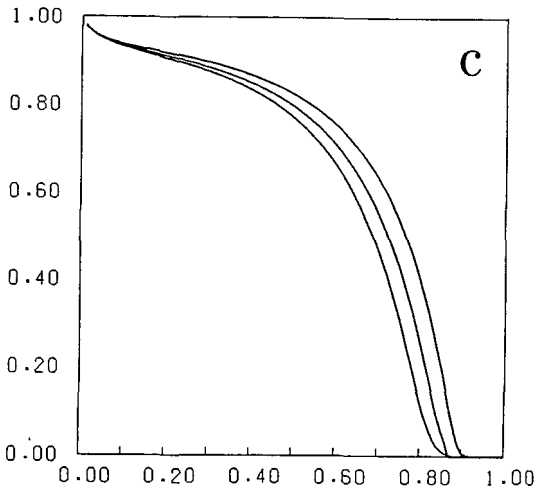
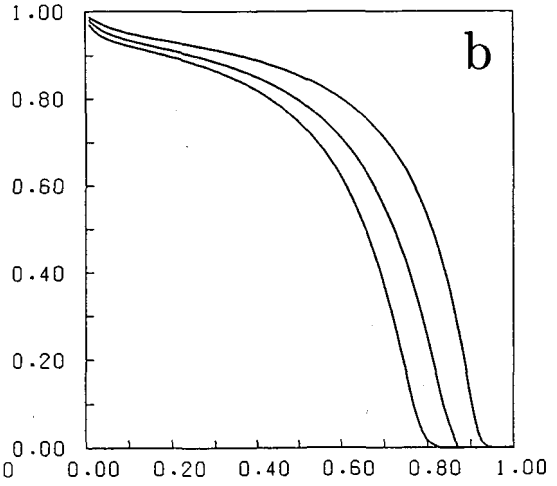
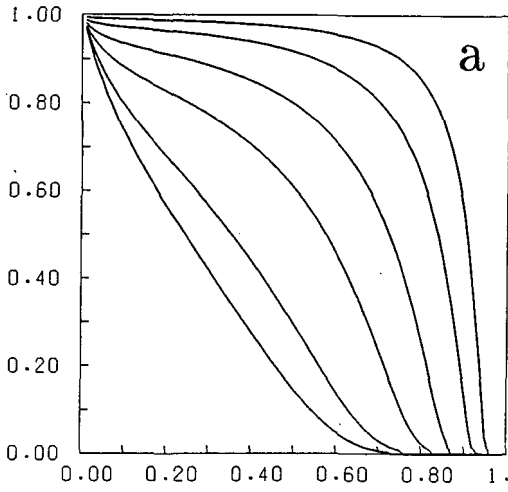


Fig.1a. Critical temperature (normalized by bulk value) of bilayers composed of superconductor and normal metal vs. ratio of super part. Values of η_0 are, from top, 0.03, 0.05, 0.1, 0.2, 0.5, and 1. Density of states and diffusion coefficient are uniform in the system.

Fig.1b. The same as Fig.1a for $\eta_0 = 0.1$ with different values of density of states in two parts; super to normal ratio is 2(top), 1(middle), or 0.5(bottom).

Fig.1c. The same as Fig.1a for $\eta_0 = 0.1$ with different values of diffusion coefficient in two parts; super to normal ratio is 2(top), 1(middle), or 0.5(bottom).

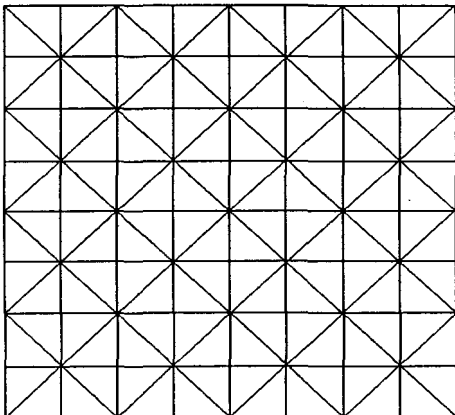


Fig.2. Division into triangular elements.

In Fig.1c we plot the behavior of the critical temperature when the diffusion coefficient in the normal part has different values compared with that in the super part. In this case, the change of the coefficient has little effect on the critical temperature. In contrast to the situation where the density of states is directly responsible for the critical temperature as in eq.(2.5) and is related to the continuity of the value of the order parameter at interfaces, the diffusion coefficient affects the order parameter only through its derivative at interfaces and has small effect on the critical behavior.

IV. TWO-DIMENSIONAL STRUCTURES

For two-dimensional structures, it is difficult to calculate the eigenfunctions of \hat{L} analytically, especially in the case of complex geometries. In order to analyze the geometrical effect, we therefore use the finite element method as described in II. We here repeat only the essential equations.

A. Division into Elements

In this analysis, we consider the structure where the superconductor is placed in the central part and the normal metal (or the weak superconductor) surrounds it, forming a square as a whole. Viewed in three dimensions, our system may be considered as a wire with a square cross section. When the density of states and the diffusion coefficient is constant throughout the system, the eigenfunctions are analytically written down and provide a reference for confirmation of the accuracy of the numerical procedures.

We divide the domain of our interest into triangular elements as shown in Fig.2. In this process, we keep all symmetry properties, such as symmetry axes and mirror planes, of the system.

B. Shape Functions

We use the third-order Hermite interpolations as shape functions. Among various possibilities, we adopt the shape functions for the element with vertices P_1 , P_2 , and P_3 given by

$$N_1 = \xi_1 (3\xi_1 - 2\xi_1^2 - 7\xi_2\xi_3), \quad (4.1)$$

$$N_2 = \xi_2 (3\xi_2 - 2\xi_2^2 - 7\xi_3\xi_1), \quad (4.2)$$

$$N_3 = \xi_3 (1 + \xi_1 + \xi_2 - 2\xi_1^2 - 2\xi_2^2 - 11\xi_1\xi_2), \quad (4.3)$$

$$N_G = 27\xi_1\xi_2\xi_3, \quad (4.4)$$

$$N_{23} = (x_{23}^2 + y_{23}^2)^{1/2} \xi_2\xi_3(\xi_2 - \xi_1), \quad (4.5)$$

$$N_{32} = (x_{23}^2 + y_{23}^2)^{1/2} \xi_2 \xi_3 (\xi_3 - \xi_1), \quad (4.6)$$

$$N_{31} = (x_{31}^2 + y_{31}^2)^{1/2} \xi_3 \xi_1 (\xi_3 - \xi_2), \quad (4.7)$$

$$N_{13} = (x_{31}^2 + y_{31}^2)^{1/2} \xi_3 \xi_1 (\xi_1 - \xi_2), \quad (4.8)$$

$$N_{12} = (x_{12}^2 + y_{12}^2)^{1/2} \xi_1 \xi_2 (\xi_1 - \xi_3), \quad (4.9)$$

$$N_{21} = (x_{12}^2 + y_{12}^2)^{1/2} \xi_1 \xi_2 (\xi_2 - \xi_3). \quad (4.10)$$

Here ξ_i is the area coordinates and $x_{ij} = x_i - x_j$, *etc.* These functions are characterized by four values, at three vertices and the center of mass P_G , together with two derivatives, along two sides, at each vertex;

$$N_i(P_j) = \delta_{ij}, \quad N_i(P_G) = \mathbf{p}_{ij} \cdot \nabla_\xi N_k(P_l) = 0, \quad (4.11)$$

$$N_G(P_G) = 1, \quad N_G(P_i) = \mathbf{p}_{ij} \cdot \nabla_\xi N_k(P_l) = 0, \quad (4.12)$$

$$\mathbf{p}_{ij} \cdot \nabla_\xi N_{ji}(P_k) = \delta_{jk}, \quad N_{ij}(P_k) = N_{ij}(P_G) = 0, \quad (4.13)$$

where

$$\mathbf{p}_{ij} = (\mathbf{P}_i - \mathbf{P}_j) / |\mathbf{P}_i - \mathbf{P}_j|, \quad (4.14)$$

and \mathbf{P}_i is the position of the vertex P_i , and $i, j, k, l = 1, 2, 3$.

C. Galerkin Equation

We express the solution as a superposition of the above shape functions for the element (l) as

$$\phi = \sum_{l=1}^n \left[N_1^{(l)} N_2^{(l)} N_3^{(l)} N_G^{(l)} N_{23}^{(l)} \dots N_{21}^{(l)} \right] \begin{bmatrix} \phi_1^{(l)} \\ \phi_2^{(l)} \\ \vdots \\ \phi_{21}^{(l)} \end{bmatrix} \quad (4.15)$$

and rewrite the original equation into the weak form by taking the inner product with the function ψ which has the same form as ϕ :

$$(\psi, (\mathcal{L} - \varepsilon)\phi) = 0. \quad (4.16)$$

Here \mathcal{L} is defined by

$$\mathcal{L} = \hat{L} / \left(\frac{\hbar D_0}{L_0^2} \right) \quad (4.17)$$

and we assume that the diffusion tensor reduces to a scalar denoting the normalized value by δ ,

$$\delta = D/D_0. \quad (4.18)$$

Taking into account the boundary condition on the normal derivative at surfaces

$$\frac{\partial}{\partial n}\phi = 0, \quad (4.19)$$

we have the Galerkin equation for our system as the superposition of the equation for the element (l)

$$[F_{ij}^{(l)}] \begin{bmatrix} \phi_1^{(l)} \\ \phi_2^{(l)} \\ \vdots \\ \phi_{21y}^{(l)} \end{bmatrix} = 0, \quad (4.20)$$

where

$$[F_{ij}^{(l)}] = [K_{ij}^{(l)}] - \varepsilon [M_{ij}^{(l)}], \quad (4.21)$$

$$K_{ij}^{(l)} = \int_{(l)} dx dy \nu \delta \nabla N_i^{(l)} \cdot \nabla N_j^{(l)}, \quad (4.22)$$

$$M_{ij}^{(l)} = \int_{(l)} dx dy \nu N_i^{(l)} N_j^{(l)}. \quad (4.23)$$

In order to perform computations, we need to evaluate the integrals $K_{ij}^{(l)}$ and $M_{ij}^{(l)}$ for each element. In the case where ν and δ can be regarded as constant in an element, it is possible to prepare a table of necessary values,

$$K_{ij}^{(l)}/\nu\delta = \int_{(l)} dx dy \nabla N_i^{(l)} \cdot \nabla N_j^{(l)}, \quad (4.24)$$

$$M_{ij}^{(l)}/\nu = \int_{(l)} dx dy N_i^{(l)} N_j^{(l)}, \quad (4.25)$$

for general purposes. We have obtained the results of these straightforward but tedious integrations by computers with the help of a software for algebraic computation. The results are given in Appendix.

D. Results

In our computations, the Debye temperature Θ_D , the effective interaction V_0 , and the density of states N_0 in the central superconducting part are assumed to be

$$\Theta_D = 300K, \quad (4.26)$$

$$V_0 N_0 = 1/3. \quad (4.27)$$

These values give the bulk critical temperature $T_c^{(bulk)} = 17K$. We also assume that

$$\theta_D = 1.2 \cdot 10^2. \quad (4.28)$$

We first compare the numerical solution for the quasi-one-dimensional case with the corresponding half-analytic solution for one-dimensional system. An example is shown in Fig.3. This comparison confirms the relation described in the subsection above and also the reliability of our numerical procedure for two-dimensional structures.

For two-dimensional cases, we perform numerical analyses for three cases of the parameter, $\eta_0 = 0.3, 0.1,$ and 0.03 . As the geometry of the structure, those shown in Fig.4 are adopted.

We present here the results for the case of uniform density of states and diffusion coefficient. The effects of the stepwise change of the latter two parameters are similar to those in one-dimensional cases: The effect of their gradual changes will be given elsewhere. The critical temperatures of these structures are shown in Fig.4.

In the case of two dimensions, the critical temperature is not a function of a single variable even for a fixed value of the parameter η_0 : It might depend on infinite degrees of freedom related to the shape of the structure. This is the very reason for the necessity of numerical analyses such as developed in II and in this paper.

On the other hand, it is also clear that the results for the critical temperature may be approximately described by a simple function in the case of either relatively large or small value of η .

When η is large, the order parameter may penetrate rather freely into the normal part and the effective interaction between electrons will be averaged over the whole structure. The critical temperature may then be determined by the ratio of the areas of the super and normal parts.

When η is small, however, the order parameter may be well localized in the super part where the effective potential is attractive, and the critical temperature may depend only on the shape of the super part measured in the scale of the length ξ . If the super part has a much smaller dimension in one direction in comparison with other directions, the critical properties may be similar to those of one-dimensional structure; directions with larger dimensions are expected to play little role in determination of the critical temperature. Thus the smallest scale of length in the structure may be most important in this situation. As an example of the smallest length for two-dimensional structures, we may take the diameter $2R$ of the largest circle which inscribes the super part.

These two values, the areal ratio of the super part σ and the diameter $2R$ of the inscribing circle, are given also in Fig.4. We now check whether these values serve as guidelines for determining the critical temperatures.

In Figs.5a, 5b, and 5c, the critical temperature normalized by the bulk value is plotted as a function of the ratio of the area of the super part in the whole structure

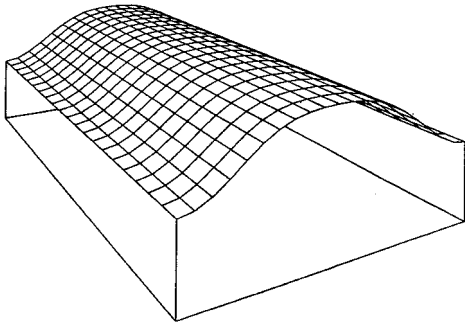


Fig.3a. Values of order parameter (divided by the interaction and density of states) numerically obtained by two-dimensional analysis for a quasi-one-dimensional structure. Central superconducting part (3/4) is symmetrically placed and $\eta_0 = 0.3$.

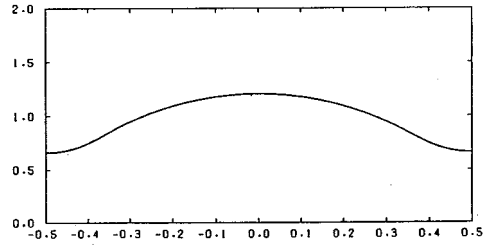
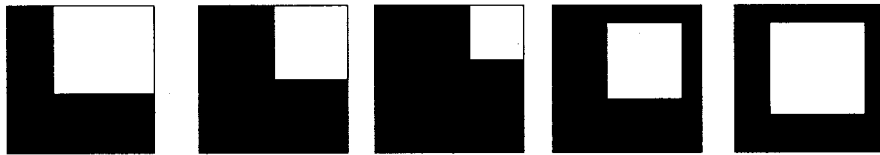


Fig.3b. The same as Fig.3a obtained by half-analytical calculation for corresponding one-dimensional structure.



η_0	A	B	C	D	E
0.3	0.200	0.110	0.036	0.0024	0.00007
0.1	0.460	0.363	0.207	0.050	0.0035
0.03	0.886	0.862	0.817	0.726	0.260
σ	0.563	0.488	0.375	0.25	0.156
2R	0.75	0.625	0.5	0.5	0.25



η_0	F	G	H	I	J
0.3	0.256	0.060	0.0008	0.0038	0.0524
0.1	0.708	0.569	0.303	0.082	0.270
0.03	0.956	0.963	0.888	0.736	0.835
σ	0.391	0.25	0.141	0.25	0.391
2R	1.25*	1.0*	0.75*	0.5	0.625

Fig.4. Critical temperatures (normalized by bulk value) of various two-dimensional structures for $\eta_0 = 0.3, 0.1$ and 0.03 . Light and dark domains are super and normal parts, and entries under σ and $2R$ are areal ratio and maximum length (in unit of side of square) of super part, respectively. (*These values are doubled due to symmetry.)



η_0	K	L	M	N
0.3	0.138	0.112	0.117	0.092
0.1	0.396	0.302	0.331	0.243
0.03	0.870	0.786	0.833	0.655
σ	0.5	0.5	0.5	0.5
2R	0.707	0.5	0.5	0.293

Fig.4. (Continued)

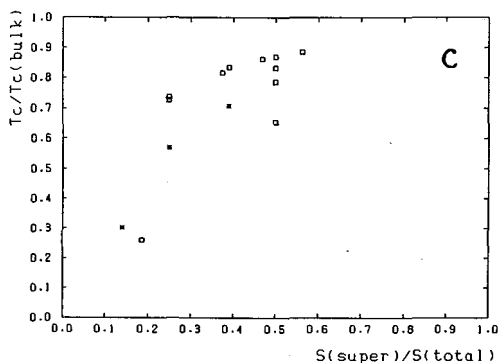
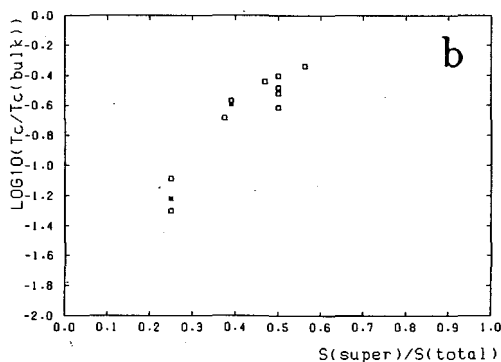
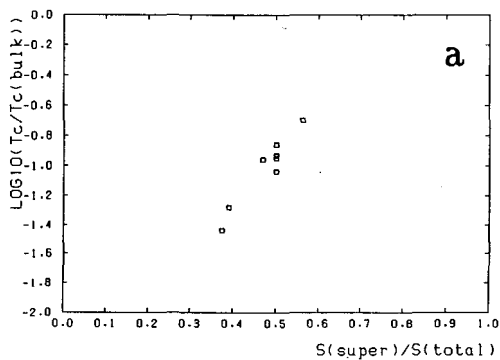


Fig.5a. Critical temperature vs. areal ratio of super part for $\eta_0 = 0.3$.

Fig.5b. The same as Fig.5a for $\eta_0 = 0.1$. Values for F, G, and H with $\eta_0 = 0.3$ are shown by *.

Fig.5c. The same as Fig.5a for $\eta_0 = 0.03$. Values for F, G, and H with $\eta_0 = 0.1$ are shown by *.

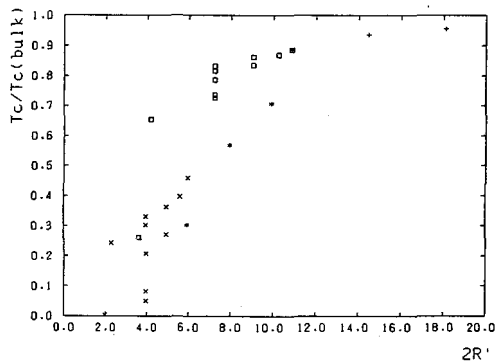


Fig.6. Critical temperature vs. maximum length normalized by ξ (at $T = T_c^{(bulk)}$) for $\eta_0 = 0.03$ [+ (F, G, H) and squares (others)] and 0.1[* (F, G, H) and x (others)].

for $\eta_0 = 0.3, 0.1,$ and 0.03 .

In structures F, G, and H, the result should be regarded as the one for a larger structure obtained by folding these structures in both directions. Since the side of the square is effectively doubled, we reduce the value of η_0 defined by eq.(2.2) by a factor 4. The critical temperatures of these structures for $\eta_0 = 0.3$ and 0.1 are plotted in Figs.5b and 5c, regarding $0.3/4 \sim 0.1$ and $0.1/4 \sim 0.03$, respectively.

We observe that the critical temperatures for $\eta_0 = 0.3$ and 0.1 are approximately determined by the areal ratio of the super part.

In the case of $\eta_0 = 0.03$, however, it is clear that we have only weak dependence on that ratio.

When η_0 is small, the critical temperature is not determined by the areal ratio. The structures A and H, with quite different values of the areal ratios and the critical temperatures at $\eta_0 = 0.3$, have similar critical temperatures at $\eta_0 = 0.03$. The same kind of observation may be made for structures C, D, I, L, and M. Noting that these groups have the values of $2R$ in common, we see that the maximum length of the super part has decisive effect on the critical temperature in these cases.

In Fig.6, we show the dependence of the normalized critical temperature on the ratio of the maximum length determined above to the value of ξ at $T = T_c^{(bulk)}$ for the cases of $\eta_0 = 0.1$ and 0.03 . We see that for sufficiently small value of η_0 the critical temperature is determined by this ratio in the first approximation.

Naturally, these guidelines are not perfect. For example, among structures B, K, L, M, and N with similar areal ratios around 0.5, K and N have the highest and the lowest critical temperatures, respectively, also for $\eta_0 = 0.3$. The maximum length is thus important even for large values of η in some cases.

In applications of superconducting microstructures, it will become necessary to predict the critical temperature of a given structure without analyzing the behavior of the order parameter in detail. Our results may be useful for this purpose.

V. CONCLUSION

We have performed numerical analyses on the critical temperatures of various superconducting microstructures in two dimensions. As a result, we have obtained their dependency on quantities characterizing the structures, the areal ratio and the appropriately defined maximum length of the superconducting part. These values may provide us with approximate guidelines to determine critical temperatures in structures of various shapes, when the characteristic length of diffusion process is either large or small enough in comparison with the scale of the structures. We also expect that the volume ratio and the maximum length may play a similar role in three dimensions.

ACKNOWLEDGMENTS

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APPENDIX A: CALCULATION OF MATRIX ELEMENTS

The values of the integrals (4.24) and (4.25) for the triangle with vertices (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) (in the counterclockwise order) are evaluated as follows. Here

$$M_{ij}/\nu = \int dx dy N_i N_j = (A1) \cdot 2S, \quad (1)$$

$$K_{ij}/\nu\delta = [(A2) + (A3)]/2S, \quad (2)$$

$$(A2)/2S = K_{ij}^x/\nu\delta = \int dx dy (\partial N_i/\partial x)(\partial N_j/\partial x), \quad (3)$$

$$(A3)/2S = K_{ij}^y/\nu\delta = \int dx dy (\partial N_i/\partial y)(\partial N_j/\partial y), \quad (4)$$

and S is the area of the triangle. The values are symmetric with respect to the suffices i and j which run from 1 to 10 corresponding to eqs.(4.1)~(4.10).

$$(i = 1, j = 1)$$

$$A1 = \frac{313}{5040}$$

$$A2 = \frac{1}{180}(49y_{31}^2 + 49y_{31}y_{23} + 199y_{23}^2)$$

$$A3 = \frac{1}{180}(49x_{31}^2 + 49x_{31}x_{23} + 199x_{23}^2)$$

$$(i = 2, j = 1)$$

$$A1 = \frac{1}{720}$$

$$A2 = \frac{1}{180}(70y_{31}^2 + 139y_{31}y_{23} + 70y_{23}^2)$$

$$A3 = \frac{1}{180}(70x_{31}^2 + 139x_{31}x_{23} + 70x_{23}^2)$$

$$(i = 2, j = 2)$$

$$A1 = \frac{313}{5040}$$

$$A2 = \frac{1}{180}(199y_{31}^2 + 49y_{31}y_{23} + 49y_{23}^2)$$

$$A3 = \frac{1}{180}(199x_{31}^2 + 49x_{31}x_{23} + 49x_{23}^2)$$

$$(i = 3, j = 1)$$

$$A1 = \frac{1}{720}$$

$$A2 = \frac{1}{180}(70y_{31}^2 + y_{31}y_{23} + y_{23}^2)$$

$$A3 = \frac{1}{180}(70x_{31}^2 + x_{31}x_{23} + x_{23}^2)$$

$$(i = 3, j = 2)$$

$$A1 = \frac{1}{720}$$

$$A2 = \frac{1}{180}(y_{31}^2 + y_{31}y_{23} + 70y_{23}^2)$$

$$A3 = \frac{1}{180}(x_{31}^2 + x_{31}x_{23} + 70x_{23}^2)$$

$$(i = 3, j = 3)$$

$$A1 = \frac{313}{5040}$$

$$A2 = \frac{1}{180}(199y_{31}^2 + 349y_{31}y_{23} + 199y_{23}^2)$$

$$A3 = \frac{1}{180}(199x_{31}^2 + 349x_{31}x_{23} + 199x_{23}^2)$$

$$(i = 4, j = 1)$$

$$A1 = \frac{3}{112}$$

$$A2 = \frac{3}{20}(-7y_{31}^2 - 7y_{31}y_{23} - 10y_{23}^2)$$

$$A3 = \frac{3}{20}(-7x_{31}^2 - 7x_{31}x_{23} - 10x_{23}^2)$$

$$(i = 4, j = 2)$$

$$A1 = \frac{3}{112}$$

$$A2 = \frac{3}{20}(-10y_{31}^2 - 7y_{31}y_{23} - 7y_{23}^2)$$

$$A3 = \frac{3}{20}(-10x_{31}^2 - 7x_{31}x_{23} - 7x_{23}^2)$$

$$(i = 4, j = 3)$$

$$A1 = \frac{3}{112}$$

$$A2 = \frac{3}{20}(-10y_{31}^2 - 13y_{31}y_{23} - 10y_{23}^2)$$

$$A3 = \frac{3}{20}(-10x_{31}^2 - 13x_{31}x_{23} - 10x_{23}^2)$$

$$(i = 4, j = 4)$$

$$A1 = \frac{81}{560}$$

$$A2 = \frac{81}{20}(y_{31}^2 + y_{31}y_{23} + y_{23}^2)$$

$$A3 = \frac{81}{20}(x_{31}^2 + x_{31}x_{23} + x_{23}^2)$$

$(i = 5, j = 1)$

$$A1 = \frac{53}{10080} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3y_{23}^2 - 25y_{23}y_{31} + 7y_{31}^2)$$

$$A3 = \frac{1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3x_{23}^2 - 25x_{23}x_{31} + 7x_{31}^2)$$

$(i = 6, j = 1)$

$$A1 = \frac{53}{10080} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{1}{180} \sqrt{x_{12}^2 + y_{12}^2} (35y_{23}^2 + 39y_{23}y_{31} + 7y_{31}^2)$$

$$A3 = \frac{1}{180} \sqrt{x_{12}^2 + y_{12}^2} (35x_{23}^2 + 39x_{23}x_{31} + 7x_{31}^2)$$

$(i = 5, j = 2)$

$$A1 = \frac{-13}{10080} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} y_{31} (7y_{23} + 2y_{31})$$

$$A3 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} x_{31} (7x_{23} + 2x_{31})$$

$(i = 6, j = 2)$

$$A1 = \frac{17}{10080} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{1}{180} \sqrt{x_{12}^2 + y_{12}^2} (14y_{23}^2 + 39y_{23}y_{31} + 22y_{31}^2)$$

$$A3 = \frac{1}{180} \sqrt{x_{12}^2 + y_{12}^2} (14x_{23}^2 + 39x_{23}x_{31} + 22x_{31}^2)$$

$(i = 5, j = 3)$

$$A1 = \frac{17}{10080} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3y_{23}^2 - 5y_{23}y_{31} - 22y_{31}^2)$$

$$A3 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3x_{23}^2 - 5x_{23}x_{31} - 22x_{31}^2)$$

$(i = 6, j = 3)$

$$A1 = \frac{-13}{10080} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{1}{180} \sqrt{x_{12}^2 + y_{12}^2} (5y_{23}^2 + 3y_{23}y_{31} - 2y_{31}^2)$$

$$A3 = \frac{1}{180} \sqrt{x_{12}^2 + y_{12}^2} (5x_{23}^2 + 3x_{23}x_{31} - 2x_{31}^2)$$

$(i = 5, j = 4)$

$$A1 = \frac{3}{1120} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{3}{20} \sqrt{x_{31}^2 + y_{31}^2} y_{31} (y_{23} - y_{31})$$

$$A3 = \frac{3}{20} \sqrt{x_{31}^2 + y_{31}^2} x_{31} (x_{23} - x_{31})$$

$(i = 6, j = 4)$

$$A1 = \frac{3}{1120} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-3}{20} \sqrt{x_{12}^2 + y_{12}^2} (2y_{23}^2 + 3y_{23}y_{31} + y_{31}^2)$$

$$A3 = \frac{-3}{20} \sqrt{x_{12}^2 + y_{12}^2} (2x_{23}^2 + 3x_{23}y_{31} + x_{31}^2)$$

$(i = 5, j = 5)$

$$A1 = \frac{1}{1260} (x_{31}^2 + y_{31}^2)$$

$$A2 = \frac{1}{180} (x_{31}^2 + y_{31}^2) (3y_{23}^2 + 3y_{23}y_{31} + 7y_{31}^2)$$

$$A3 = \frac{1}{180} (x_{31}^2 + y_{31}^2) (3x_{23}^2 + 3x_{23}x_{31} + 7x_{31}^2)$$

$(i = 6, j = 5)$

$$A1 = \frac{1}{5040} \sqrt{x_{31}^2 + y_{31}^2} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-1}{36} \sqrt{x_{31}^2 + y_{31}^2} \sqrt{x_{12}^2 + y_{12}^2} y_{31} (y_{23} + y_{31})$$

$$A3 = \frac{-1}{36} \sqrt{x_{31}^2 + y_{31}^2} \sqrt{x_{12}^2 + y_{12}^2} x_{31} (x_{23} + x_{31})$$

$(i = 6, j = 6)$

$$A1 = \frac{1}{1260}(x_{12}^2 + y_{12}^2)$$

$$A2 = \frac{1}{180}(x_{12}^2 + y_{12}^2)(7y_{23}^2 + 11y_{23}y_{31} + 7y_{31}^2)$$

$$A3 = \frac{1}{180}(x_{12}^2 + y_{12}^2)(7x_{23}^2 + 11x_{23}x_{31} + 7x_{31}^2)$$

 $(i = 7, j = 5)$

$$A1 = \frac{-1}{5040}\sqrt{x_{31}^2 + y_{31}^2}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{-1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}y_{23}y_{31}$$

$$A3 = \frac{-1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}x_{23}x_{31}$$

 $(i = 7, j = 1)$

$$A1 = \frac{-13}{10080}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{-1}{180}\sqrt{x_{23}^2 + y_{23}^2}y_{23}(2y_{23} + 7y_{31})$$

$$A3 = \frac{-1}{180}\sqrt{x_{23}^2 + y_{23}^2}x_{23}(2x_{23} + 7x_{31})$$

 $(i = 7, j = 6)$

$$A1 = \frac{-1}{10080}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{12}^2 + y_{12}^2}y_{23}(y_{23} + y_{31})$$

$$A3 = \frac{-1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{12}^2 + y_{12}^2}x_{23}(x_{23} + x_{31})$$

 $(i = 7, j = 2)$

$$A1 = \frac{53}{10080}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(7y_{23}^2 - 25y_{23}y_{31} + 3y_{31}^2)$$

$$A3 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(7x_{23}^2 - 25x_{23}x_{31} + 3x_{31}^2)$$

 $(i = 7, j = 7)$

$$A1 = \frac{1}{1260}(x_{23}^2 + y_{23}^2)$$

$$A2 = \frac{1}{180}(x_{23}^2 + y_{23}^2)(7y_{23}^2 + 3y_{23}y_{31} + 3y_{31}^2)$$

$$A3 = \frac{1}{180}(x_{23}^2 + y_{23}^2)(7x_{23}^2 + 3x_{23}x_{31} + 3x_{31}^2)$$

 $(i = 7, j = 3)$

$$A1 = \frac{17}{10080}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(22y_{23}^2 + 5y_{23}y_{31} - 3y_{31}^2)$$

$$A3 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(22x_{23}^2 + 5x_{23}x_{31} - 3x_{31}^2)$$

 $(i = 8, j = 1)$

$$A1 = \frac{17}{10080}\sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{12}^2 + y_{12}^2}(22y_{23}^2 + 39y_{23}y_{31} + 14y_{31}^2)$$

$$A3 = \frac{1}{180}\sqrt{x_{12}^2 + y_{12}^2}(22x_{23}^2 + 39x_{23}x_{31} + 14x_{31}^2)$$

 $(i = 7, j = 4)$

$$A1 = \frac{3}{1120}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{-3}{20}\sqrt{x_{23}^2 + y_{23}^2}y_{23}(y_{23} - y_{31})$$

$$A3 = \frac{-3}{20}\sqrt{x_{23}^2 + y_{23}^2}x_{23}(x_{23} - x_{31})$$

 $(i = 8, j = 2)$

$$A1 = \frac{53}{10080}\sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{12}^2 + y_{12}^2}(7y_{23}^2 + 39y_{23}y_{31} + 35y_{31}^2)$$

$$A3 = \frac{1}{180}\sqrt{x_{12}^2 + y_{12}^2}(7x_{23}^2 + 39x_{23}x_{31} + 35x_{31}^2)$$

$(i = 8, j = 3)$

$$A1 = \frac{-13}{10080} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-1}{180} \sqrt{x_{12}^2 + y_{12}^2} (2y_{23}^2 - 3y_{23}y_{31} - 5y_{31}^2)$$

$$A3 = \frac{-1}{180} \sqrt{x_{12}^2 + y_{12}^2} (2x_{23}^2 - 3x_{23}x_{31} - 5x_{31}^2)$$

$(i = 8, j = 8)$

$$A1 = \frac{1}{1260} (x_{12}^2 + y_{12}^2)$$

$$A2 = \frac{1}{180} (x_{12}^2 + y_{12}^2) (7y_{23}^2 + 11y_{23}y_{31} + 7y_{31}^2)$$

$$A3 = \frac{1}{180} (x_{12}^2 + y_{12}^2) (7x_{23}^2 + 11x_{23}x_{31} + 7x_{31}^2)$$

$(i = 8, j = 4)$

$$A1 = \frac{3}{1120} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-3}{20} \sqrt{x_{12}^2 + y_{12}^2} (y_{23}^2 + 3y_{23}y_{31} + 2y_{31}^2)$$

$$A3 = \frac{-3}{20} \sqrt{x_{12}^2 + y_{12}^2} (x_{23}^2 + 3x_{23}x_{31} + 2x_{31}^2)$$

$(i = 9, j = 1)$

$$A1 = \frac{17}{10080} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3y_{23}^2 + 11y_{23}y_{31} - 14y_{31}^2)$$

$$A3 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3x_{23}^2 + 11x_{23}x_{31} - 14x_{31}^2)$$

$(i = 8, j = 5)$

$$A1 = \frac{-1}{10080} \sqrt{x_{31}^2 + y_{31}^2} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} \sqrt{x_{12}^2 + y_{12}^2} y_{31} (y_{23} + y_{31})$$

$$A3 = \frac{-1}{180} \sqrt{x_{31}^2 + y_{31}^2} \sqrt{x_{12}^2 + y_{12}^2} x_{31} (x_{23} + x_{31})$$

$(i = 9, j = 2)$

$$A1 = \frac{-13}{10080} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{1}{180} \sqrt{x_{31}^2 + y_{31}^2} y_{31} (7y_{23} + 5y_{31})$$

$$A3 = \frac{1}{180} \sqrt{x_{31}^2 + y_{31}^2} x_{31} (7x_{23} + 5x_{31})$$

$(i = 8, j = 6)$

$$A1 = \frac{1}{2016} (x_{12}^2 + y_{12}^2)$$

$$A2 = \frac{1}{36} (x_{12}^2 + y_{12}^2) (y_{23} + y_{31})^2$$

$$A3 = \frac{1}{36} (x_{12}^2 + y_{12}^2) (x_{23} + x_{31})^2$$

$(i = 9, j = 3)$

$$A1 = \frac{53}{10080} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3y_{23}^2 + 31y_{23}y_{31} + 35y_{31}^2)$$

$$A3 = \frac{1}{180} \sqrt{x_{31}^2 + y_{31}^2} (3x_{23}^2 + 31x_{23}x_{31} + 35x_{31}^2)$$

$(i = 8, j = 7)$

$$A1 = \frac{1}{5040} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-1}{36} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2} y_{23} (y_{23} + y_{31})$$

$$A3 = \frac{-1}{36} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2} x_{23} (x_{23} + x_{31})$$

$(i = 9, j = 4)$

$$A1 = \frac{3}{1120} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{-3}{20} \sqrt{x_{31}^2 + y_{31}^2} y_{31} (y_{23} + 2y_{31})$$

$$A3 = \frac{-3}{20} \sqrt{x_{31}^2 + y_{31}^2} x_{31} (x_{23} + 2x_{31})$$

$$(i = 9, j = 5)$$

$$A1 = \frac{1}{2016}(x_{31}^2 + y_{31}^2)$$

$$A2 = \frac{1}{36}(x_{31}^2 + y_{31}^2)y_{31}^2$$

$$A3 = \frac{1}{36}(x_{31}^2 + y_{31}^2)x_{31}^2$$

$$(i = 10, j = 1)$$

$$A1 = \frac{-13}{10080}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}y_{23}(5y_{23} + 7y_{31})$$

$$A3 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}x_{23}(5x_{23} + 7x_{31})$$

$$(i = 9, j = 6)$$

$$A1 = \frac{-1}{10080}\sqrt{x_{31}^2 + y_{31}^2}\sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-1}{180}\sqrt{x_{31}^2 + y_{31}^2}\sqrt{x_{12}^2 + y_{12}^2}y_{31}(y_{23} + y_{31})$$

$$A3 = \frac{-1}{180}\sqrt{x_{31}^2 + y_{31}^2}\sqrt{x_{12}^2 + y_{12}^2}x_{31}(x_{23} + x_{31})$$

$$(i = 10, j = 2)$$

$$A1 = \frac{17}{10080}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(14y_{23}^2 - 11y_{23}y_{31} - 3y_{31}^2)$$

$$A3 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(14x_{23}^2 - 11x_{23}x_{31} - 3x_{31}^2)$$

$$(i = 9, j = 7)$$

$$A1 = \frac{-1}{10080}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}y_{23}y_{31}$$

$$A3 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}x_{23}x_{31}$$

$$(i = 10, j = 3)$$

$$A1 = \frac{53}{10080}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(35y_{23}^2 + 31y_{23}y_{31} + 3y_{31}^2)$$

$$A3 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}(35x_{23}^2 + 31x_{23}x_{31} + 3x_{31}^2)$$

$$(i = 9, j = 8)$$

$$A1 = \frac{-1}{5040}\sqrt{x_{31}^2 + y_{31}^2}\sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{31}^2 + y_{31}^2}\sqrt{x_{12}^2 + y_{12}^2}y_{31}(y_{31} + y_{23})$$

$$A3 = \frac{1}{180}\sqrt{x_{31}^2 + y_{31}^2}\sqrt{x_{12}^2 + y_{12}^2}x_{31}(x_{31} + x_{23})$$

$$(i = 10, j = 4)$$

$$A1 = \frac{3}{1120}\sqrt{x_{23}^2 + y_{23}^2}$$

$$A2 = \frac{-3}{20}\sqrt{x_{23}^2 + y_{23}^2}y_{23}(2y_{23} + y_{31})$$

$$A3 = \frac{-3}{20}\sqrt{x_{23}^2 + y_{23}^2}x_{23}(2x_{23} + x_{31})$$

$$(i = 9, j = 9)$$

$$A1 = \frac{1}{1260}(x_{31}^2 + y_{31}^2)$$

$$A2 = \frac{1}{180}(x_{31}^2 + y_{31}^2)(3y_{23}^2 + 3y_{23}y_{31} + 7y_{31}^2)$$

$$A3 = \frac{1}{180}(x_{31}^2 + y_{31}^2)(3x_{23}^2 + 3x_{23}x_{31} + 7x_{31}^2)$$

$$(i = 10, j = 5)$$

$$A1 = \frac{-1}{10080}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}y_{23}y_{31}$$

$$A3 = \frac{1}{180}\sqrt{x_{23}^2 + y_{23}^2}\sqrt{x_{31}^2 + y_{31}^2}x_{23}x_{31}$$

(i = 10 , j = 6)

$$A1 = \frac{-1}{5040} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{1}{180} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2} y_{23} (y_{23} + y_{31})$$

$$A3 = \frac{1}{180} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2} x_{23} (x_{23} + y_{31})$$

(i = 10 , j = 7)

$$A1 = \frac{1}{2016} (x_{23}^2 + y_{23}^2)$$

$$A2 = \frac{1}{36} (x_{23}^2 + y_{23}^2) y_{23}^2$$

$$A3 = \frac{1}{36} (x_{23}^2 + y_{23}^2) x_{23}^2$$

(i = 10 , j = 8)

$$A1 = \frac{-1}{10080} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2}$$

$$A2 = \frac{-1}{180} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2} y_{23} (y_{23} + y_{31})$$

$$A3 = \frac{-1}{180} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{12}^2 + y_{12}^2} x_{23} (x_{23} + x_{31})$$

(i = 10 , j = 9)

$$A1 = \frac{1}{5040} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{31}^2 + y_{31}^2}$$

$$A2 = \frac{1}{36} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{31}^2 + y_{31}^2} y_{23} y_{31}$$

$$A3 = \frac{1}{36} \sqrt{x_{23}^2 + y_{23}^2} \sqrt{x_{31}^2 + y_{31}^2} x_{23} x_{31}$$

(i = 10 , j = 10)

$$A1 = \frac{1}{1260} (x_{23}^2 + y_{23}^2)$$

$$A2 = \frac{1}{180} (x_{23}^2 + y_{23}^2) (7y_{23}^2 + 3y_{23}y_{31} + 3y_{31}^2)$$

$$A3 = \frac{1}{180} (x_{23}^2 + y_{23}^2) (7x_{23}^2 + 3x_{23}x_{31} + 3x_{31}^2)$$