

Some Theoretical Aspects on Optimal Toll-Rate and Scale of Urban Expressway

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Synopsis

In this paper, the optimal toll-rate and scale of urban expressway is discussed economically. The criterion used is the total surplus of expressway, which is defined as the sum of consumer's surplus of expressway users and producer's surplus on the side of expressway administrator. The total surplus is equivalent to the difference of the value of total saved travel times of expressway users and the total cost required to construct and administrate expressway systems.

The flow-dependent travel times is assumed, and traffic demand is induced through equilibrium method. On the other hand, for the management of expressway to be capable, the accounting condition that the fare revenue must repay the total cost should be satisfied. Under this accounting condition, the optimal rate and scale which maximize the total surplus are requested, and their properties are examined.

The analysis is practiced for both cases of flow-independent cost function and flow-dependent cost function. One remarkable result is that the maximum feasible scale under the accounting condition does not provide a maximum for total surplus.

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1. Introduction

As the principle of deciding toll-rate of urban expressway in our nation, the principle of repayment and the principle of fairness and propriety ought to be applied. The principle of repayment means that the fare recompenses the total cost required to construct and administrate expressway systems. Meanwhile, the principle of fairness and propriety means that the fare should not only not exceed the benefit obtained by using the expressway, but also be decided considering user's ability of burden for fare and the balance with other transits freightages. In deciding rate practically, traffic demands, benefits and costs connected with expressway systems are calculated in detail.

By the way, the problems of toll-rate of urban expressway have been treated theoretically through welfare economics by some authors. Some of them concerning to this study may be summarized as follows. Sasaki proposed an operations research model to decide the uniform rate region of urban expressway [1]. His study was followed by Yamada, in which a theory to decide optimum rate and scale of urban expressway was developed [2]. Yamada's results are that the optimum rate is equivalent to the value of saved travel time obtained by using an xpressway just as long as the average trip length, and the optimal scale should be the maximum feasible one within the bounds of possibility of repayment. Iida propoesd the scale which maximizes the number of users under the accounting condition as the optimum scale, and he calculated the optimal scale of Kitakyusyu urban expressway system actually [3]. Myojin and Asai tried an expansion of Yamada's model using congestion costs [4].

In this study, extensions of Yamada's model to more general cases are practiced. In Yamada's paper, the travel time is assumed to be constant for traffic flow. But actually, travel time increases with the increase of traffic flow. Meanwhile, the traffic demand on expressway is a function of rate and saved travel time. Therefore, there should exists an equilibrium traffic demand, which we uses in this paper for analysis. Next, in Yamada's paper the total fare revenue is constrained such that it should be equivalent to the total cost. This condition is expanded in this paper such that the fare revenue should not be less than the total cost. By this expansion of constraint, the total surplus may be expected to increase, and produced gains may be made use for countermoves against traffic accidents and traffic noise problems. This is not contrary to the principle of repayment in a wide meaning.

The main assumptions of this study are as follows. The toll-rate is supposed to be constant for any trip length on expressway, that is to say a uniform rate is supposed. Besides, the interval of collecting fares is assumed to be one interval. This means the problem is analyzed statically. The scale is defined as the total length of an expressway system, and the total cost is supposed to be a function of demand flow and scale. A more important assumption is that there exists a finite solution for optimal rate and scale. Because if finite solution does not exist, the discussion is meaningless.

In the following, traffic demand on expressway is induced by a simple model which was modified Yamada's model. The total surplus and accounting condition are formulated by use of equilibrium demand function. The conditions for a finite optimal solution to exist are made clear. Then the optimal rate and scale which maximize total surplus under the accounting condition are requested, and their properties are examined.

The analysis is practiced for two cases. First, a case in which the total cost is assumed to be a function of scale only, namely the case of flow-independent cost function is treated. Then we deal with a more general case that the total cost is a function of both demand flow and scale. One of interest of this study is whether the maximum feasible scale under the accounting condition is the optimal or not. This is examined in the paper in connection with the assumptions made for producer's surplus

2. Equilibrium Traffic Demand on Urban Expressway

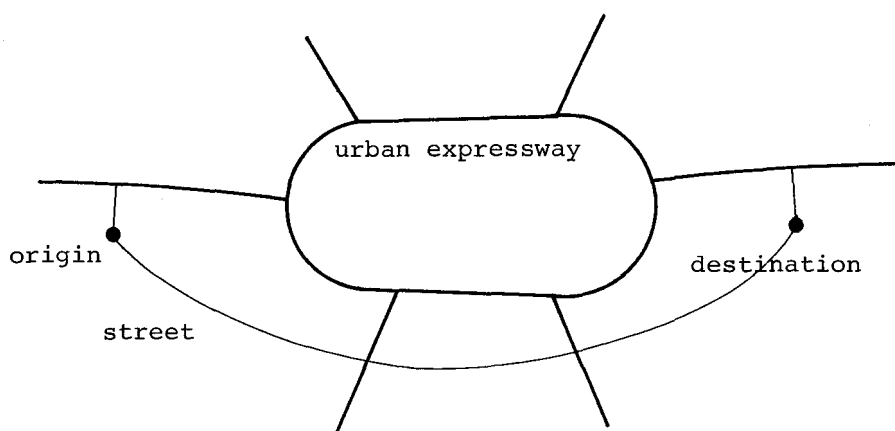


Fig.1 Urban Expressway and Street

In traveling from an arbitrary origin to other arbitrary destination in urban area, we assume the trip lengths are equivalent in both cases of using expressway route and street route. Besides, we assume the lengths of access road to and from expressway are constant for any origin and destination pairs. Let denote Q_a the total trips in the urban area and q the total trips which use expressway route. Moreover, let denote t_e, t_s the travel times required to run a unit length on expressway and street respectively. Travel times are assumed to be functions of total flow on expressway, such that $\partial t_e / \partial q > 0, \partial t_s / \partial q < 0$.

From the assumption that trip lengths are equivalent in both cases of using expressway route and street route, the saved travel time produced by using expressway of a tripmaker with trip length l is expressed by

$$T(l) = (l - 2l_1)(t_s - t_e) \tag{1}$$

where l_1 is the length of access to and from expressway. The difference of t_s and t_e represents the saved travel time produced by using expressway just as long as a unit length, and is denoted by t . Needless to say, t is a monotone decreasing function of flow q and the function $t = t(q)$ is called the service function in this paper.

Under a given scale of expressway, the traffic demand seems to be settled by service level t and rate p . Of course, under an arbitrary rate p , the demand is a monotone increasing function of service level t . Then the demand will balance at the intersection of service function and demand function. Such equilibrium demand may be obtained uniquely for every rates. We call the relation between equilibrium demand q and rate p the equilibrium demand function.

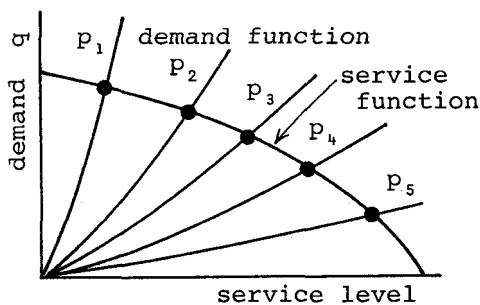


Fig.2 Demand functions for Each Rate and Service Function

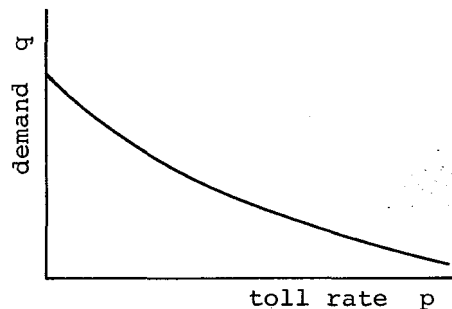


Fig.3 Equilibrium Demand Function

Now, the equilibrium demand function is induced. We assume that a trip maker chooses expressway route, if the value of saved travel time exceeds the rate. While, the trip length in urban area seems to follow negative exponential distribution

$$f(\ell) = \mu e^{-\mu\ell}, \quad (2)$$

where $1/\mu$ is the average of trip length. Then, expressway route is used by trip makers with trip length $\ell > 2\ell_1 + p/\omega t$. Where ω is the value of times per hour, which is supposed to be constant in this paper. Accordingly, the traffic demand on expressway is induced as

$$q = Q_a \int_{\ell > 2\ell_1 + p/\omega t} f(\ell) d\ell = Q_a e^{-\mu(2\ell_1 + \frac{p}{\omega t})}. \quad (3)$$

The demand at $p=0$ is called divergence object flow in this paper, and is denoted by Q . The divergence object flow Q is supposed to be a function of scale s which is defined as the total length of expressway network, namely $Q = Q(s)$. Then, the demand function is represented by $q = Q \exp(-\mu p/\omega t)$.

The equilibrium demand is given by intersections of service function and demand functions. That is the solution for demand q of the equation

$$q = Q e^{-\left(\frac{\mu p}{\omega t(q)}\right)}. \quad (4)$$

But equation (4) can not be solved for demand q as far as function $t(q)$ is not specialized. Therefore, instead of requesting equilibrium demand function $q = f(p)$, we request the inverse function $p = f^{-1}(q)$. This function is given by

$$p = \frac{1}{\mu} \omega t(q) \log\left(\frac{Q}{q}\right). \quad (5)$$

The differential coefficient of rate p on inverse demand function (5) is represented by

$$\frac{dp}{dq} = \frac{\omega}{\mu} t'(q) \log\frac{Q}{q} - \frac{\omega}{\mu q} t(q). \quad (6)$$

Since $t(q) > 0$, $t'(q) < 0$, $0 < q < Q$ is valid naturally so far as expressway is used practically, the differential coefficient dp/dq takes negative sign. As a result, $dq/dp < 0$ holds on equilibrium demand function. This means the equilibrium traffic demand on expressway decreases surely when toll-rate p increases. Although toll-rate p should be used essentially as the variable of the problem, we use demand flow on expressway q , because toll-rate p corresponds to flow q one to one on equilibrium demand function.

3. Total Surplus of Urban Expressway and Accounting Condition

3.1 Formulation

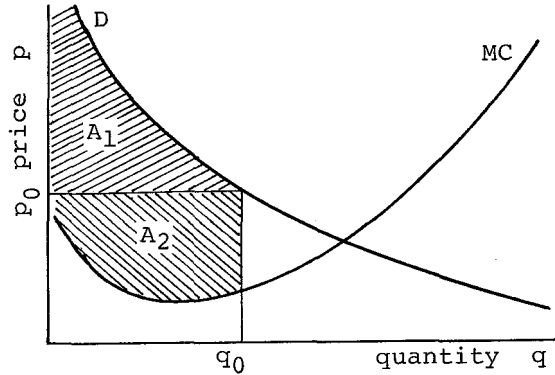


Fig. 4 Consumer's Surplus and Producer's Surplus

In Fig.4, D represents the demand curve, and MC represents the marginal cost curve for an arbitrary goods. When the goods is produced by q_0 and is sold at price p_0 , the area A_1 represents the sum of the difference between the price which a consumer is willing to pay and the price payed actually. In this meaning, the area A_1 is called consumer's surplus. Meanwhile, the area A_2 is equivalent to the difference between total incomes obtained by producing the goods and total costs consumed to produce the goods. Hence, the area A_2 is called producer's surplus. The sum of consumer's surplus and producer's surplus is called total surplus or social surplus. So far as the demand curve is monotone decreasing for price p , consumer's surplus increases evidently, as price p_0 decreases or quantity q_0 increases. Producer's surplus is a measure of wealthy of consumers. While, total surplus is a measure of wealthy on a view point of national economy. It is said that the factors of production of the goods are used most suitably, when total surplus is maximum. That is the reason why we define the optimal rate and scale are those that maximize total surplus.

In case of rate problems of urban expressway, the traffic demand on expressway corresponds to quantity, and the rate corresponds to price. In this study, it is assumed that expressway route is used when the value of saved travel time exceeds the rate. A trip maker with trip length l has will to pay as much as the value of saved travel time $(l-2l_1)wt$. Accordingly, the surplus of this user is equivalent to $(l-2l_1)wt-p$. The sum of surplus of expressway users F_c , that is to say consumer's surplus is represented by

$$F_c = Q_a \int_{(l-2l_1)wt-p > 0} \{(l-2l_1)wt-p\} f(l) dl = \frac{1}{\mu} wtq, \tag{7}$$

where, $\omega t/\mu$ is equivalent to the value of saved travel time obtained by using expressway just as long as the average trip length l/μ . The average length which users run on expressway is equivalent to $l/\mu+p/\omega t$. But the value of saved travel time obtained by running expressway as long as additional length $p/\omega t$ is equivalent to the rate payed, so that they valance in consumer's surplus.

On the other hand, on the side of supplier of expressway service, they have fare revenue pq as against total cost C required to construct and administrate expressway systems. Accordingly, producer's surplus F_p is represented by

$$F_p = pq - C$$

$$= \frac{1}{\mu} \omega t q \log\left(\frac{Q}{q}\right) - C, \tag{8}$$

where, C is a function of demand q and scale s , namely $C = C(q, s)$. Then, total surplus F_t is expressed as

$$F_t = F_c + F_p$$

$$= \frac{1}{\mu} \omega t q + \frac{1}{\mu} \omega t q \log\left(\frac{Q}{q}\right) - C. \tag{9}$$

By the way, the total cost of expressway ought to be repayed by fare revenue. Then, the inequality

$$p q - C \geq 0 \tag{10}$$

must be satisfied. Using equilibrium demand function, this accounting condition can be written by

$$\frac{1}{\mu} \omega t q \log\left(\frac{Q}{q}\right) - C \geq 0. \tag{11}$$

3.2 Some assumptions for Consumer's Surplus and Producer's Surplus

To simplify discussions, we make some assumptions. First, it is assumed that both consumer's surplus F_c and producer's surplus F_p are concave for flow q and scale s . If $t'(q) < 0$ and $t''(q) < 0$, then consumer's surplus F_c is concave, but for producer's surplus to be concave, a more complicated condition is required. A necessary and sufficient condition for producer's surplus F_p to be concave is that the following Hessian matrix of F_p is negative definite for the region of variables where the condition (11) is satisfied:

$$H = \begin{bmatrix} \frac{\omega}{\mu} \{qt'' \log\left(\frac{Q}{q}\right) + 2t' \log\left(\frac{Q}{q}\right) - 2t' - \frac{t}{q}\} - C''_{qq} & \frac{\omega Q'}{\mu Q} (t + qt') - C''_{qs} \\ \frac{\omega Q'}{\mu Q} (t + qt') - C''_{qs} & \frac{\omega}{\mu} qt \frac{Q'' Q - Q'^2}{Q^2} - C''_{ss} \end{bmatrix} \tag{12}$$

Where t' , t'' , Q' , Q'' , C''_{qq} , C''_{qs} and C''_{ss} represent $t'=dt/dq$, $t''=d^2t/dq^2$, $Q'=dQ/ds$, $Q''=d^2Q/ds^2$, $C''_{qq}=\partial^2C/\partial q^2$, $C''_{qs}=\partial^2C/\partial q\partial s$ and $C''_{ss}=\partial^2C/\partial s^2$ respectively. These convexity conditions are rather strict but they seem to be satisfied practically.

Then, a more significant condition is supposed. That is producer's surplus $F_p=pq-C$ has a finite maximum. It assures the problem has a finite optimal solution. Moreover, a practical condition is also supposed. That is there exists flow q and scale s which satisfy

$$F_p = pq - C > 0. \tag{13}$$

Then a region which satisfies the accounting condition (10) and is not a point exists. If function $F_p=F_p(q,s)$ contacts at a point with q - s plane, then the accounting condition is satisfied only at a point of contact. In this case, there are no room to discuss optimal rate and scale, and expressway seems not to be constructed actually. For that reason, we exclude this case from our discussion. It is needless to say, that the case which have no solution of the inequality $pq - C \geq 0$ is also excluded from the discussion.

If flow-independent travel time is supposed, then the condition for flow q and scale s to exist such that $F_p=pq-C>0$ can be induced as

$$C < \frac{\omega t}{\mu} \cdot \frac{Q}{e}, \tag{14}$$

because $0 < q \log(Q/q) < Q/e$ is valid. This condition demands existence of a scale at which total cost is less than the value of saved travel time of users $q=Q/e$ obtained by using eexpressway just as long as the average trip length l/μ .

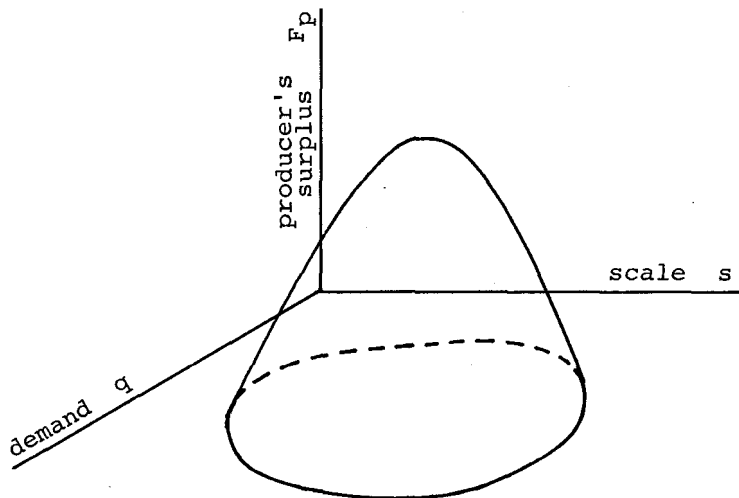


Fig.5 Producer's Surplus $F_p = F_p(q,s)$

4. Maximizing Total Surplus under Flow-Dependent Cost Function

4.1 Necessary and Sufficient Condition for Maximum

Now we suppose that total cost C is a function of scale s only. This implies that the service level of expressway is not assured of a fixed one by supplier but is entrusted to, so to speak, the market equilibrium on the side of users. Therefore the saved travel time for a unit length is assumed to be a function of demand q , namely $t=t(q)$. Then total surplus F_t is represented by

$$F_t = \frac{1}{\mu} \omega q t(q) + \frac{1}{\mu} \omega q t(q) \log\left(\frac{Q(s)}{q}\right) - C(s). \quad (15)$$

While, the accounting condition is expressed as

$$\frac{\omega}{\mu} q t(q) \log\left(\frac{Q(s)}{q}\right) - C(s) \geq 0. \quad (16)$$

In spite of maximizing F_t , $-F_t$ is minimized under the accounting condition (16). From convexity assumptions for user's surplus and producer's surplus, the problem is a convex non-linear programming. Let λ be Lagrange's multiplier, then Lagrangean function $\phi(q, s, \lambda)$ is represented by

$$\begin{aligned} \phi(q, s, \lambda) = & -\frac{\omega}{\mu} q t(q) - \frac{\omega}{\mu} q t(q) \log\left(\frac{Q(s)}{q}\right) + C(s) \\ & + \lambda \left\{ -\frac{\omega}{\mu} q t(q) \log\left(\frac{Q(s)}{q}\right) + C(s) \right\}. \end{aligned} \quad (17)$$

From Kuhn-Tucker's theorem, the necessary and sufficient condition for $-F_t$ to be minimum is that there exist demand q and multiplier $\lambda \geq 0$ which satisfy following equations:

$$\begin{aligned} \frac{\partial \phi}{\partial q} = & -\frac{\omega}{\mu} \{q t'(q) + t(q) \log\left(\frac{Q}{q}\right) + q t'(q) \log\left(\frac{Q}{q}\right)\} \\ & - \frac{\omega}{\mu} \lambda \{t(q) \log\left(\frac{Q}{q}\right) + q t'(q) \log\left(\frac{Q}{q}\right) - t(q)\} = 0 \end{aligned} \quad (18)$$

$$\frac{\partial \phi}{\partial s} = -\frac{\omega}{\mu} \cdot \frac{g}{Q} t(q) Q'(s) + C'(s) + \lambda \left\{ -\frac{\omega}{\mu} \cdot \frac{g}{Q} t(q) Q'(s) + C'(s) \right\} = 0 \quad (19)$$

$$\frac{\partial \phi}{\partial \lambda} = -\frac{\omega}{\mu} q t(q) \log\left(\frac{Q(s)}{q}\right) + C(s) \leq 0 \quad (20)$$

$$\lambda \left\{ -\frac{\omega}{\mu} q t(q) \log\left(\frac{Q(s)}{q}\right) + C(s) \right\} = 0 \quad (21)$$

The properties of optimal flow and optimal scale which maximize these conditions are discussed in the followings.

4.2 Optimal Rate under Flow-Independent Saved travel Times

First, we discuss such a case that saved travel times are flow-independent. This is a very simplified model. In this case, $t(q)$ is constant so that it is denoted by t . Besides $t'(q)=0$ is valid, then equation (18) becomes

$$-\frac{\omega}{\mu} t \log\left(\frac{Q}{q}\right) - \lambda \frac{\omega}{\mu} \{t \log\left(\frac{Q}{q}\right) - t\} = 0. \tag{22}$$

By the way, the solution for q of a equation

$$t \log\left(\frac{Q}{q}\right) - t = 0 \tag{23}$$

gives demand flow q_p , which maximize producer's surplus under a given scale s . As $t > 0$ is valid in a range $0 < q < Q$, q_p is given by $q_p = Q/e$. But at $q = q_p$, $\partial \phi / \partial q = -t\omega/\mu < 0$ holds, then total surplus F_t is not maximum at $q = q_p$. Hence, the optimum flow q_t which maximize total surplus is greater than Q/e . Then Lagrange's multiplier λ is represented by

$$\lambda = \frac{\log\left(\frac{Q}{q}\right)}{1 - \log\left(\frac{Q}{q}\right)}. \tag{24}$$

Since $Q/e < q_t < Q$ is valid, multiplier λ takes positive sign. This concludes that at the maximum of total surplus F_t , the accounting condition (16) takes equality sign, namely $pq = C$ is valid. Equation $pq = C$ has two solutions for q , and the optimum solution is equivalent to larger one naturally.

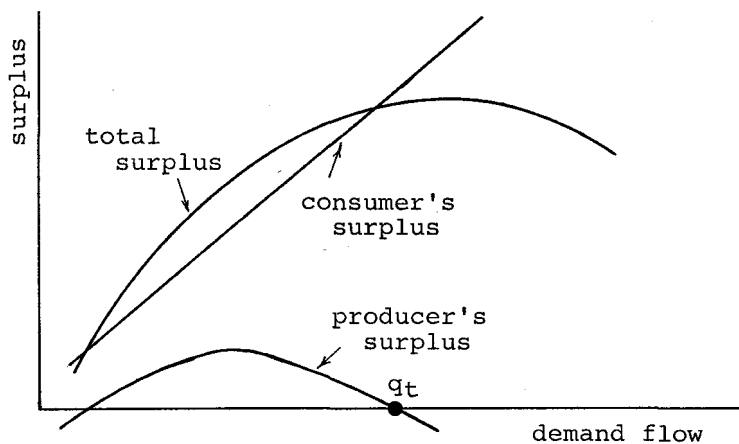


Fig.6 Consumer's, Producer's and Total Surplus in Case of Flow-Independent Saved Travel Times

The optimal flow q_t and optimal rate p_t are represented by

$$q_t = Qe^{-\left(\frac{\lambda}{1+\lambda}\right)}, \quad (25)$$

$$p_t = \frac{\lambda}{1+\lambda} \cdot \frac{\omega t}{\mu} \quad (26)$$

Then multiplier λ is a solution of equation

$$\frac{\omega}{\mu} tQ \frac{\lambda}{1+\lambda} e^{-\left(\frac{\lambda}{1+\lambda}\right)} = c. \quad (27)$$

Equation (27) has a solution such that $\lambda > 0$ surely, from the assumption $C < (Q/e) (\omega t/\mu)$.

In addition, as $\lambda > 0$ is valid,

$$p_t < \frac{1}{\mu} \omega t$$

holds. This means that the optimal rate is smaller than the value of saved travel time obtained by an user through using expressway just as long as the average trip length.

4.3 Optimal Rate under Flow-Dependent Saved Travel Times

Next, the case of flow-dependent saved travel times is treated. In this case, the optimal solution must satisfy

$$qt'(q) + t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) + \lambda \{t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) - t(q)\} = 0. \quad (28)$$

But at a point an equation

$$t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) - t(q) = 0$$

is satisfied, an inequality

$$t(q) + qt'(q) = \frac{t(q)}{\log\left(\frac{Q}{q}\right)} > 0$$

is valid, because $0 < q < Q$ and $t(q) > 0$. Whence, at this point

$$\frac{\partial \phi}{\partial q} = -\frac{\omega}{\mu} \{t(q) + qt'(q)\} < 0$$

holds. This means that the optimal flow q_t under a fixed scale is greater than flow q_p at which the fare revenue is maximum. Then, multiplier λ is expressed as

$$\lambda = -\frac{qt'(q) + t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right)}{t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) - t(q)}. \quad (29)$$

Since producer's surplus F_p is supposed to be concave and have a finite maximum, an inequality

$$\frac{\partial}{\partial q}(F_p) = \frac{\omega}{\mu} \{t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) - t(q)\} < 0 \tag{30}$$

holds at $q=q_t$. Besides it is supposed that there exist demand q and scale s which satisfy $pq-C > 0$, then the problem has a finite optimal solution.

The properties of optimal solution depend on the sign of multiplier. In case of $\lambda=0$, total surplus has a maximum in a region of $pq > C$. The optimal flow is a solution for q of equation

$$qt'(q) + t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) = 0. \tag{31}$$

Rearranging this equation, the optimal rate p_t is expressed as

$$p_t = \frac{\omega t(q)}{\mu} \cdot \frac{-\eta_{tq}}{1+\eta_{tq}} > \frac{C}{q}, \tag{32}$$

where η_{tq} is the elasticity of saved travel time with respect to flow on expressway.

Meanwhile in case of $\lambda > 0$, the total surplus takes maximum value on $pq = C$. In this case, an inequality

$$qt'(q) + t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) > 0$$

is valid at $q=q_t$. Then the optimal rate, which is equivalent to the cost per user, have a property such that

$$p_t = \frac{C}{q} > \frac{\omega t(q)}{\mu} \cdot \frac{-\eta_{tq}}{1+\eta_{tq}}. \tag{33}$$

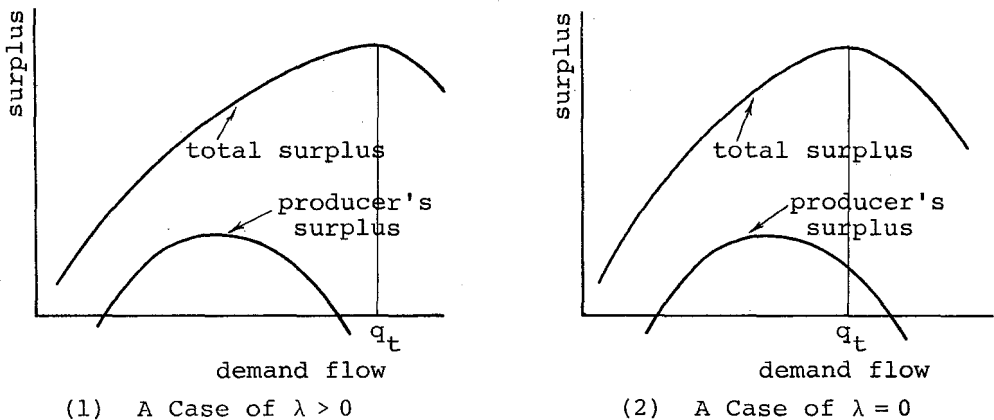


Fig.7 Two Cases of the Maximum of Total Surplus in Case of Flow-Dependent Saved Travel Times

4.4 Properties of Optimal Scale

Next, equation (19) which is closely related to optimal scale is considered. Changing the equation,

$$\frac{\partial \phi}{\partial s} = (1+\lambda) \left\{ -\frac{\omega}{\mu} \cdot \frac{q}{Q} t(q) Q'(s) + C'(s) \right\} = 0 \quad (34)$$

is formed. The Lagrange's multiplier is non-negative, so that

$$-\frac{\omega}{\mu} \cdot \frac{q}{Q} t(q) Q'(s) + C'(s) = 0 \quad (35)$$

is obtained. This is equivalent to $\partial(pq-C)/\partial s = 0$, which means that at the optimal solution, producer's surplus $pq-C$ ought to be maximized for scale s under a constant rate or demand.

Equation (35) can be expressed as

$$\frac{C'(s)}{Q'(s)} \frac{q}{Q} = \frac{\omega}{\mu} t(q), \quad (36)$$

then this is identical with the condition induced by Yamada [2], who derived the condition such that consumer's surplus is maximized under the equality accounting condition using flow-independent saved travel times. Yamada explained equation (36) as follows: "Scale of expressway should be decided such that marginal cost per marginal demand caused by expansion of scale is equivalent to the value of saved travel time obtained by an user through using expressway just as long as the average trip length".

This condition demands that if scale is expanded a little at the optimal scale and rate is regulated such that flow is constant, then the increase of fare revenue should be equivalent to the increase of total cost. At this moment, the amount of total surplus is ensured not to be changed. Now scale is expanded by Δs , the increase of rate Δp is

$$\Delta p = \frac{\omega}{\mu} \cdot \frac{t(q)}{Q} \cdot Q'(s) \Delta s. \quad (37)$$

Accordingly, the increase of fare revenue is expressed by

$$\Delta(pq) = q\Delta p = \frac{\omega}{\mu} \cdot t(q) Q'(s) \cdot \frac{q}{Q} \Delta s. \quad (38)$$

Then equation (35) is formed if $\Delta(pq)$ is equivalent to the increase of cost $C'(s)\Delta s$.

5. Some Examinations on Optimal Solution

5.1 Comparison of Flows and Rates Maximizing Consumer's Surplus, Producer's Surplus and Total Surplus

As was stated previously, the optimal flow q_t which maximizes total surplus is greater than the flow q_p at which producer's surplus takes a maximum. Then, the optimal flow q_t is made comparison with flow q_c which maximizes consumer's surplus.

At $q = q_t$,

$$\frac{\partial}{\partial q}(F_t) = \frac{\omega}{\mu} \{qt'(q) + qt'(q) \log\left(\frac{Q}{q}\right) + t(q) \log\left(\frac{Q}{q}\right)\} \geq 0$$

is valid. Then,

$$t(q) + qt'(q) \geq -\frac{qt'(q)}{\log\left(\frac{Q}{q}\right)} > 0$$

holds. While, $\partial F_c / \partial q = \omega \{t(q) + qt'(q)\} / \mu$, and F_c is supposed to be concave, so that at $q = q_t$, $\partial F_c / \partial q > 0$ is valid. Therefore, $q_t < q_c$ is valid. On the facts as described above,

$$q_p < q_t < q_c \tag{39}$$

can be concluded. On equilibrium demand functions, $dq/dp < 0$ is always valid, then $p_p > p_t > p_c$ can be also concluded.

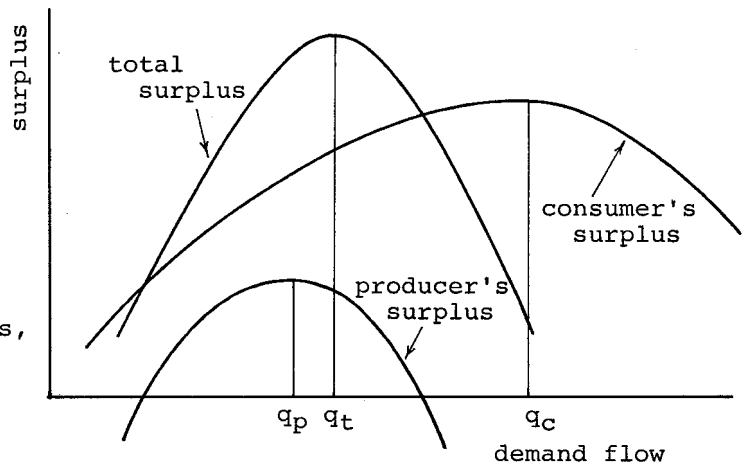


Fig.8 Optimal Flows Maximizing Consumer's, Producer's and Total Surplus

5.2 Comparison of Optimal Solution with the Case of Equality Accounting Condition

If it is supposed that the fare revenue must be equivalent to total cost, the problem becomes maximization of consumer's surplus

$$F_c = \frac{\omega}{\mu} qt(q) \tag{40}$$

subject to equality accounting condition

$$\frac{\omega}{\mu} qt(q) \log\left(\frac{Q}{q}\right) = C. \quad (41)$$

In case of flow-independent travel times, the optimal solution under inequality accounting condition was cleared to exist on equality condition $pq=C$. Therefore the optimal solutions of both problems coincide perfectly.

Meanwhile, in case of flow-dependent travel times, the equation (41) has two solutions for flow q .

Let q_1 denote the smaller solution and q_2 the larger one. In the two, the optimal flow which maximizes consumer's surplus is evidently the larger one $q=q_2$. While the region of feasible solution in case of inequality constraint is $q_1 \leq q \leq q_2$. Therefore, the optimal flow under inequality constraint is equal or smaller than optimal flow under equality constraint. This means that the optimal rate under inequality constraint is equal or greater than optimal rate under equality constraint. Where, the case that both solutions coincide is limited to the case that total surplus takes a maximum on $pq=C$ under inequality constraint.

The necessary condition of first order for consumer's surplus to be maximum under equality constraint is given by

$$t(q) + qt'(q) + \lambda \{ t(q) \log\left(\frac{Q}{q}\right) + qt'(q) \log\left(\frac{Q}{q}\right) - t(q) \} = 0 \quad (42)$$

$$\lambda \left\{ \frac{\omega}{\mu} \cdot \frac{q}{Q} t(q) Q'(s) - C'(s) \right\} = 0 \quad (43)$$

$$\frac{\omega}{\mu} qt(q) \log\left(\frac{Q}{q}\right) - C(s) = 0 \quad (44)$$

Then three cases occur in compliance with the sign of $\partial F_C / \partial q$ at $q=q_2$. If $\partial F_C / \partial q < 0$ (case 1), then $\lambda < 0$ is valid. If $\partial F_C / \partial q = 0$ (case 2), namely

$$t(q) + qt'(q) = 0, \quad (45)$$

then $\lambda = 0$ is obtained. Lastly if $\partial F_C / \partial q > 0$ (case 3), then $\lambda > 0$ holds.

In case 1 and case 3, condition

$$\frac{\omega}{\mu} \cdot \frac{q}{Q} t(q) Q'(s) - C'(s) = 0 \quad (46)$$

must be satisfied. On the contrary in case 2, $\lambda = 0$ holds, so that equation (43) is valid without condition (46).

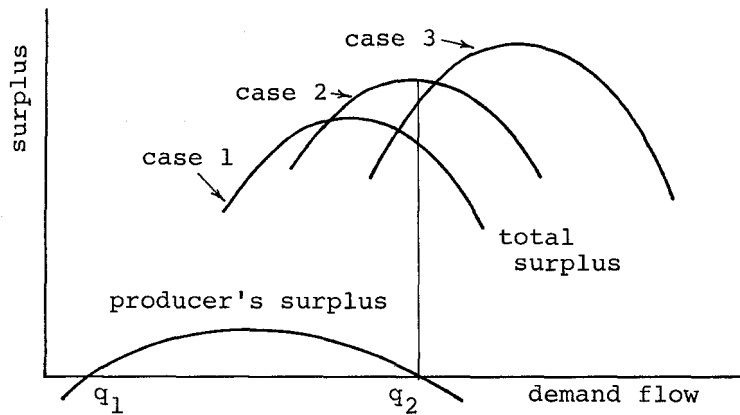


Fig.9 Three Cases in Compliance with the Sign of $\partial F_C/\partial q$ at $q=q_2$

5.3 Examination on Maximum Feasible Scale

When scale of expressway is expanded to the limit under accounting condition, producer's surplus curve for q contacts with q axis at the maximum. The accounting condition is valid only at the point of contact $q=q^*$. Then, equalities

$$pq - C = 0 \tag{47}$$

$$\frac{\partial}{\partial q} (pq - C) = 0 \tag{48}$$

holds at $q = q^*$.

While for total surplus to be maximum at this point, conditions (18) and (19) should also be satisfied. That is variables q , s and multiplier λ must satisfy these four conditions. These hold true only in a very special case. Equation (19) is equivalent to

$$\frac{\partial}{\partial s} (pq - C) = 0 \tag{49}$$

Then conditions (47), (48) and (49) demand that function $F_p(q,s)=pq-C$ contacts with $q-s$ plane at the maximum. This case was excluded from our discussion for the reason that demand and scale are decided uniquely so that there is no room to refer to the optimality of total surplus. Then we can conclude that in a range of our subject of investigation, the maximum feasible scale does not provide the maximum of total surplus.

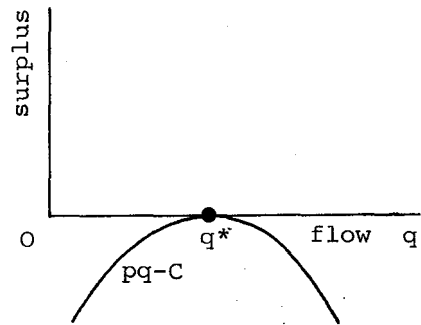


Fig.10 Consumer's Surplus in Case of Maximum Feasible Scale

6. Maximizing Total Surplus under Flow-Dependent Cost Function

6.1 Necessary and Sufficient Condition for Maximum

Now, we analyze the problem on the basis that expressway should be provided at a level above fixed one. To construct and administrate expressway such that traffic congestion does not arise, the total cost required will increase with the increase of traffic demand. Then, total cost is assumed to be a function of demand flow q and scale s , namely $C = C(q, s)$.

Producer's surplus for a given scale s is expressed by

$$F_p = \int_0^q \left\{ p - \frac{\partial C(\xi, s)}{\partial \xi} \right\} d\xi = pq - C(q, s) \tag{50}$$

The service level is assured a fixed one in this case, so that saved travel time is supposed to be constant for flow q . Then total surplus is represented as

$$F_t = \alpha q + \alpha q \log\left(\frac{Q(s)}{q}\right) - C(q, s), \tag{51}$$

where $\alpha = \omega t / \mu$. The accounting condition is written by

$$\alpha q \log\left(\frac{Q(s)}{q}\right) - C(q, s) \geq 0. \tag{52}$$

In the same manner as stated previously, producer's surplus $pq - C$ is supposed to be a concave function for flow q and scale s , and have a finite maximum. The necessary and sufficient condition for producer's surplus to be concave is that Hessian matrix

$$H = \begin{bmatrix} -\frac{\alpha}{q} - C''_{qq} & \frac{\alpha Q'}{Q} - C''_{qs} \\ \frac{\alpha Q'}{Q} - C''_{qs} & \frac{\alpha Q''}{Q} - C''_{ss} \end{bmatrix} \tag{53}$$

is negative definite in the range of variables where accounting condition (52) is satisfied. It is also assumed that there exist flow q and scale s such that

$$\alpha q \log\left(\frac{Q(s)}{q}\right) - C(q, s) > 0 \tag{54}$$

This is equivalent to the condition that there exists a scale s which satisfies

$$C\left(\frac{Q}{e}, s\right) < \alpha \cdot \frac{Q}{e}. \tag{55}$$

Let denote λ be Lagrange's multiplier, then we have Lagrangean function

$$\phi(q, s, \lambda) = -\alpha q - \alpha q \log\left(\frac{Q}{q}\right) + C + \lambda \{-\alpha q \log\left(\frac{Q}{q}\right) + C\} \quad (56)$$

The necessary and sufficient condition for total surplus to be maximum is that there exist demand flow q , scale s and multiplier $\lambda \geq 0$ which satisfy following equations:

$$\frac{\partial \phi}{\partial q} = -\alpha \log\left(\frac{Q}{q}\right) + C'_q + \lambda \{\alpha - \alpha \log\left(\frac{Q}{q}\right) + C'_q\} = 0 \quad (57)$$

$$\frac{\partial \phi}{\partial s} = (\lambda + 1) \{C'_s - \alpha \frac{q}{Q} Q'\} = 0 \quad (58)$$

$$\frac{\partial \phi}{\partial \lambda} = -\alpha q \log\left(\frac{Q}{q}\right) + C \leq 0 \quad (59)$$

$$\lambda \{-\alpha q \log\left(\frac{Q}{q}\right) + C\} = 0 \quad (60)$$

6.2 Properties of Optimal Solution

Properties of optimal solution under flow-dependent cost function is discussed.

At a point such that

$$\frac{\partial F_p}{\partial q} = -\alpha + \alpha \log\left(\frac{Q}{q}\right) - C'_q = 0,$$

$\partial \phi / \partial q = -\alpha < 0$ is obtained, so that optimal flow is greater than the flow which maximize producer's surplus. Besides at the optimal flow $q = q_t$, inequality $\partial F_p / \partial q = -\alpha + \alpha \log(Q/q) - C'_q < 0$ is valid. Then we have

$$\lambda = \frac{\alpha \log\left(\frac{Q}{q}\right) - C'_q}{\alpha - \alpha \log\left(\frac{Q}{q}\right) + C'_q}. \quad (61)$$

At this moment, two cases will occur in accordance with the sign of partial differential coefficient of total surplus. In case of $\alpha \log(Q/q) - C'_q > 0$, $\lambda > 0$ holds true. Then total surplus takes a maximum on $pq = C$. Therefore optimal rate p_t has a property such that

$$p_t = \frac{C}{q_t} > C'_q. \quad (62)$$

While in case of $\alpha \log(Q/q) - C'_q = 0$, we have $\lambda = 0$. Accordingly, total surplus takes a maximum in the region $pq > C$. Then optimal rate has a nature such that

$$p_t = C'_q > \frac{C}{q_t}. \quad (63)$$

Now condition (58) is examined. Since $\lambda \geq 0$ is valid, condition (58) demands

$$C'_s - \alpha \frac{Q}{Q} Q'(s) = 0 . \quad (64)$$

This is as same as is induced in the case of flow-independent cost function. The meanings of this condition were stated before.

If equality accounting condition $pq = C$ is considered, then optimal rate and scale is given as the solution of following equations:

$$-\alpha \log \left(\frac{Q}{q} \right) + C'_q + \lambda \{ \alpha - \alpha \log \left(\frac{Q}{q} \right) + C'_q \} = 0 \quad (65)$$

$$(\lambda + 1) \{ C'_s - \alpha \frac{Q}{Q} Q'(s) \} = 0 \quad (66)$$

$$\alpha q \log \left(\frac{Q}{q} \right) - C = 0 \quad (67)$$

Where multiplier λ is not limited to be non-negative. In this case, at the maximum of producer's surplus, $\alpha - \alpha \log(Q/q) + C'_q = 0$ is valid, then $\partial \phi / \partial q = -\alpha < 0$ holds. So that total surplus does not take a maximum at this point, and at the maximum of total surplus, $-\alpha + \alpha \log(Q/q) - C'_q < 0$ is valid. Accordingly,

$$\lambda = \frac{\alpha \log \left(\frac{Q}{q} \right) - C'_q}{\alpha - \alpha \log \left(\frac{Q}{q} \right) + C'_q} > -1 \quad (68)$$

is induced.

As a result, optimal flow q_t and optimal scale s_t are solutions of equations

$$\alpha \frac{Q}{Q} Q'(s) - C'_s = 0 , \quad (69)$$

$$\alpha q \log \left(\frac{Q}{q} \right) - C = 0 . \quad (70)$$

It is needless to say that within limits of our discussion, the maximum feasible scale under accounting condition does not give the optimal solution either in case of flow-dependent cost function. This can be proved in the same manner as was stated previously.

6.3 Explanation of Optimal Solution Using Graphs

The properties of optimal solution may be explained rationally using graphs. In Fig. 11, D, MC and AC represent equilibrium demand curve $p = f^{-1}(q)$, marginal cost curve $MC = \partial C / \partial q$ and average cost curve $AC = C/q$ respectively. If any constraint is not imposed, the intersection of demand curve and marginal cost curve P gives the maximum of total surplus naturally. But when accounting condition is considered,

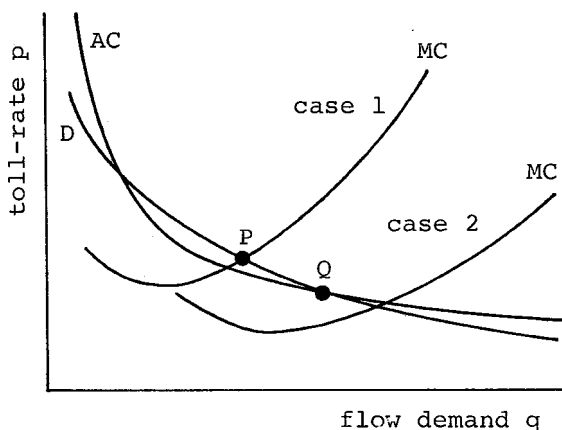


Fig.11 Equilibrium Demand, Marginal Cost and Average Cost Function

for point P to provide the maximum, rate p corresponding to point P should be greater than the average cost (case 1).

On the other hand, if the flow corresponding to the intersection of demand curve and marginal cost exceeds the limit of accounting condition, total surplus takes a maximum at the limit of accounting condition, namely at the intersection of demand curve and average cost curve point Q (case 2). In this case, corresponding marginal cost should be smaller than the optimal rate.

7. Concluding Remarks

So far, we discussed on optimal rate and scale of urban expressway on the basis of maximizing total surplus. Through the analysis, some results were obtained. The summary is as follows:

- (1) Total surplus and accounting condition were formed by expanding Yamada's model.
- (2) The conditions for a maximum of total surplus to be obtained were cleared.
- (3) Necessary and sufficient conditions for total surplus to be maximum were induced.
- (4) Properties concerning to optimal flow, optimal rate and optimal scale in various cases were examined, and their meanings were revealed.
- (5) The maximum feasible scale was cleared not to provide the optimal solution.

But these results involves various kinds of problem, and it is only significant from theoretical point of view.

In modeling traffic demand on expressway, many assumptions were

made. Most of them were first proposed by Yamada, except the idea of equilibrium traffic demand. But they contain some unreal assumptions. Those are the fact that spatial distribution of traffic demand and spatial arrangement of expressway in urban area were ignored. It is also the problem that time process on expansion of expressway network and growth of demand were neglected.

Other important problems are those concerning to the convexity of producer's surplus. A rather strict condition were supposed for producer's surplus. But they may be replaced to a weak condition that producer's surplus is pseudoconcave for flow and scale. This may be satisfied almost in any cases. Even if the condition is so replaced, the analysis developed in this paper is also valid.

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