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On Conformal Mapping onto Circular-Radial Slit Covering Surfaces of Annular and Circular Types

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In paper 7) we concerned ourselves with the conformal mapping onto circular-radial slit covering surfaces over the whole plane and its extremal property. In the present paper we shall concern ourselves with the conformal mapping onto circular-radial slit covering surfaces of annular and circular types and their extremal properties (Theorems 1.1 and 2.1). Especially the extremal property with respect to the radial slits is new.

The results are stated only for the case of the planar domain of finite connectivity. The method suggests the possibility of an extension to the case of a domain of infinite connectivity or an open Riemann surface of finite genus. We shall concern ourselves with this problem in the subsequent paper.

§ 1. Circular-radial slit covering surface of annular type.

1. Definitions. Let B be a domain on the z-plane of which the boundary C consists of a finite number of continua $C_1, \cdots, C_N$ ($N \geqq 2$). Partition the boundary C into three disjoint sets of its components

$$C' = \sum_{j=1}^{\lambda} C_j,$$

$$C'' = \sum_{j=1}^{\mu} C_{\lambda+j}$$

and

$$C''' = \sum_{j=1}^{\nu} C_{\lambda+\mu+j}$$

$$(\lambda \geqq 2,\ \mu \geqq 0,\ \nu \geqq 0,\ \lambda+\mu+\nu = N),$$

where $C'' = \phi$ or $C''' = \phi$ is permitted. Let $w = f(z)$ be a single-valued regular function on B. The rotation number $\nu_k(f)$ of the image of a generic boundary component C_k about $w=0$ under f is defined by

$$\nu_k(f) \equiv \frac{1}{2\pi}\int_{C_k^*} d \arg f,$$

where C_k^* is an analytic Jordan curve homotopic to C_k in B and $\nu_k(f)$ is an integer not depending on a particular choice of C_k^*.

Let z_0 be an arbitrarily preassigned point in B and let ν_j ($j=1, \cdots, \lambda$) be arbitrarily preassigned non-zero integers under the condition

$$(1.1) \qquad \sum_{j=1}^{\lambda} \nu_j = 0.$$

Let $\mathfrak{F}$ be the class of single-valued regular functions $w=f(z)$ on B which have the properties:

(a) f has no zero point in B;

(b) $\nu_j(f) = \nu_j \quad (j=1, \cdots, \lambda)$,
$\nu_j(f) = 0 \quad (j=\lambda+1, \cdots, N)$;

(c) $D_B(\lg|f|) < +\infty$,

where by $D_B(\lg|f|)$ we denote the Dirichlet integral of $\lg|f|$ over B;

(d) f satisfies the normalization condition:

$$f(z_0) = 1.$$

We can easily see that $\mathfrak{F} \neq \phi$.

Let $\{\Omega_{jn}\}_{n=1}^{\infty}$ ($j=1, \cdots, N$) be sequences of ends defining C_j respectively; i. e. $C_j \subset \overline{\Omega_{jn}}$ for every n, $\overline{\Omega_{j,n+1}} \subset \Omega_{jn}$ and $\bigcap_{n=1}^{\infty} \Omega_{jn} = \phi$. Let $\mathfrak{G}$ be the subclass of $\mathfrak{F}$ which consists of functions $f(z)$ of $\mathfrak{F}$ satisfying the condition:

(e) An arbitrary branch of $\arg f$ is constant on each component C_j ($j=\lambda+\mu+1, \cdots, N$), which means that

$$\bigcap_{n=1}^{\infty} \overline{\arg f(\Omega_{jn})}$$

is reduced to a real value.

Let $\mathfrak{H}$ be the subclass of $\mathfrak{F}$ which consists of functions $f(z)$ of $\mathfrak{F}$ satisfying the condition:

(f) $|f|$ is constant on each component C_j $(j=\lambda+1, \cdots, \lambda+\mu)$, which means that

$$\bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|}$$

is reduced to a real value.

Set

$$m_j(f) \equiv \inf \bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|},$$

$$M_j(f) \equiv \sup \bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|} \qquad \text{if } \nu_j(f)>0,$$

and

$$m_j(f) \equiv \sup \bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|},$$

$$M_j(f) \equiv \inf \bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|} \qquad \text{if } \nu_j(f)<0$$

$(j=1, \cdots, \lambda)$. Let $\mathfrak{G}'$ be the subclass of $\mathfrak{G}$ which consists of functions $f(z)$ of $\mathfrak{G}$ satisfying the condition:

(g) $$D_B(\lg|f|) \leqq 2\pi \sum_{j=1}^{\lambda} \nu_j \lg M_j(f),$$

where by $D_B(\lg|f|)$ we denote the Dirichlet integral of $\lg|f|$ over B.

Let $\mathfrak{H}'$ be the subclass of $\mathfrak{H}$ which consists of functions $f(z)$ of $\mathfrak{H}$ satisfying the condition:

(h) $$D_B(\lg|f|) \leqq 2\pi \sum_{j=1}^{\lambda} \nu_j \lg m_j(f).$$

Let $\{B_n\}_{n=1}^{\infty}$ be an exhaustion of B such that the boundary ∂B_n of B_n consists of analytic Jordan curves C_{jn} $(j=1, \cdots, N)$ homotopic to C_j respectively, and let $C'_n = \sum_{j=1}^{\lambda} C_{jn}$, $C''_n = \sum_{j=1}^{\mu} C_{\lambda+j,n}$ and $C'''_n = \sum_{j=1}^{\nu} C_{\lambda+\mu+j,n}$. If $f \in \mathfrak{G}$ satisfies the condition:

(i) $$\int_{C''} \lg|f| d \arg f \equiv \lim_{n\to\infty} \int_{C''_n} \lg|f|\, d \arg f \leqq 0,$$

then f belongs to the class $\mathfrak{G}'$. If $f \in \mathfrak{H}$ satisfies the conditions:

(j) $|f| = \text{const.} (=m_j(f)=M_j(f))$ on each component C_j $(j=1, \cdots, \lambda)$;

(k) $$\int_{C'''} \lg|f|\, d \arg f \equiv \lim_{n\to\infty} \int_{C'''_n} \lg|f|\, d \arg f \leqq 0,$$

then f belongs to the class $\mathfrak{H}'$. We shall see that the classes $\mathfrak{G}, \mathfrak{H}, \mathfrak{G}'$ and $\mathfrak{H}'$ are not vacuous (cf. Remark of **3**).

2. Theorem. By a *function of the class* $\mathfrak{F}$ *mapping onto a circular-radial slit covering surface of annular type* we mean the function φ of the class $\mathfrak{F}$ which satisfies the conditions:

(l) An arbitrary branch of $\arg \varphi$ is constant on each component C_j $(j=\lambda+\mu+1, \cdots, N)$;

(m) $|\varphi|$ is constant on each component C_j $(j=1, \cdots, \lambda+\mu)$.

For the function φ

$$D_B(\lg|\varphi|) = 2\pi \sum_{j=1}^{\lambda} \nu_j \lg m_j(\varphi) = 2\pi \sum_{j=1}^{\lambda} \nu_j \lg M_j(\varphi)$$

holds. The image covering surface by φ is called a *circular-radial slit covering surface of annular type*.

Theorem 1.1. (i) *For each class* $\mathfrak{F}$ *there exists one and only one function* φ *of the class* $\mathfrak{F}$ *mapping onto a circular-radial slit covering surface of annular type*;

(ii) *The function* φ *is the only element which simultaneously belongs to* $\mathfrak{G}$ *and* $\mathfrak{H}$;

(iii) *For every* $f \in \mathfrak{G}$ *the inequality*

$$D_B(\lg|\varphi|) \leqq D_B(\lg|f|)$$

holds and thus for every $f \in \mathfrak{G}'$ *the inequality*

$$\prod_{j=1}^{\lambda} M_j(\varphi)^{\nu_j} \leqq \prod_{j=1}^{\lambda} M_j(f)^{\nu_j}$$

holds. In the both inequalities the equality signs appear if and only if $f \equiv \varphi$;

(iv) *For every* $f \in \mathfrak{H}'$ *the inequalities*

$$\prod_{j=1}^{\lambda} m_j(\varphi)^{\nu_j} \geqq \prod_{j=1}^{\lambda} m_j(f)^{\nu_j}$$

and thus

$$D_B(\lg|\varphi|) \geqq D_B(\lg|f|)$$

hold. In the both inequalities the equality signs appear if and only if $f \equiv \varphi$.

Proof. (i) and (iii) have been proved for the case of an infinitely connected domain in Theorems 5.2 and 7.1 of 8) respectively. For the conformity we shall state their proofs again.

The domain B can always be conformally mapped onto the domain by a univalent function Φ of which the boundary consists of analytic Jordan curves. Thus we may assume that so is the domain B.

Construction of φ *in* (i). It is easy to find a solution u of the boundary value problem satisfying the conditions:

(A) u is single-valued harmonic on B;

(B) u is constant on each boundary component C_j $(j=1,\cdots,\lambda+\mu)$ with the constant so chosen that

$$\int_{C_j}\frac{\partial u}{\partial n}ds=-2\pi\nu_j \quad (j=1,\cdots,\lambda),$$

$$\int_{C_j}\frac{\partial u}{\partial n}ds=0 \quad (j=\lambda+1,\cdots,\lambda+\mu),$$

where $\partial/\partial n$ denotes the inner normal derivative on C_j and ds the line element of C_j;

(C) $\frac{\partial u}{\partial n}=0$ along C_j $(j=\lambda+\mu+1,\cdots,N)$;

(D) $u(z_0)=0$.

Let u^* be a conjugate harmonic function of u uniquely determined up to multiples of 2π under the condition

$$u^*(z_0)\equiv 0 \quad (\mathrm{mod}\ 2\pi)$$

and set $\varphi(z)=\exp(u+iu^*)$. Then it is easily verified that $\varphi(z)$ is the function satisfying the property of (i) up to the uniqueness.

Proof of (iii). Let f be an arbitrary element of $\mathfrak{G}$, and set $U=\lg|f|$, $u=\lg|\varphi|$ and $h=U-u$. Then we have that

$$(1.2)\quad \begin{aligned} &D_B(\lg|f|)-D_B(\lg|\varphi|)\\ &\quad=2D_B(u,h)+D_B(h).\end{aligned}$$

We shall verify that

$$(1.3)\quad D_B(u,h)=0.$$

Let $\{B_n\}_{n=1}^{\infty}$ be an exhaustion of B such that C''' is a portion of the boundary ∂B_n of B_n for all n and $\partial B_n-C'''$ consists of analytic Jordan curves C_{jn} $(j=1,\cdots,\lambda+\mu)$ homotopic to C_j respectively. Let $u_n(z)$ for each $n=1,2,\cdots$ be the function on B_n which satisfies the conditions:

(A) u_n is single-valued harmonic on B_n;

(B) $u_n=c_j$ on each component C_{jn} $(j=1,\cdots,\lambda+\mu)$, where c_j $(j=1,\cdots,\lambda+\mu)$ are the constant values which $u(z)$ takes on C_j respectively;

(C) $\frac{\partial u_n}{\partial n}=0$ along C'''.

Set $u_n(z)=c_j$ $(j=1,\cdots,\lambda+\mu)$ on each ring domain of $B-\overline{B}_n$ adjacent to C_{jn} respectively. Then we can easily see that $\{u_n\}_{n=1}^{\infty}$ uniformly converges to u on B and thus

$$(1.4)\quad \lim_{n\to\infty}D_B(u-u_n)=0.$$

Since

$$\int_{C_{jn}}\frac{\partial h}{\partial n}ds=\int_{C_{jn}}\frac{\partial U}{\partial n}ds-\int_{C_{jn}}\frac{\partial u}{\partial n}ds=0 \quad (j=1,\cdots,\lambda+\mu)$$

and

$$\frac{\partial h}{\partial n}=0 \quad \text{along } C''',$$

we find that

$$(1.5)\quad D_{B_n}(u_n,h)=-\int_{\partial B_n}u_n\frac{\partial h}{\partial n}ds=0 \quad \text{for every } n.$$

Further by the Schwarz inequality

$$(1.6)\quad \begin{aligned}&|D_B(u,h)-D_{B_n}(u_n,h)|\\ &\quad\leqq\sqrt{D_B(u-u_n)D_B(h)}\end{aligned}$$

holds. Our assertion (1.3) follows from (1.4), (1.5) and (1.6). Consequently by (1.2) and (1.3) we have that

$$D_B(\lg|f|)-D_B(\lg|\varphi|)=D_B(h)\geqq 0.$$

The equality sign in the last inequality appears if and only if $h\equiv$const.$=0$ and thus $f\equiv\varphi$, because of the normalization condition (d) of **1.**

Proof of (iv). Let f be an arbitrary element of $\mathfrak{H}'$, and set $U=\lg|f|$, $u=\lg|\varphi|$ and $h=u-U$. Then we have that

$$(1.7)\quad \begin{aligned}&2\pi\Big(\sum_{j=1}^{\lambda}\nu_j\lg m_j(\varphi)-\sum_{j=1}^{\lambda}\nu_j\lg m_j(f)\Big)\\ &=D_B(\lg|\varphi|)-2\pi\sum_{j=1}^{\lambda}\nu_j\lg m_j(f)\\ &=D_B(h)+2D_B(U,u)-D_B(U)\\ &\quad-2\pi\sum_{j=1}^{\lambda}\nu_j\lg m_j(f).\end{aligned}$$

We shall verify that

$$(1.8)\quad D_B(U,u)\geqq 2\pi\sum_{j=1}^{\lambda}\nu_j\lg m_j(f).$$

Let $\{B_n\}_{n=1}^{\infty}$ be an exhaustion of B such that C'' is a portion of the boundary ∂B_n of B_n and $\partial B_n-C''$ consists of analytic Jordan curves C_{jn} $(j=1,\cdots,\lambda,\lambda+\mu+1,\cdots,N)$ homotopic to C_j respectively. Let B_{nk} be a subdomain of B surrounded by $C_n'=\sum_{j=1}^{\lambda}C_{jn}$, C'' and $C_k'''=\sum_{j=\lambda+\mu+1}^{N}C_{jk}$ $(k>n)$. Thus $B_n\subset B_{nk}$, and C_n' and C'' are common portions of the boundaries of B_n and B_{nk}. Let $v_{nk}(z)$ for each pair of n and k $(k>n)$ be the function on B_{nk} which satisfies the conditions:

(A) v_{nk} is single-valued harmonic on B_{nk};

(B) $v_{nk}=$ const. on each component C_{jn} ($j=1,\cdots,\lambda$) and on each component C_j ($j=\lambda+1,\cdots,\lambda+\mu$) with the constant so chosen that

$$\int_{C_{jn}}\frac{\partial v_{nk}}{\partial n}ds=-2\pi\nu_j \quad (j=1,\cdots,\lambda),$$

$$\int_{C_j}\frac{\partial v_{nk}}{\partial n}ds=0 \qquad (j=\lambda+1,\cdots\lambda+\mu);$$

(C) $\dfrac{\partial v_{nk}}{\partial n}=0$ along C_k''';

(D) $v_{nk}(z_0)=0$.

For $l>k$ the equation

$$D_{B_{nk}}(v_{nk},v_{nl})=-\int_{\partial B_{nk}}v_{nl}\frac{\partial v_{nk}}{\partial n}ds$$

$$=-\int_{\partial B_{nl}}v_{nl}\frac{\partial v_{nl}}{\partial n}ds=D_{B_{nl}}(v_{nl})$$

implies that

$$D_{B_{nk}}(v_{nl}-v_{nk})\leqq D_{B_{nk}}(v_{nk})-D_{B_{nl}}(v_{nl}).$$

Thus $D_{B_{nk}}(v_{nk})$ is monotone decreasing with k and v_{nk} strongly converges to a function v_n on $\boldsymbol{B}_n\equiv\bigcup_{k=n+1}^{\infty}B_{nk}$ as $k\to\infty$:

(1.9) $\lim\limits_{k\to\infty}D_{B_{nk}}(v_{nk}-v_n)=0.$

The function v_n satisfies the conditions:

(A) v_n is single-valued harmonic on $\boldsymbol{B}_n$;

(B) $v_n=$ const. on each component C_{jn} ($j=1,\cdots,\lambda$) and on each component C_j ($j=\lambda+1,\cdots,\lambda+\mu$) with the constant so chosen that

$$\int_{C_{jn}}\frac{\partial v_n}{\partial n}ds=-2\pi\nu_j \quad (j=1,\cdots,\lambda),$$

$$\int_{C_j}\frac{\partial v_n}{\partial n}ds=0 \qquad (j=\lambda+1,\cdots,\lambda+\mu);$$

(C) $\dfrac{\partial v_n}{\partial n}=0$ along C''';

(D) $v_n(z_0)=0$.

For $n>m$ the equation

$$D_{\boldsymbol{B}_m}(v_m,v_n)=-\int_{\partial\boldsymbol{B}_m}v_m\frac{\partial v_n}{\partial n}ds$$

$$=-\int_{\partial\boldsymbol{B}_m}v_m\frac{\partial v_m}{\partial n}ds=D_{\boldsymbol{B}_m}(v_m)$$

implies that

$$D_{\boldsymbol{B}_m}(v_n-v_m)\leqq D_{\boldsymbol{B}_n}(v_n)-D_{\boldsymbol{B}_m}(v_m).$$

Thus $D_{\boldsymbol{B}_n}(v_n)$ is monotone increasing with n. Furthermore the equation

$$D_{\boldsymbol{B}_n}(v_n,u)=-\int_{\partial\boldsymbol{B}_n}v_n\frac{\partial u}{\partial n}ds$$

$$=-\int_{\partial\boldsymbol{B}_n}v_n\frac{\partial v_n}{\partial n}ds=D_{\boldsymbol{B}_n}(v_n)$$

implies that

$$D_{\boldsymbol{B}_n}(u-v_n)\leqq D_B(u)-D_{\boldsymbol{B}_n}(v_n).$$

Thus v_n strongly converges to u on B as $n\to\infty$:

(1.10) $\lim\limits_{n\to\infty}D_{\boldsymbol{B}_n}(u-v_n)=0.$

By (1.9), (1.10) and the inequality

$$\sqrt{D_{B_{nk}}(u-v_{nk})}\leqq\sqrt{D_{\boldsymbol{B}_n}(u-v_n)}+\sqrt{D_{B_{nk}}(v_n-v_{nk})},$$

we see that there exists a subsequence $\{k_n\}_{n=1}^{\infty}$ of $\{k\}_{k=1}^{\infty}$ such that

(1.11) $\lim\limits_{n\to\infty}D_{B_{nk_n}}(u-v_{nk_n})=0.$

Further the inequality

$$|D_B(U,u)-D_{B_{nk_n}}(U,v_{nk_n})|$$

$$\leqq|D_{B_{nk_n}}(U,u-v_{nk_n})|+|D_{B-B_{nk_n}}(U,u)|$$

$$\leqq\sqrt{D_B(U)D_{B_{nk_n}}(u-v_{nk_n})}+|D_{B-B_{nk_n}}(U,u)|$$

holds and thus

(1.12) $\lim\limits_{n\to\infty}D_{B_{nk_n}}(U,v_{nk_n})=D_B(U,u).$

If we note that

$$\int_{C_{jn}}\frac{\partial v_{nk}}{\partial n}ds=2\pi\nu_j \quad (j=1,\cdots,\lambda),$$

$U=$ const. on each C_j ($j=\lambda+1,\cdots,\lambda+\mu$),

$$\int_{C_j}\frac{\partial v_{nk}}{\partial n}ds=0, \qquad \int_{C_j}\frac{\partial U}{\partial n}ds=0$$

$$(j=\lambda+1,\cdots,\lambda+\mu)$$

and

$$\frac{\partial v_{nk}}{\partial n}=0 \qquad \text{along } C_k''',$$

then we find that

(1.13) $D_{B_{nk}}(U,v_{nk})=-\displaystyle\int_{\partial B_{nk}}U\frac{\partial v_{nk}}{\partial n}ds$

$$=-\sum_{j=1}^{\lambda}\int_{C_{jn}} U\frac{\partial v_{nk}}{\partial n}ds \geqq 2\pi\sum_{j=1}^{\lambda}\nu_j \mu_{jn}(U),$$

where $\mu_{jn}(U)$ $(j=1,\cdots,\lambda)$ are defined by

$$\mu_{jn}(U)\equiv\min_{C_{jn}} U \qquad \text{if } \nu_j>0$$

and

$$\mu_{jn}(U)\equiv\max_{C_{jn}} U \qquad \text{if } \nu_j<0.$$

We can easily see that

$$(1.14)\quad \lim_{n\to\infty}\mu_{jn}(U)=\lg m_j(f) \quad (j=1,\cdots,\lambda).$$

Our assertion (1.8) follows from (1.12), (1.13) and (1.14). Consequently, by (1.7), (1.8) and the condition (h) of **1** we have that

$$(1.15)\quad \begin{aligned} &2\pi\Big(\sum_{j=1}^{\lambda}\nu_j \lg m_j(\varphi)-\sum_{j=1}^{\lambda}\nu_j \lg m_j(f)\Big)\\ &\geqq D_B(h)+2\pi\sum_{j=1}^{\lambda}\nu_j \lg m_j(f)-D_B(\lg|f|)\\ &\geqq 0. \end{aligned}$$

The equality sign in the second inequality of (1.15) appears if and only if $h\equiv$ const. $=0$ and thus $f\equiv\varphi$, because of the normalization condition (d) of **1**. Then the equality sign in the first inequality of (1.15) also appears.

Proof of the uniqueness in (i). Let $\hat{\varphi}$ be another element of $\mathfrak{F}$ with the same circular-radial slit mapping property of annular type as φ. Then by (iii) and (iv) we have that

$$D_B(\lg|\hat{\varphi}|)=D_B(\lg|\varphi|)$$

and thus

$$\hat{\varphi}\equiv\varphi.$$

Now (ii) is evident.

Remark. The second inequality of (iii) of Theorem 1.1 does not generally hold for $f\in\mathfrak{G}$. Also the both inequalities of (iv) of Theorem 1.1 do not generally hold for $f\in\mathfrak{H}$. They are shown by simple examples (cf. Examples of 7)).

3. Corollaries. We should note that in Theorem 1.1 the case $C''=\phi$ or $C'''=\phi$ is permitted. Then we have the following corollary.

Corollary 1.1. (i) *For each class* $\mathfrak{F}$ *there exists one and only one function* ψ *of the class* $\mathfrak{F}$ *mapping onto a circular slit covering surface of annular type;*

(ii) *For every* $f\in\mathfrak{F}$ *the inequality*

$$D_B(\lg|\psi|)\leqq D_B(\lg|f|)$$

holds and thus for every $f\in\mathfrak{F}$ *satisfying* (g) *of* **1** *the inequality*

$$\prod_{j=1}^{\lambda}M_j(\psi)^{\nu_j}\leqq\prod_{j=1}^{\lambda}M_j(f)^{\nu_j}$$

holds. In the both inequalities the equality signs appear if and only if $f\equiv\psi$;

(iii) *For each class* $\mathfrak{F}$ *there exists one and only one function* χ *of the class* $\mathfrak{F}$ *mapping onto a radial slit covering surface of annular type;*

(iv) *For every* $f\in\mathfrak{F}$ *satisfying* (h) *of* **1**, *the inequalities*

$$\prod_{j=1}^{\lambda}m_j(\chi)^{\nu_j}\geqq\prod_{j=1}^{\lambda}m_j(f)^{\nu_j}$$

and thus

$$D_B(\lg|\chi|)\geqq D_B(\lg|f|)$$

hold. In the both inequalities the equality signs appear if and only if $f\equiv\chi$.

Remark. Let D_j $(j=\lambda+1,\cdots,N)$ be the complement continua of B adjacent to C_j respectively and let

$$B^1=B+\sum_{j=1}^{\mu}D_{\lambda+j} \quad\text{and}\quad B^2=B+\sum_{j=1}^{\nu}D_{\lambda+\mu+j}.$$

Let $\mathfrak{F}(B^1)$ and $\mathfrak{F}(B^2)$ be the class $\mathfrak{F}$ defined for the domains B^1 and B^2 respectively in place of B. Apply the consequences (iii) and (i) of Corollary 1.1 to $\mathfrak{F}(B^1)$ and $\mathfrak{F}(B^2)$ respectively. Then we see that the restrictions to the domain B of the functions $\chi\in\mathfrak{F}(B^1)$ and $\psi\in\mathfrak{F}(B^2)$ of Corollary 1.1 belong to $\mathfrak{G}$ and $\mathfrak{H}$ respectively. Furthermore it is easily verified that the functions χ and ψ also belong to $\mathfrak{G}'$ and $\mathfrak{H}'$ respectively. The above construction method is available for each domain conformally equivalent to B in place of B. Therefore we know that the both classes $\mathfrak{G}'$ and $\mathfrak{H}'$ have infinite numbers (in continuum potency) of elements other than the function φ of Theorem 1.1.

Consider the class $\mathfrak{F}$ of the case $\lambda=2$, $\nu_1=1$, $\nu_2=-1$ in **1** and let $\mathfrak{F}_1$, $\mathfrak{G}_1$ and $\mathfrak{H}_1$ be the subclasses of $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ respectively consisting of *univalent* functions $f(z)$. Then $\mathfrak{G}_1\subset\mathfrak{G}'$ and further $f\in\mathfrak{H}'$ for $f\in\mathfrak{H}_1$ satisfying (j) of **1.** Thus we have the following corollary of

Theorem 1.1.

COROLLARY 1.2. (i) *For the class* $\mathfrak{F}$ *of the case* $\lambda=2$, $\nu_1=1$, $\nu_2=-1$ *there exists one and only one function* φ *of the class* $\mathfrak{F}$ *mapping onto a circular-radial slit covering surface of annular type (circular-radial slit annulus)*;

(ii) *The function* φ *is the only element which simultaneously belongs to* $\mathfrak{G}$ *and* $\mathfrak{H}$;

(iii) *For every* $f\in\mathfrak{G}_1$ *the inequalities*

$$D_B(\lg|\varphi|)\leqq D_B(\lg|f|)$$

and thus

$$\frac{M_1(\varphi)}{M_2(\varphi)}\leqq\frac{M_1(f)}{M_2(f)}$$

hold. In the both inequalities the equality signs appear if and only if $f\equiv\varphi$;

(iv) *For every* $f\in\mathfrak{H}_1$ *satisfying* (j) *of* **1**, *the inequalities*

$$\frac{m_1(\varphi)}{m_2(\varphi)}\geqq\frac{m_1(f)}{m_2(f)}$$

and thus

$$D_B(\lg|\varphi|)\geqq D_B(\lg|f|)$$

hold. In the both inequalities the equality signs appear if and only if $f\equiv\varphi$.

If $C'''=\phi$ (or $C''=\phi$) in (iii) (or (iv) resp.) of Corollary 1. 2, then $\mathfrak{G}_1=\mathfrak{F}_1$ (or $\mathfrak{H}_1=\mathfrak{F}_1$ resp.) and the present consequences are reduced to the well-known classical results (cf. 1), 9) and 10)).

§ 2. Circular-radial slit covering surface of circular type.

1. Definitions. Let B be a domain on the z-plane of which the boundary C consists of a finite number of continua $C_1,\cdots,C_N$ ($N\geqq 1$). Partition the boundary C into three disjoint sets of its components

$$C'=\sum_{j=1}^{\lambda}C_j,$$

$$C''=\sum_{j=1}^{\mu}C_{\lambda+j}$$

and

$$C'''=\sum_{j=1}^{\nu}C_{\lambda+\mu+j}$$

$$(\lambda\geqq1,\ \mu\geqq0,\ \nu\geqq0,\ \lambda+\mu+\nu=N),$$

where $C''=\phi$ or $C'''=\phi$ is permitted. Let z_j ($j=1,\cdots,\iota$; $\iota\geqq1$) be arbitrarily preassigned ι points in B, and let $w=f(z)$ be a single-valued regular function on B which has the only zeros z_j ($j=1,\cdots,\iota$). The rotation number $\nu_k(f)$ of the image of a generic boundary component C_k about $w=0$ under f is defined by

$$\nu_k(f)\equiv\frac{1}{2\pi}\int_{C_k^*}d\arg f,$$

where C_k^* is an analytic Jordan curve homotopic to C_k in $B-\{z_j\}_{j=1}^{\iota}$ and $\nu_k(f)$ is an integer not depending on a particular choice of C_k^*.

Let n_j ($j=1,\cdots,\iota$) and ν_j ($j=1,\cdots,\lambda$) be arbitrarily preassigned positive integers under the condition

$$\sum_{j=1}^{\iota}n_j=\sum_{j=1}^{\lambda}\nu_j. \tag{2.1}$$

Let $\mathfrak{F}$ be the class of single-valued regular functions $w=f(z)$ on B which have the properties:

(a) f has only zeros z_j ($j=1,\cdots,\iota$) with their order n_j respectively;

(b) $\nu_j(f)=\nu_j$ $(j=1,\cdots,\lambda)$,
$\nu_j(f)=0$ $(j=\lambda+1,\cdots,N)$;

(c) $$\left|\int_C\lg|f|\,d\arg f\right|<+\infty,$$

where the line integral means $\lim_{n\to\infty}\int_{\partial B_n}\lg|f|\,d\arg f$ with an exhaustion $\{B_n\}_{n=1}^{\infty}$ of B;

(d) f satisfies the normalization condition

$$\lim_{z\to z_1}\frac{f(z)}{(z-z_1)^{n_1}}=\frac{1}{n_1!}f^{(n_1)}(z_1)=1.$$

We can easily see that $\mathfrak{F}\neq\phi$.

Let $\{\Omega_{jn}\}_{n=1}^{\infty}$ ($j=1,\cdots,N$) be sequences of ends defining C_j respectively; i. e. $C_j\subset\overline{\Omega_{jn}}$ for every n, $\overline{\Omega_{j,n+1}}\subset\Omega_{jn}$ and $\bigcap_{n=1}^{\infty}\Omega_{jn}=\phi$. Let $\mathfrak{G}$ be the subclass of $\mathfrak{F}$ which consists of functions $f(z)$ of $\mathfrak{F}$ satisfying the condition:

(e) An arbitrary branch of $\arg f$ is constant on each component C_j ($j=\lambda+\mu+1,\cdots,N$), which means that

$$\bigcap_{n=1}^{\infty}\overline{\arg f(\Omega_{jn})}$$

is reduced to a real value.

Let $\mathfrak{H}$ be the subclass of $\mathfrak{F}$ which consists of functions $f(z)$ of $\mathfrak{F}$ satisfying the conditions:

(f) $|f|$ is constant on each component C_j

$(j=\lambda+1, \cdots, \lambda+\mu)$, which means that

$$\bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|}$$

is reduced to a real value.

Set

$$m_j(f) \equiv \inf \bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|},$$

$$M_j(f) \equiv \sup \bigcap_{n=1}^{\infty} \overline{|f(\Omega_{jn})|} \quad (j=1, \cdots, \lambda).$$

Let $\mathfrak{G}'$ be the subclass of $\mathfrak{G}$ which consists of functions $f(z)$ of $\mathfrak{G}$ satisfying the condition:

(g) $\int_C \lg|f| \, d \arg f \leqq 2\pi \sum_{j=1}^{\lambda} \nu_j \lg M_j(f).$

Let $\mathfrak{H}'$ be the subclass of $\mathfrak{H}$ which consists of functions $f(z)$ of $\mathfrak{H}$ satisfying the condition:

(h) $\int_C \lg|f| \, d \arg f \leqq 2\pi \sum_{j=1}^{\lambda} \nu_j \lg m_j(f).$

Let $\{B_n\}_{n=1}^{\infty}$ be an exhaustion of B such that the boundary ∂B_n of B_n consists of analytic Jordan curves C_{jn} $(j=1, \cdots, N)$ homotopic to C_j respectively and such that $\{z_j\}_{j=1}^{\iota} \subset B_1$, and let $C_n' = \sum_{j=1}^{\lambda} C_{jn}$, $C_n'' = \sum_{j=1}^{\mu} C_{\lambda+j,n}$ and $C_n''' = \sum_{j=1}^{\nu} C_{\lambda+\mu+j,n}$. If $f \in \mathfrak{G}$ satisfies the condition:

(i) $$\int_{C''} \lg|f| \, d \arg f \equiv \lim_{n\to\infty} \int_{C_n''} \lg|f| \, d \arg f \leqq 0,$$

then f belongs to the class $\mathfrak{G}'$. If $f \in \mathfrak{H}$ satisfies the conditions:

(j) $|f| = \text{const.} (=m_j(f)=M_j(f))$ on each component C_j $(j=1, \cdots, \lambda)$;

(k) $$\int_{C'''} \lg|f| \, d \arg f \equiv \lim_{n\to\infty} \int_{C_n'''} \lg|f| \, d \arg f \leqq 0,$$

then f belongs to the class $\mathfrak{H}'$. We shall see that the classes $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{G}'$ and $\mathfrak{H}'$ are not vacuous (cf. Remark of **3**).

2. Theorem. By a *function of the class* $\mathfrak{F}$ *mapping onto a circular-radial slit covering surface of circular type* we mean the function φ of the class $\mathfrak{F}$ which satisfies the conditions:

(l) An arbitrary branch of $\arg \varphi$ is constant on each component C_j $(j=\lambda+\mu+1, \cdots, N)$;

(m) $|\varphi|$ is constant on each component C_j $(j=1, \cdots, \lambda+\mu)$.

For the function φ

$$\int_{\partial B} \lg|\varphi| \, d \arg \varphi = 2\pi \sum_{j=1}^{\lambda} \nu_j \lg m_j(\varphi) = 2\pi \sum_{j=1}^{\lambda} \nu_j \lg M_j(\varphi)$$

holds. The image covering surface by φ is called a *circular-radial slit covering surface of circular type*.

Let

$$J(f) = \int_C \lg|f| \, d \arg f - 2\pi \sum_{j=1}^{\iota} n_j \lg |f^{[n_j]}(z_j)|, \tag{2.2}$$

and

$$m(f) = \frac{\prod_{j=1}^{\lambda} m_j(f)^{\nu_j}}{\prod_{j=1}^{\iota} |f^{[n_j]}(z_j)|^{n_j}}, \quad M(f) = \frac{\prod_{j=1}^{\lambda} M_j(f)^{\nu_j}}{\prod_{j=1}^{\iota} |f^{[n_j]}(z_j)|^{n_j}} \tag{2.3}$$

for $f \in \mathfrak{F}$, where

$$f^{[n_j]}(z_j) \equiv \lim_{z\to z_j} \frac{f(z)}{(z-z_j)^{n_j}} = \frac{1}{n_j!} f^{(n_j)}(z_j) \quad (j=1, \cdots, \iota).$$

Generally $M(f) \geqq m(f)$. $2\pi \lg M(f) \geqq J(f)$ for $f \in \mathfrak{G}'$, $2\pi \lg m(f) \geqq J(f)$ for $f \in \mathfrak{H}'$ and further $2\pi \lg m(\varphi) = 2\pi \lg M(\varphi) = J(\varphi)$ for the function φ.

THEOREM 2. 1. (i) *For each class* $\mathfrak{F}$ *there exists one and only one function* φ *of the class* $\mathfrak{F}$ *mapping onto a circular-radial slit covering surface of circular type;*

(ii) *The function* φ *is the only element which simultaneously belongs to* $\mathfrak{G}$ *and* $\mathfrak{H}$;

(iii) *For every* $f \in \mathfrak{G}$ *the inequality*

$$J(\varphi) \leqq J(f)$$

holds and thus for every $f \in \mathfrak{G}'$ *the inequality*

$$M(\varphi) \leqq M(f)$$

holds. In the both inequalities the equality signs appear if and only if $f \equiv \varphi$;

(iv) *For every* $f \in \mathfrak{H}'$ *the inequalities*

$$m(\varphi) \geqq m(f)$$

and thus

$$J(\varphi) \geqq J(f)$$

hold. In the both inequalities the equality signs appear if and only if $f\equiv\varphi$.

Proof. The domain B can always be conformally mapped onto the domain of which the boundary consists of analytic Jordan curves, by a univalent function Φ satisfying the conditions $\Phi(z_1)=z_1$ and $\Phi'(z_1)=1$. Thus we may assume that so is the domain B. In fact, by the mapping the functional $J(f)$ varies only an additive quantity

$$2\pi\sum_{j=2}^{\iota} n_j^2 \lg |\Phi'(z_j)|$$

and the functionals $m(f)$ and $M(f)$ vary only a multiplicative constant

$$\prod_{j=2}^{\iota} |\Phi'(z_j)|^{n_j^2}$$

independent of a particular choice of $f\in\mathfrak{F}$.

Construction φ *in* (i). It is easy to find a solution u of the boundary value problem satisfying the conditions:

(A) u is single-valued harmonic on $B-\{z_j\}_{j=1}^{\iota}$ and has logarithmic singularities

$$u(z)=n_1 \lg|z-z_1|+o(1) \quad \text{at } z_1$$

and

$$u(z)=n_j \lg|z-z_j|+O(1) \quad \text{at } z_j \quad (j=2,\cdots,\iota);$$

(B) u is constant on each boundary component C_j $(j=1,\cdots,\lambda+\mu)$ with the constant so chosen that

$$\int_{C_j}\frac{\partial u}{\partial n}ds=-2\pi\nu_j \quad (j=1,\cdots,\lambda),$$

$$\int_{C_j}\frac{\partial u}{\partial n}ds=0 \qquad (j=\lambda+1,\cdots,\lambda+\mu);$$

(C) $\dfrac{\partial u}{\partial n}=0$ along C_j $(j=\lambda+\mu+1,\cdots,N)$.

Let u^* be a conjugate harmonic function of u uniquely determined up to multiples of 2π under the condition

$$\lim_{z\to z_1}(u^*(z)-n_1\arg(z-z_1))\equiv 0 \pmod{2\pi},$$

and set $\varphi(z)=\exp(u+iu^*)$. Then it is easily verified that $\varphi(z)$ is the function satisfying the property of (i) up to the uniqueness.

Proof of (iii). Let f be an arbitrary element of $\mathfrak{G}$ and let

$$B_r=B-\sum_{j=1}^{\iota}\{|z-z_j|\leqq r\},$$

where r should be chosen suitably sufficiently small. Then, the image curves of $\{|z-z_j|=r\}$ $(j=1,\cdots,\iota)$ under f surround about $w=0$ m_j-times $(j=1,\cdots,\iota)$ respectively and lie between circumferences

$$|w|=r^{n_j}|f^{[n_j]}(z_j)|(1+\delta(r))$$

and

$$|w|=r^{n_j}|f^{[n_j]}(z_j)|(1-\delta(r)) \quad (j=1,\cdots,\iota)$$

respectively, where the positive number $\delta(r)$ does not depend on a particular choice of $f\in\mathfrak{G}$ and

$$\lim_{r\to 0}\delta(r)=0.$$

Therefore, using the Green formula, we have that

$$\begin{aligned} J(f)&=D_{B_r}(\lg|f|)\\ &\quad+\sum_{j=1}^{\iota}\int_{|z-z_j|=r}\lg|f|\,d\arg f\\ &\quad-2\pi\sum_{j=1}^{\iota}n_j\lg|f^{[n_j]}(z_j)|\end{aligned}$$

$$(2.4)\quad \begin{aligned}&=D_{B_r}(\lg|f|)+2\pi\sum_{j=1}^{\iota}n_j\lg|r^{n_j}f^{[n_j]}(z_j)|\\ &\quad-2\pi\sum_{j=1}^{\iota}n_j\lg|f^{[n_j]}(z_j)|+O(\delta(r))\\ &=D_{B_r}(\lg|f|)+2\pi\sum_{j=1}^{\iota}n_j^2\lg r+O(\delta(r)).\end{aligned}$$

Set $U=\lg|f|$, $u=\lg|\varphi|$ and $h=U-u$. Then we have that

$$\begin{aligned}J(f)-J(\varphi)&=D_{B_r}(\lg|f|)-D_{B_r}(\lg|\varphi|)\\ &\qquad+O(\delta(r))\\ &=D_{B_r}(U)-D_{B_r}(u)+O(\delta(r))\\ &=2D_{B_r}(u,h)+D_{B_r}(h)+O(\delta(r)),\end{aligned}$$

which yields, by $r\to 0$,

$$(2.5)\quad J(f)-J(\varphi)=2D_B(u,h)+D_B(h).$$

By the method similar to (iii) of Theorem 1 of 7) we can verify that

$$(2.6)\qquad D_B(u,h)=0.$$

By (2.5) and (2.6) we have that

$$J(f)-J(\varphi)=D_B(h)\geqq 0.$$

The equality sign in the last inequality appears if and only if $h\equiv$ const.$=0$ and thus $f\equiv\varphi$, because of the normalization condition (d) of **1**.

Proof of (iv). Let f be an arbitrary element of $\mathfrak{H}'$ and set $U=\lg|f|$, $u=\lg|\varphi|$ and $h=u-U$. By (2.4) we have that

$$J(\varphi)-J(f)=D_{B_r}(u)-D_{B_r}(U)+O(\delta(r))$$
$$=2D_{B_r}(U,h)+D_{B_r}(h)+O(\delta(r)),$$

which yields, by $r\to 0$,

$$(2.7)\quad J(\varphi)-J(f)=2D_B(U,h)+D_B(h).$$

We shall show that

$$(2.8)\quad D_B(U,h)\geqq 2\pi\sum_{j=1}^{\lambda}\nu_j\lg m_j(f)-\int_{\partial B}\lg|f|\,d\arg f.$$

Let $\{B_n\}_{n=1}^{\infty}$ be an exhaustion of B such that C'' is a portion of the boundary ∂B_n of B_n, $\partial B_n-C''$ consists of analytic Jordan curves C_{jn} ($j=1,\cdots,\lambda,\lambda+\mu+1,\cdots,N$) homotopic to C_j respectively and such that $\bigcup_{j=1}^{\iota}\{z_j\}\subset B_1$. Let B_{nk} ($k\geqq n$; $B_{nn}=B_n$) be a subdomain of B surrounded by $C'_n=\sum_{j=1}^{\lambda}C_{jn}$, C'' and $C'''_k=\sum_{j=\lambda+\mu+1}^{N}C_{jk}$. Thus $B_n\subset B_{nk}$, and C'_n and C'' are common portions of the boundaries of B_n and B_{nk}. Let $v_{nk}(z)$ for each pair of n and k ($k\geqq n$) be the function on B_{nk} which satisfies the conditions:

(A) v_{nk} is single-valued harmonic on $B_{nk}-\{z_j\}_{j=1}^{\iota}$ and has logarithmic singularities

$$v_{nk}(z)=n_1\lg|z-z_1|+o(1)\quad \text{at } z_1$$

and

$$v_{nk}(z)=n_j\lg|z-z_j|+O(1)\quad \text{at } z_j\quad (j=1,\cdots,\ );$$

(B) $v_{nk}=$const. on each component C_{jn} ($j=1,\cdots,\lambda$) and on each component C_j ($j=\lambda+1,\cdots,\lambda+\mu$) with the constant so chosen that

$$\int_{C_{jn}}\frac{\partial v_{nk}}{\partial n}\,ds=-2\pi\ _j\quad (j=1,\cdots,\lambda),$$

$$\int_{C_j}\frac{\partial v_{nk}}{\partial n}\,ds=0\qquad (j=\lambda+1,\cdots,\lambda+\mu);$$

(C) $$\frac{\partial v_{nk}}{\partial n}=0\qquad \text{along } C'''_k.$$

Extend v_{nk} to B by setting $v_{nk}=0$ on $B-\overline{B}_{nk}$.

For $l\geqq k>n$ the equation

$$D_{B_{nk}}(v_{nk}-v_{nn},v_{nl}-v_{nn})$$
$$=-\int_{\partial B_n}(v_{nl}-v_{nn})\frac{\partial}{\partial n}(v_{nk}-v_{nn})\,ds$$
$$-\int_{C'''_k-C'''_n}v_{nl}\frac{\partial v_{nk}}{\partial n}\,ds$$
$$=\int_{C'''_n}v_{nn}\frac{\partial v_{nk}}{\partial n}\,ds=D_{B_{nk}}(v_{nk}-v_{nn})$$

implies that

$$D_{B_{nk}}(v_{nl}-v_{nk})\leqq D_{B_{nl}}(v_{nl}-v_{nn})-D_{B_{nk}}(v_{nk}-v_{nn}).$$

Thus $D_{B_{nk}}(v_{nk}-v_{nn})$ is monotone increasing with k. Let v_{n0} be the function on B_n which satisfies the conditions:

(A) v_{n0} is single-valued harmonic on $B_n-\{z_j\}_{j=1}^{\iota}$ and has the same logarithmic singularities as v_{nn} at z_j ($j=1,\cdots,\iota$);

(B) $v_{n0}=0$ on ∂B_n.

Since, on setting $v_{n0}\equiv 0$ on $B-\overline{B}_n$,

$$D_{B_{nk}}(v_{n0}-v_{nn},v_{nk}-v_{nn})=\int_{C'''_n}v_{nn}\frac{\partial v_{nk}}{\partial n}\,ds$$
$$=D_{B_{nk}}(v_{nk}-v_{nn}),$$

we find that

$$D_{B_{nk}}(v_{nk}-v_{n0})=D_{B_n}(v_{n0}-v_{nn})-D_{B_{nk}}(v_{nk}-v_{nn}).$$

Hence $D_{B_{nk}}(v_{nk}-v_{nn})$ is uniformly bounded and

$$v_n=\lim_{k\to\infty}v_{nk}$$

exists on $\boldsymbol{B}_n\equiv\bigcup_{k=n}^{\infty}B_{nk}$ with

$$(2.9)\quad \lim_{k\to\infty}D_{B_{nk}}(v_n-v_{nk})=0.$$

The function v_n satisfies the conditions:

(A) v_n is single-valued harmonic on $\boldsymbol{B}_n-\{z_j\}_{j=1}^{\iota}$ and has the logarithmic singularities

$$v_n(z)=n_1\lg|z-z_1|+o(1)\quad \text{at } z_1$$

and

$$v_n(z)=n_j\lg|z-z_j|+O(1)\quad \text{at } z_j\quad (j=1,\cdots,\ );$$

(B) $v_n=$const. on each component C_{jn} ($j=1,\cdots,\lambda$) and on each component C_j ($j=\lambda+1,\cdots,\lambda+\mu$) with the constant so chosen that

$$\int_{C_{jn}} \frac{\partial v_n}{\partial n} ds = -2\pi\nu_j \quad (j=1, \cdots, \lambda),$$

$$\int_{C_j} \frac{\partial v_n}{\partial n} ds = 0 \qquad (j=\lambda+1, \cdots, \lambda+\mu);$$

(C) $\frac{\partial v_n}{\partial n}=0$ along C'''.

Extend v_n to B by setting $v_n=0$ on $B-\overline{B}_n$. For $n \geqq m$ the equation

$$D_{B_m}(v_m-v_1, v_n-v_1)$$

$$= -\int_{\partial B_1}(v_m-v_1)\frac{\partial(v_n-v_1)}{\partial n}ds - \int_{C'_m-C'_1} v_m \frac{\partial v_n}{\partial n} ds$$

$$= \int_{C'_1} v_m \frac{\partial v_1}{\partial n} ds - \int_{C'_m} v_m \frac{\partial v_m}{\partial n} ds = D_{B_m}(v_m-v_1)$$

implies that

$$D_{B_m}(v_n-v_m) \leqq D_{B_n}(v_n-v_1) - D_{B_m}(v_m-v_1).$$

Thus $D_{B_n}(v_n-v_1)$ is monotone increasing with n. Furthermore the equation

$$D_{B_n}(v_n-v_1, u-v_1) = D_{B_n}(v_n-v_1)$$

implies that

$$D_{B_n}(u-v_n) \leqq D_B(u-v_1) - D_{B_n}(v_n-v_1).$$

Thus v_n strongly converges to u on B as $n\to\infty$:

(2.10) $\lim_{n\to\infty} D_{B_n}(u-v_n)=0.$

By (2.9), (2.10) and the inequality

$$\sqrt{D_{B_{nk}}(u-v_{nk})} \leqq \sqrt{D_{B_n}(u-v_n)} + \sqrt{D_{B_{nk}}(v_n-v_{nk})},$$

we see that there exists a subsequence $\{k_n\}_{n=1}^{\infty}$ of $\{k\}_{k=1}^{\infty}$ such that

(2.11) $\lim_{n\to\infty} D_{B_{nk_n}}(u-v_{nk_n})=0.$

Set $h_n=v_{nk_n}-U$. Then by (2.11) we can easily see that

(2.12) $\lim_{n\to\infty} D_{B_{nk_n}}(U, h_n) = D_B(U, h).$

When we note the boundary behaviors of v_{nk} and U, we find that

$$D_{B_{nk_n}}(U, h_n) = -\int_{\partial B_{nk_n}} U \frac{\partial v_{nk_n}}{\partial n} ds$$

(2.13)
$$+\int_{\partial B_{nk_n}} U\frac{\partial U}{\partial n} ds$$

$$\geqq 2\pi\sum_{j=1}^{\lambda} \nu_j \mu_{jn}(U) + \int_{\partial B_{nk_n}} U\frac{\partial U}{\partial n} ds,$$

where $\mu_{jn}(U)$ $(j=1, \cdots, \lambda)$ are defined by

$$\mu_{jn}(U) = \min_{C_{jn}} U$$

respectively. We can easily see that

(2.14) $\lim_{n\to\infty} \mu_{jn}(U) = \lg m_j(f) \quad (j=1, \cdots, \lambda).$

Our assertion (2.8) follows from (2.12), (2.13) and (2.14). Consequently, by (2.2), (2.3), (2.7), (2.8) and the condition (h) of **1** we have that

(2.15)
$$\begin{aligned} &2\pi(\lg m(\varphi) - \lg m(f)) \\ &= J(\varphi) - J(f) - \Big(2\pi\sum_{j=1}^{\lambda}\nu_j \lg m_j(f) \\ &\qquad - \int_C \lg|f|\, d\arg f\Big) \\ &\geqq D_B(h) + 2\pi\sum_{j=1}^{\lambda}\nu_j \lg m_j(f) \\ &\qquad - \int_C \lg|f|\, d\arg f \geqq 0. \end{aligned}$$

The equality sign in the last inequality of (2.15) appears if and only if $h\equiv$const.$=0$ and thus $f\equiv\varphi$, because of the normalization condition (d) of **1**. Then the equality sign in the first inequality of (2.15) also appears.

The uniqueness in (i) will be obvious by (iii) and (iv). (ii) is also evident.

Remark. The second inequality of (iii) of Theorem 2.1 does not generally hold for $f\in\mathfrak{G}$. Also the both inequalities of (iv) of Theorem 2.1 do not generally hold for $f\in\mathfrak{H}$. They are shown by simple examples (cf. Examples of 7)).

3. Corollaries. We should note that in Theorem 2.1 the case $C''=\phi$ or $C'''=\phi$ is permitted. Then we have the following corollary.

Corollary 2.1. (i) *For each class $\mathfrak{F}$ there exists one and only one function ψ of the class $\mathfrak{F}$ mapping onto a circular slit covering surface of circular type*;

(ii) *For every $f\in\mathfrak{F}$ the inequality*

$$J(\psi) \leqq J(f)$$

holds and thus for every $f \in \mathfrak{F}$ satisfying (g) *of* **1** *the inequality*

$$M(\psi) \leqq M(f)$$

holds. In the both inequalities the equality signs appear if and only if $f \equiv \psi$;

(iii) *For each class $\mathfrak{F}$ there exists one and only one function χ of the class $\mathfrak{F}$ mapping onto a radial slit covering surface of circular type;*

(iv) *For every $f \in \mathfrak{F}$ satisfying* (h) *of* **1**, *the inequalities*

$$m(\chi) \geqq m(f)$$

and thus

$$J(\chi) \geqq J(f)$$

hold. In the both inequalities the equality signs appear if and only if $f \equiv \chi$.

REMARK. Let D_j ($j=\lambda+1, \cdots, N$) be the complement continua of B adjacent to C_j respectively and let

$$B^1 = B + \sum_{j=1}^{\mu} D_{\lambda+j} \text{ and } B^2 = B + \sum_{j=1}^{\nu} D_{\lambda+\mu+j}.$$

Let $\mathfrak{F}(B^1)$ and $\mathfrak{F}(B^2)$ be the classes $\mathfrak{F}$ defined for the domains B^1 and B^2 respectively in place of B. Apply the consequences (iii) and (i) of Corollary 2.1 to $\mathfrak{F}(B^1)$ and $\mathfrak{F}(B^2)$ respectively. Then we see that the restrictions to the domain B of the functions $\chi \in \mathfrak{F}(B^1)$ and $\psi \in \mathfrak{F}(B^2)$ of Corollary 2.1 belong to $\mathfrak{G}$ and $\mathfrak{H}$ respectively. Furthermore it is easily verified that the functions χ and ψ also belong to $\mathfrak{G}'$ and $\mathfrak{H}'$ respectively. The above construction method is available for each domain conformally equivalent to B in place of B. Therefore we know that the both classes $\mathfrak{G}'$ and $\mathfrak{H}'$ have infinite numbers (in continuum potency) of elements other than the function φ of Theorem 2.1.

Consider the class $\mathfrak{F}$ of the case $\lambda = \iota = 1$, $\nu_1 = n_1 = 1$ in **1**, and let $\mathfrak{F}_1$, $\mathfrak{G}_1$ and $\mathfrak{H}_1$ be the subclasses of $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ respectively consisting of *univalent* functions $f(z)$. Then $\mathfrak{G}_1 \subset \mathfrak{G}'$ and further $f \in \mathfrak{H}'$ for $f \in \mathfrak{H}_1$ satisfying (j) of **1**. Thus we have the following corollary of Theorem 2.1.

COROLLARY 2.2. (i) *For the class $\mathfrak{F}$ of the case $\lambda = \iota = 1$, $\nu_1 = n_1 = 1$ there exists one and only one function φ of the class $\mathfrak{F}$ mapping onto a circular-radial slit covering surface of circular type (circular-radial slit disk);*

(ii) *The function φ is the only element which simultaneously belongs to $\mathfrak{G}$ and $\mathfrak{H}$;*

(iii) *For every $f \in \mathfrak{G}_1$ the inequalities*

$$J(\varphi) \leqq J(f)$$

and thus

$$M_1(\varphi) \leqq M_1(f)$$

hold. In the both inequalities the equality signs appear if and only if $f \equiv \varphi$;

(iv) *For every $f \in \mathfrak{H}_1$ satisfying* (j) *of* **1**, *the inequalities*

$$m_1(\varphi) \geqq m_1(f)$$

and thus

$$J(\varphi) \geqq J(f)$$

hold. In the both inequalities the equality signs appear if and only if $f \equiv \varphi$.

If $C''' = \phi$ (or $C'' = \phi$) in (iii) (or (iv) resp.) of Corollary 2.2, then $\mathfrak{G}_1 = \mathfrak{F}_1$ (or $\mathfrak{H}_1 = \mathfrak{F}_1$ resp.) and the present consequences are reduced to the well-known classical results (cf. 1), 9) and 10)).

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