Memoirs of the School of Engineering Okayama University Vol. 10, No. 2, January 1976

A Study on Non-steady Groundwater Flow in a Semi-confined Aquifer

Iichiro KONO* and J. B. SELLMEIJER**

(Received December 16, 1975)

Synopsis

This paper deals with the groundwater flow in a semi-confined aquifer causing the phenomena of consolidation and free surface lowering. Since the main effect of consolidation has taken place before noticeable lowering of the free surface, one may solve each phenomenon on its own. The real solution may be obtained by the principle of superposition. However, the solution for lowering the free surface is delayed due to the cosolidation by a certain timelapse, depending on the place-coordinates.

1. Introduction

The groundwater flow in conection with consolidation will be studied in a sandy permeable aquifer, which is situated on an impermeable base and overlained by a less permeable clay layer with a free surface. The flow in the sandy layer will be supposed to be mainly horizontal. Since the permeability of clay is much less than that of sand, the flow in the clay layer will be mainly vertical.

It will be possible to describe the flow in the sandy layer by an equation containing the average head over the height of that layer as unknown. The influence of the clay layer is expressed by its parameters. In addition, a relationship between the average head over the height

* Department of Civil Engineering

** Visiting Researcher, Delft Technological University, The Netherland

of the sand and the height of the free surface in the clay can be derived.

It will appear that the contribution of the flow from the clay layer consists of two parts; the cotribution of the free surface and that of the consolidation of the layer. Both terms have the form of the Boulton integral [see (1)] and cause a delay in the flow phenomenon in the sandy layer.

As lowering of the free surface of the clay layer is a much slower phenomenon than consolidation, it is possible separate both phenomena. One may solve each on its own. The real solution can be approximately obtained by the superposition of both solutions. However the solution for lowering of the free surface in the clay is shifted by a certain timelapse, depending on the place-coordinates.Though in practical terms this is of hardly any influence, it is interesting from a theoretical viewpoint. Consolidation here results just in a certain delay, which turns out to be a very interesting function of the place-coordinates.

In order to derive the above mentioned equations it is necessary to work with a constant vertical resistance in the clay instead of with a constant permeability. This has its main influence during the initial stage of the lowering of the free surface in the clay, after the greatest effect of consolidation has taken place. Its importance depends on the values of the derivatives of the height of the free surface in the clay with respect to the place and time coordinates.

As an example of the use of the derived equations to solve a practical problem the case of pumping up water in the center of a circular island is studied. The results show that consolidation is a matter of days/months, while lowering of the free surface in the clay calls for months/years. The maximum delay caused by consolidation is of order of the ratio of a combination of the coefficients of storage of the sandy and clayey layers to the porosity of the clay; this ratio having a small value for normal soil.

2. Analytical Description

The groundwater flow will be studied in a sandy permeable aquifer which is situated on an impermeable base and overlain by a less permeable clay layer with a free water table. A cross section of this system is shown in Fig.1. H is the height of the sandy layer, h_0 is a representative level of the head of the free surface in the clay, for example the lebel in the initial stage. Since it will be supposed that the area of the flow phenomenon is broad in propotion to the height $H+h_0$ (the vertical shear stress due to the flow by consolidation must be negligible), the flow in the sandy as well as in the clayey will conform to the simplified consolidation equation,

$$\frac{\partial\phi}{\partial t} = C\Delta_3\phi \qquad (1)$$

where, C: consolidation coefficient, G: shear modulus of soil,
K_w: compression modulus of water, ϕ : head,
ρ : density of water, g: accelaration of gravity,
k : permeability coefficient, n: porosity of soil,
K ; compression modulus of soil, t; time elapse,
x,y and z : place-coordinate,

$$\frac{1}{C}=\frac{\rho g}{k}\left(\frac{1}{K+4G/3} + \frac{n}{K_W}\right) \qquad \Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Equation (1) will be applied on the flow in both layers. In the plane the layers have in common, $z=-h_0$, the flow is related by two conditions of continuity: one for the specific discharge, the other for the head. By those conditions the influence which the flow in the clay has on the flow in the sand can be expressed in the flow variable of the sandy layer. One obtains an equation to be solved under the boundary conditions for the configuration to be studied.

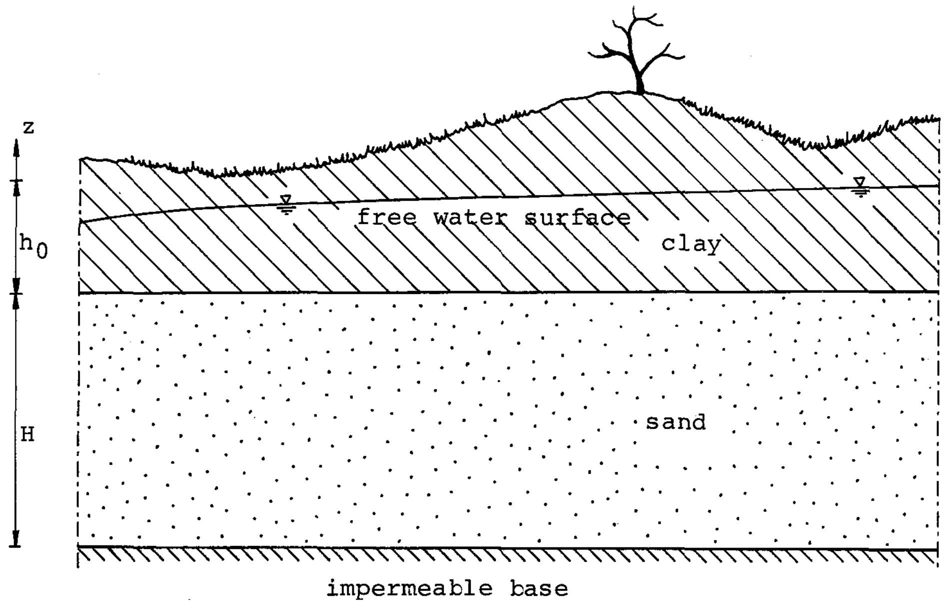

Fig.1 General configuration

Since the flow in the sandy layer is supposed to be mainly horizontal, it is useful to work with the average head over the height of this layer,

$$\tilde{\phi} = \frac{1}{H}\int_{-H-h_0}^{-h_o} \phi \, dz$$

To introduce $\tilde{\phi}$ in equation (1) one may integrate with respect to z over the height H.
As H is constant and the base is impermeable, one obtains,

$$\frac{1}{C_s}\frac{\partial\tilde{\phi}}{\partial t} = \Delta_2\tilde{\phi} + \frac{1}{H}\left(\frac{\partial\phi}{\partial z}\right)_{z_s=-h_0} \qquad (2)$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The index s refers to the sand; the index c refers to the clay. Therefore one needs to know the relation between the flow in the sand and the flow in the clay. The two conditions of continuity for the specific discharge as well as the head are,

$$k_s\left(\frac{\partial\phi}{\partial z}\right)_{z_s=-h_0} = k_c\left(\frac{\partial\phi}{\partial z}\right)_{z_c=-h_o} \qquad (3)$$

$$(\phi)_{z_s=-h_0} = (\phi)_{z_c=-h_0} \sim \tilde{\phi}$$

Since the flow in the clay layer will be mainly vertical, one may simplify equation (1), if applied to this layer, as

$$\frac{\partial\phi}{\partial t} = C_c\frac{\partial^2\phi}{\partial z^2} \qquad (4)$$

with boundary conditions for $z=-h_0$, and the free surface to be denoted by z=h [see Hantush (2)].

$$\begin{cases} z=-h_0 \\ \phi = \tilde{\phi} \end{cases} \qquad \begin{cases} z=h \\ \phi = h \end{cases} \qquad \frac{\partial\phi}{\partial z} = -\frac{n_c}{k_c}\frac{\partial h}{\partial t}$$

At the time t=0 no flow will be supposed. Everywhere in the field the head has then the same value, $\phi=0$. It is not easy to solve equation (4) under these conditions. However, an analytical solution is possible for the case of a constant vertical resistance in the clay rather than a constant permeability.

$$\frac{k_c}{h+h_0} = \frac{k_c'}{h_0} \qquad k_c/(h+h_0) \;:\; \text{vertical resistance}$$

Furthermore one must assume,

$$C_c = \left(\frac{h+h_0}{h_0}\right)^2 C'_c \qquad C'_c\text{: constant}$$

But this has no practical importance, as it will turn out later that consolidation loses its influence before noticeable lowering of the free surface in the clay occurs. In order to obtain constant boundaries as time elapses, one may work with a variable, depending on the height of the free surface,

$$Z = \frac{h_0}{h+h_0}(z+h_0)$$

One has to solve now a well known type of partial differential equation,

$$\frac{\partial \phi}{\partial t} = C'_c \frac{\partial^2 \phi}{\partial Z^2} \tag{5}$$

with initial and boundary conditions,

$$\begin{cases} t=0 \\ \phi=0 \end{cases} \qquad \begin{cases} Z=0 \\ \phi=\tilde{\phi} \end{cases} \qquad \begin{cases} Z=h_0 \\ \phi=h \end{cases} \qquad \frac{\partial \phi}{\partial Z} = -\frac{n_c}{k'_c}\cdot\frac{\partial h}{\partial t}$$

It is possible to solve this equation by the theory of Laplace transformations. After the transformation of the equation (5) and its boundary conditions one obtains a linear second order differential equation in Z, with three boundary conditions, two of them necessary to solve the differential equation and one necessary to eliminate the unknown head of the free surface. From the solution of the differential equation and the third extra condition the following can be derived,

$$\left(\frac{\partial \bar{\phi}}{\partial Z}\right)_{Z=0} = -\frac{sh\left(h_o\sqrt{\frac{p}{C'_c}}\right) + \frac{n_c}{S'_c}h_o\sqrt{\frac{p}{C'_c}}\,ch\left(h_o\sqrt{\frac{p}{C'_c}}\right)}{ch\left(h_o\sqrt{\frac{p}{C'_c}}\right) + \frac{n_c}{S'_c}h_o\sqrt{\frac{p}{C'_c}}\,sh\left(h_o\sqrt{\frac{p}{C'_c}}\right)}\sqrt{\frac{p}{C'_c}}\,\Psi \tag{6}$$

$$X = \left(ch\left(h_o\sqrt{\frac{p}{C'_c}}\right) + \frac{n_c}{S'_c}h_o\sqrt{\frac{p}{C'_c}}\,sh\left(h_o\sqrt{\frac{p}{C'_c}}\right)\right)^{-1}\Psi \tag{7}$$

where,

$\bar{\phi} = L(\phi) = \int_0^\infty \phi \exp(-pt)\,dt$

$\Psi = L(\tilde{\phi}) = \int_0^\infty \tilde{\phi} \exp(-pt)\,dt$

$X = L(h) = \int_0^\infty h \exp(-pt)\,dt$

L: Laplace transformation, S: coefficient of storage

$S'_c = k'_c h_0/C'_c \qquad S_s = k_s H/C_s$

It is possible to find the inverse transform of (6). This can be achieved by the convolution theorem of Laplace transformations and the expansion theorem of Heaviside. The result is,

$$\left(\frac{\partial\phi}{\partial Z}\right)_{Z=0} = -\frac{1}{h_0}\int_0^{\infty}\frac{\partial\tilde{\phi}}{\partial\lambda}2\sum_{m=0}^{\infty}\frac{P_m^2(n_c/S_c')^2+1}{P_m^2(n_c/S_c')^2+(n_c/S_c')+1}\exp\left\{-P_m^2\frac{C_c'}{h_0^2}(t-\lambda)\right\}d\lambda \quad (8)$$

where,

P_m is the root of $\frac{n_c}{S_c}P_m = \mathrm{ctg}(P_m)$ with $m\pi \leq P_m \leq (m+1)\pi$

AS for soil and especially clay the ratio n_c/S_c' has a large value. Therefore it holds approximately that,

$$P_0^2 = \frac{S_c'}{n_c} \qquad P_m^2 = m^2\pi^2 \qquad M \geq 1$$

and one may write formula (8) as the following,

$$\left(\frac{\partial\phi}{\partial Z}\right)_{Z=0} = -\frac{1}{h_0}\int_0^t\frac{\partial\tilde{\phi}}{\partial\lambda}\exp\left\{-\frac{k_c'}{n_c h_0}(t-\lambda)\right\}d\lambda - \frac{2}{h_0}\int_0^t\frac{\partial\tilde{\phi}}{\partial\lambda}\sum_{m=0}^{\infty}\exp\left\{-m^2\pi^2\frac{C_c'}{h_0}(t-\lambda)\right\}d\lambda \quad (9)$$

The result shows that for normal clay ($n_c \gg S_c'$) the flow from the clay to the sand consists of two integrals of the Boulton type [see (1)]. The first one represents the contribution of lowering of the free surface and can also be derived for the case in which consolidation is ignored. The second integral represents the contribution of the consolidation and can also be derived for the case in which the free surface in the clay is constant. In that case, since the discharge from the clay to the sand is not limited, one may think the n_c in formula (9) to be infinitely large.

Since it has turned out that the problem is composed of two types of phenomena, it is sensible to study first the structure of (6). It is unknown from the theory of Laplace transformations that large values of its parameter, in this case t. For large values of p it follows from equation (7) that the head of the free surface in the clay hardly changes. Therefore for the first stage it can be written,

$$C_c' = C_c \qquad k_c' = k_c \qquad S_c' = S_c \qquad Z = z + h_0$$

and hence one may omit, in the behaviour of equation (6) for large values of p, the dash in the consolidation coefficient and use the normal vertical place-coordinate,

$$\left(\frac{\partial\bar{\phi}}{\partial Z}\right)_{Z_c=-h_0} = -h_0\sqrt{\frac{p}{C_c}}\,\mathrm{ctgh}\left(h_0\sqrt{\frac{p}{C_c}}\right)\frac{\Psi}{h_0} \quad (10)$$

Moreover, in the case of a constant head in the clay instead of a free water table formula (10) is the exact solution for the type of problem studied in this paper.

It is interesting to know to what extent equation (10) will approximately the real solution and whether it will lead to a specific situation from which the influence of the free surface can be taken into account. With this purpose in mind, one may consider values of S_c'/n_c and S_s/n_c , which are small enough that formula (10) will also hold for not so large values of p, so that one may write approximately,

$$-\frac{k_c}{k_s}\,\frac{1}{H}\left(\frac{\partial\bar{\phi}}{\partial Z}\right)_{Z_s=-h_0}+L\left\{\frac{1}{C_s}\,\frac{\partial\tilde{\phi}}{\partial t}\right\}\sim\frac{k_c}{k_s}\,\frac{1}{h_0 H}\left\{1+\frac{h_0}{k_c}\left(S_c+\frac{1}{3}S_c\right)p\right\}\Psi\,\frac{k_c}{k_s}\,\frac{1}{h_0 H}\Psi \tag{11}$$

From (2),(3) and the invers transformation of (11) it then follows approximately,

$$\Delta^2\tilde{\phi}=\frac{k_c}{k_s}\,\frac{1}{h_0 H}\tilde{\phi} \tag{20}$$

Formura (12) is the steady flow equation after consolidation alone. Therefore for values of $(S_s+S_c/3)/n_c$, which are small enough, the process of consolidation of the clayey as well as the sandy layer loses its importance before lowering of the free surface is noticeable.

Conversely small values of the parameter p in (6) and (7) give information for large values of the time. For small values of p (6) behaves through it were,

$$\left(\frac{\partial\bar{\phi}}{\partial Z}\right)_{Z=0}=-\frac{1+S_c'/n_c}{P+k_c'/n_c h_0}\,\frac{p\Psi}{h_o}\ \sim\ \frac{-1}{p+k_c'/n_c h_0}\,\frac{p\Psi}{h_0} \tag{13}$$

It follows from (7) that it holds for the head of the free surface in this cace,

$$X=-\frac{1}{p+k_c'/n_c h_0}\cdot\frac{k_c'}{n_c h_0}\Psi \tag{14}$$

Moreover, in the cace of the influence of consolidation not being taken into account, equation (13) and (14) are the exact solution for the studied problem.

Since we know now the structure of the problem, we can study it as a whole. It follows from (2),(3) and (9) that the average head over the height of the sandy layer conforms to the following equation for small values of the parameter S_c'/n_c' ,

$$\Delta_2\tilde{\phi}=\frac{1}{C_s}\cdot\frac{\partial\tilde{\phi}}{\partial t}+\frac{k_s'}{k_c}\cdot\frac{1}{h_0 H}\left\{\int_0^t\frac{\partial\tilde{\phi}}{\partial\lambda}\exp\left\{-\frac{k_c'}{n_c h_0}(t-\lambda)\right\}d\lambda+\int_0^t\frac{\partial\tilde{\phi}}{\partial\lambda}2\sum_{m=0}^{\infty}\exp\left\{-m^2\pi^2\frac{C_c'}{h_0^2}(t-\lambda)\right\}d\lambda\right\} \tag{15}$$

It followed from the Laplace transformation that in the first stage consolidation prevailed. Therefore we may write,

$$\tilde{\phi} = \tilde{\phi}_c + \tilde{\phi}_*$$

where $\tilde{\phi}_c$: the solution for consolidation alone,

$\tilde{\phi}_*$: the influence of lowering of the free surface in the clay on the consolidation.

As the first term and the second integral of the right part of formula (15) are mainly reflected by $\tilde{\phi}_c$, it can be shown that those terms applied on $\tilde{\phi}_*$ have negrigible influence. Therefore one may write approximately by reducing the equation for $\tilde{\phi}_c$ from (15),

$$\Delta\tilde{\phi}_* = \int_0^{\mathcal{T}} \frac{\partial\tilde{\phi}_*}{\partial\lambda}\exp\{-(\mathcal{T}-\lambda)\}d\lambda + \int_0^{\mathcal{T}} \frac{\partial\tilde{\phi}_c}{\partial\lambda}\exp\{-(\mathcal{T}-\lambda)\}d\lambda - \tilde{\phi}_c \tag{17}$$

where,

$$\Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} \qquad \xi = x\sqrt{\frac{k_c'}{k_s}\frac{1}{h_0 H}} \qquad \eta = y\sqrt{\frac{k_c'}{k_s}\frac{1}{h_0 H}} \qquad \mathcal{T} = \frac{k_c'}{n_c h_0}t$$

It is useful to rewrite (17) by partial integration of the second integral and by introducing the steady flow equation after only consolidation (12),

$$\Delta\tilde{\phi}_* = \int_0^{\mathcal{T}} \frac{\partial\tilde{\phi}_*}{\partial\lambda}\exp\{-(\mathcal{T}-\lambda)\}d\lambda - \tilde{\phi}_0\{1-\exp(-\mathcal{T})\} + \int_0^{\mathcal{T}} (\tilde{\Phi}_0 - \tilde{\Phi}_c)\exp\{-(\mathcal{T}-\lambda)\}d\lambda \tag{18}$$

where $\tilde{\phi}_0$ is the solution of (12).

If $\mathcal{T}$ is so small that consolidation still prevails, then it holds that $\tilde{\phi}_* \sim 0$. But if $\mathcal{T}$ is not particularly small, so that consolidation has already become less important, then for those values it holds $\tilde{\Phi}_c \sim \tilde{\Phi}_0$ and it can be shown that the second integral of (18) will behave like the area between the curves $\tilde{\Phi}_0$ and $\tilde{\Phi}_c$ as functions of $\mathcal{T}$ times the function $\exp(-\mathcal{T})$. The area between the curves $\tilde{\Phi}_c$ and $\tilde{\Phi}_0$ can be found from their Laplace transformations. It follows from (10) and the transformed equations (2),(3) that the transformed head for consolidation alone, $L(\tilde{\Phi}_c) = \Psi_c$, conforms to the equation,

$$\Delta_2\Psi_c = \frac{k_c}{k_s}\frac{1}{h_0 H}\{\frac{S_s}{S_c}h_0^2\frac{P}{C_c} + h_0\sqrt{\frac{P}{C_c}}\,\mathrm{ctgh}(h_0\sqrt{\frac{P}{C_c}})\}\Psi_c \tag{19}$$

The solution of this linear second order differential equation will be of the form,

$$\Psi_c = \frac{1}{P}f\left\{(\xi, \eta, \xi_a, \xi_b, \eta_a, \eta_b)\sqrt{\frac{S_s}{S_c}h_0^2\frac{P}{C_c} + h_0\sqrt{\frac{P}{C_c}}\,\mathrm{ctgh}(h_0\sqrt{\frac{P}{C_c}})}\right\} \tag{20}$$

where a and b denote the boundaries.
The transformed solution of the steady flow equation after consolidation alone, $L(\widetilde{\phi}_0) = \psi_0$, follows from (20) by taking the limit $p \to 0$,

$$\psi_0 = \frac{1}{p} f(\xi, \eta, \xi_a, \xi_b, \eta_a, \eta_b)$$

The area between the curves $\widetilde{\phi}_0$ and $\widetilde{\phi}_c$ as functions of η is then equal to,

$$\frac{k_c}{n_c h_0} \lim_{p \to 0} (\psi_c - \psi_0) = -\frac{S_s + S_c/3}{n_c} \left(\frac{\xi}{2} f_\xi + \frac{\eta}{2} f_\eta + \frac{\xi_a}{2} f + \frac{\xi_b}{2} f + \frac{\eta_a}{2} f_{\eta_a} + \frac{\eta_b}{2} f_{\eta_b}\right) \quad (21)$$

where f_ξ etc. are the partial derivatives with respect to ξ etc.

The resulting equation (21) is very interesting. For if one determines the transformed solution for only the lowering of the free surface in the clay from (13) and the transformed equations (2) and (3) (the head in this case to be denoted by $\widetilde{\phi}_s$ with transformation $L(\widetilde{\phi}_s) = \psi_s$), one will obtain,

$$\Delta_2 \psi_s = \frac{k_c'}{k_s} \frac{1}{h_0 H} \frac{p}{p + k_c'/n_c h_0} \psi_s \quad (22)$$

where of course the consolidation term for the sandy layer is omitted. The solution of this linear second order differential equation, applied on the same problem as consolidation before, will also have the form (20),

$$\psi_s = \frac{1}{p} f\left\{(\xi, \eta) \xi_a, \xi_b, \eta_a, \eta_b) \sqrt{\frac{1}{p + k_c'/n_c h_0}}\right\} \quad (23)$$

The derivative of $\widetilde{\phi}_s$ with respect to η for the limit $\eta \to 0$ can be found by determination of the behaviour of $\psi_s - \psi_0$ for large values of p,

$$\lim_{p \to \infty} (\psi_s - \phi_0) = \lim_{p \to \infty} -\frac{k_c'}{n_c h_0} \frac{1}{p^2} \left(\frac{\xi}{2} f_\xi + \frac{\eta}{2} f_\eta + \frac{\xi_a}{2} f_{\xi_a} + \frac{\xi_b}{2} f_{\xi_b} + \frac{\eta_a}{2} f_{\eta_a} + \frac{\eta_b}{2} f_{\eta_b}\right)$$

It follows from the invers transform of this result that the delivative of $\widetilde{\phi}_s$ with respect to η for the limit $\eta \to 0$, being denoted by $\widetilde{\phi}_1$ becomes,

$$\widetilde{\phi}_1 = \{-\left(\frac{\xi}{2} f_\xi + \frac{\eta}{2} f_\eta + \frac{\xi_a}{2} f_{\xi_a} + \frac{\xi_b}{2} f_{\xi_b} + \frac{\eta_a}{2} f_{\eta_a} + \frac{\eta_b}{2} f_{\eta_b}\right) \quad (24)$$

It turns out from (21) and (24) that the area between the curves $\widetilde{\phi}_0, \widetilde{\phi}_c$ as functions of $\widetilde{\phi}_c$ is proportional to the derivative of $\widetilde{\phi}_c$ with respect to η for the limit $\eta \to 0$,

$$\int_0^{\infty}(\tilde{\phi}_o-\tilde{\phi}_c)\,d\eta = \frac{S}{n_c}\tilde{\Psi}_1 \tag{25}$$

where $S = S_s + S_c/3$

This is a interesting result. Moreover it is not possible to determine the derivative of $\tilde{\phi}_s$ in according to the more advanced method of Laplace transformations because the function $\tilde{\phi}_s$ is discontinuous for the time t=0 .

One may now write for the auxiliary function $\tilde{\phi}_*$ by means of (18) and (25),

$$\Delta\tilde{\phi}_* = \int_0^{\eta}\frac{\partial\tilde{\phi}_*}{\partial\lambda}\exp\{-(\eta-\lambda)\}d\lambda - \tilde{\phi}_0\{1-\exp(-\eta)\} + \frac{S}{n_c}\tilde{\phi}_1\exp(-\eta) \tag{26}$$

with the important condition $\tilde{\phi}_* \sim 0$ for very small values of η. Therefore to solve equation (26) one may outline the problem by assuming $\tilde{\phi}_* = 0$ for $0 \leq \eta \leq \eta_0$, where η_0 is a fixed time ($\eta_0 \ll 1$). One may then rewrite equation (26), applying the subsituation $\mu = \lambda + \eta_0$

$$\Delta\tilde{\phi}_* = \int_0^{\eta-\eta_0}\frac{\partial\tilde{\phi}_*}{\partial\mu}\exp\{-(\eta-\eta_0)-\mu\}d\mu - \tilde{\phi}_0\{1-\exp[-(\eta-\eta_0)]\}$$
$$+\{-\tilde{\phi}_0\{\exp(\eta_0)-1\} + \frac{S}{n}\tilde{\phi}_1\}\exp(-\eta) \tag{27}$$

In order to eliminate the last term of this equation one may propose that the introduced time η_0 satisfies,

$$\eta \sim \exp(\eta_0) - 1 = \frac{S}{n}\tilde{\phi}_1/\tilde{\phi}_0 \tag{28}$$

The rewritten equation (27) becomes then,

$$\Delta(\tilde{\phi}_*+\tilde{\phi}_0) = \int_0^{\eta-\eta_0}\frac{\partial(\tilde{\phi}_*+\tilde{\phi}_0)}{\partial\mu}\exp\{-\{(\eta-\eta_0)-\mu\}\}d\mu + \tilde{\phi}_0\exp-(\eta-\eta_0) \tag{29}$$

If one studies what kind of differential equation the function $\tilde{\phi}_s$ has to satisfy, one will notice that this is exactly the same type as the equation (29), provided $\eta > 0$. Therefore the solution of (29) will have the same form as $\tilde{\phi}_s$, but as a function of $\eta - \eta_0$ instead of η. This means that one may write equation (16) as follows,

$$\tilde{\phi} = \tilde{\phi}_c(\eta) \qquad 0 \leq \eta \leq \eta_0$$
$$\tilde{\phi} = \tilde{\phi}_c(\eta) + \tilde{\phi}_s(\eta - \eta_0) - \tilde{\phi}_0 \qquad \eta > \eta_0 \tag{30}$$
$$\eta_0 = \frac{S\tilde{\phi}_1}{n_c\tilde{\phi}_0} = \frac{S}{n_c}\lim_{\eta\to 0}\frac{\partial\tilde{\phi}_s(\eta)}{\partial\eta}/\tilde{\phi}_s(\eta)$$

where $S = S_s + S_c/3$

One has now succeeded in simplifying the problem studied for small values of the ratio S/n_c. It has turnd out that the solution of complex problems containing consolidation as well as lowering of the free surface can be achieved by superposition of the individual solution for consolidation and lowering of the free surface. However the solution for lowering of the free surface has to be shifted over a certain timelapse, depending on the value of the function $\tilde{\phi}_s$ and of its derivative, both for the limit $t \to 0$. As the consideration takes place mainly in the first stage, it will generally result in a certain delay in lowering of the free surface in the clay.

3. Practical Example

The theory of the problems of configuration as in Fig.1 has been studied in the previous section. In this section this theory will be applied to the following situation. This will concern a cylindrical layer system with radius r_1. In the center of it a pump tube is set up, having a circular cross section. The boundary faces free water with a constant level. At time t=0, when no flow is supposed, pumping will be started over the full height of the sandy layer. The total discharge is constant and will be denoted by Q (see Fig.2).

As the problem is circularly symmetrical, it is appropriate to work with polar coordinates,

$$r^2 = x^2 + y^2 \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

where r is the distance from the center.
The outer boundary condition for the sandy layer then becomes,

$$r = r_1 \quad ; \quad \tilde{\phi}_c = 0$$

As the radius of the pump tube is relatively very small, one may outline the inner boundary condition as the following,

$$\lim_{r \to 0} r \frac{\partial \tilde{\phi}_c}{\partial r} = \frac{Q}{2\pi k_s H}$$

It follows from the theory in the previous section that one can obtain the solution of this problem by solving it both for consolidation alone and for lowering of the free surface alone. As consolidation will occur in the first stage we will start with the study of this mechanism. Here one may apply equation (15) but as the lowering of the free surface will not be taken into account, one must think the n_c in this equation to be infinitely large. In this way one may rewrite equation (15) as follows,

$$\frac{\partial \tilde{\phi}_c}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \tilde{\phi}_c}{\partial \rho} = \frac{S_s}{S_c}\frac{\partial \tilde{\phi}_c}{\partial \tau} + \tilde{\phi}_c + \int_0^\tau \frac{\partial \tilde{\phi}_c}{\partial \lambda} 2\sum_{m=0}^{\infty} \exp\{-m^2\pi^2(\tau-\lambda)\} d\lambda \tag{31}$$

where

$$\rho = r\sqrt{\frac{k_c}{k_s}\frac{1}{h_0 H}} \qquad \tau = \frac{C_c}{h_0^2}t = \frac{k_c}{S_c h_0}t$$

One now has to solve equation (31) under the boundary conditions,

$$\lim_{\rho\to 0} \rho\frac{\partial \tilde{\phi}_c}{\partial \rho} = \frac{Q}{2\pi k_s H} \qquad \rho = \rho_1 \ ; \ \tilde{\phi}_c = 0 \qquad \left(\rho_1 = r_1\sqrt{\frac{k_c}{k_s}\frac{1}{h_0 H}}\right)$$

It will be useful to determine in advance the steady flow solution after the consolidation phenomenon has occured. This solution has to satisfy the equation,

$$\frac{\partial^2 \tilde{\phi}_0}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi_0}{\partial \rho} = \phi_0$$

with the solution under the above mentioned boundary conditions being,

$$\tilde{\phi}_0 = -\frac{Q}{2\pi k_s H}\, K_0(\rho) - \frac{K_0(\rho_1)}{I_0(\rho_1)}\, I_0(\rho) \tag{32}$$

where $I_\alpha(\rho)$ and $K_\alpha(\rho)$ denote the modified Bessel functions of order α, of the first and second kind respectively. In order to obtain boundary conditions equal to zero, one may subtract the steady flow equation from equation (31) and solve the problem for the variable,

$$\frac{Q}{2\pi k_s H}\phi = \tilde{\phi}_c - \tilde{\phi}_0 \tag{33}$$

One then has to solve an equation of exactly the same type as (31),

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho} = \frac{S_s}{S_c}\frac{\partial \phi}{\partial \tau} + \phi + \int_0^\tau \frac{\partial \phi}{\partial \lambda}\, 2\sum_{m=1}^{\infty} \exp\{-m^2\pi^2(\tau-\lambda)\} d\lambda$$

but under the boundary conditions,

$$\lim_{\rho\to 0} \rho\frac{\partial \phi}{\partial \rho} = 0 \qquad \rho = \rho_1 \qquad \phi = 0$$

However the initial condition becomes

$$\tau = 0 \qquad \phi = K_0(\rho) - \frac{K_0(\rho_1)}{I_0(\rho_1)} I_0(\rho)$$

As the boundary conditions of equation (34) are zero one may apply the separation axiom. The solution of (34) can be supposed to be a function R of ρ times

a function T of τ . In this way one will obtain by introducing an arbitrary constant two equations : one a linear second order differential equation for $R(\rho)$, the other a mixed differential integral equation for $T(\tau)$. The differential equation for $R(\rho)$ can be solved under the boundary conditions for the sandy layer.
The result is,

$$R = AJ_0\left\{\frac{\rho}{\rho_1} j_{0,n}\right\} \tag{35}$$

where A is an arbitrary constant and $J_\alpha(x)$ is a Bessel function of order α of the first kind, in which the zero's are denoted by $j_{\alpha,n}$.
By solving the equation for $R(\rho)$, one also finds the value of the introduced arbitrary constant for the purpose of applying the separation axiom. Thus one is able to determine the general solution for the mixed differential integral equation for $T(\tau)$. This can be done by applying the theory of Laplace transformation. The result is,

$$T = \sum_{m=1}^{\infty} \frac{2(1+j^2_{0,n}/\rho_1^2)T_0}{P_m^2(1+S_s/S_c) + j^2_{0,n}/\rho_1^2 + P_m^2 ctg^2 P_m} \exp(-P_m^2 T) \tag{36}$$

where P_m is the m-th root of $S_s/S_c \cdot P_m - j^2_{0,n}/\rho_1^2 P_m = ctg(P_m)$ $\quad P_m > 0$

T_0 is the value of T for $\tau = 0$

Apparently one may write,

$$\sum_{m=1}^{\infty} \frac{2(1+ j^2_{0,n}/\rho_1^2)}{P_m^2(1+S_s/S_c) + j^2_{o,n}/\rho_1^2 + P_m^2 ctg^2 P_m} = 1$$

Bessel functions of real variable and order have an infinite number of zero's. Therefore it follows from (35) and (36) that the possible solution of ϕ can have the form,

$$\phi = \sum_{n=1}^{\infty} a_n J_0\left\{\frac{\rho}{\rho_1} j_{0,n}\right\} \sum_{m=1}^{\infty} \frac{2(1 + j^2_{0,n}/\rho_1^2)}{P_m^2(1+S_s/S_c) + j^2_{0,n}/\rho_1^2 + P_m^2 ctg^2 P_m} \exp(-P_m^2 \tau) \tag{37}$$

where AT_0 is denoted by a_n . Formula (37) satisfies the differential equation (34) and the boundary conditions for the sandy layer. The initial condition still has to be satisfied. Therefore the coefficients a_n have to be chosen in such a way that the following folds,

$$K_0(\rho) - \frac{K_0(\rho_1)}{I_0(\rho_1)} I_0(\rho) = \sum_{n=1}^{\infty} a_n J_0\left\{\frac{\rho}{\rho_1} j_{0,n}\right\} \tag{38}$$

The right side of (38) is a Fourier-Bessel expansion. It follows from the theory of Fourier-Bessel expansions that it then holds for the coefficients a_n that

$$a_n = \frac{2}{J_1^2(j_{0,n})}\int_0^1 \lambda\{K_0(\rho_1\lambda) - \frac{K_0(\rho_1)}{I_0(\rho_1)} I_0(\rho_1\lambda)\} J_0\{\lambda j_{0,n}\}d\lambda$$

Later in this section, when lowering of the free surface is studied, it will turn out that it also holds that,

$$K_0(\rho) - \frac{K_0(\rho_1)}{I_0(\rho_1)} I_0(\rho) = \sum_{n=1}^{\infty} \frac{2}{J_1^2(j_{0,n})} \frac{1}{\rho_1^2+j_{0,n}^2} J_0\{\frac{\rho}{\rho_1}j_{0,n}\}$$

As this equation is identical with (38) we apparently may write,

$$a_n = \frac{2}{J_1^2(j_{o,n})} \frac{1}{\rho_1^2+j_{0,n}^2} \tag{39}$$

and

$$\int_0^1 \lambda\{K_0(\rho_1\lambda) - \frac{K_0(\rho_1)}{I_0(\rho_1)} I_0(\rho_1\lambda)\ J_0\{\lambda j_{0,n}\}d\lambda = \frac{1}{\rho_1^2 + j_{0,n}^2}$$

The problem is solved now. By use of (32),(33),(36),(37),(38) and (39) one may write for the solution of the average head over the height of the sandy layer.

$$\tilde{\phi}_c = -\frac{Q}{2\pi k_s H} \sum_{n=1}^{\infty} \frac{2}{\rho_1^2 J_1^2(j_{0,n})} J_0\{\frac{\rho}{\rho_1}j_{0,n}\} \sum_{m=1}^{\infty} \frac{2(1-\exp(-P_m^2\tau))}{P_m^2(1+S_s/S_c)+j_{0,n}^2/\rho_1^2+P_m^2 ctg^2 P_m} \tag{40}$$

where P_m is the m-th root of $S_s/S_c \cdot P_m - J_{0,n}^2/\rho_1^2 P_m = ctgP_m$, $P_m > 0$

$$\sum_{m=1}^{\infty} \frac{2(1+J_{0,n}^2/\rho_1^2)}{P_m^2(1+S_s/S_c)+j_{0,n}^2/\rho_1^2 + P_m^2 ctg^2 P_m} = 1$$

It is possible to simplify equation (40) by applying the approximations,

$$j_{0,n} \sim \pi(n-1/4) \quad ; \quad J_1^2(j_{0,n}) \sim 2/\pi^2(n-1/4)$$

These approximations hold especially very well for the larger values of the integer n. One may now write, allowing an absolute error of about 1% of the unity of $2\pi k_s H/Q \cdot \tilde{\phi}_c$

$$\tilde{\phi}_c = -\frac{Q}{2\pi k_s H} \sum_{n=1}^{\infty} \frac{\pi}{\rho_1} (n-1/4) J_0\{\rho \frac{\pi}{\rho_1}(n-1/4)\} \sum_{m=1}^{\infty} \frac{2(1-\exp(-P_m^2\tau))}{P_m^2(1+S_s/S_c)+\frac{\pi^2}{\rho_1^2}(n-1/4)^2+P_m^2 ctg^2 P_m} \frac{\pi}{\rho_1} \tag{41}$$

where P_m is the m-th root of $S_s/S_c \cdot P_m - \frac{\pi^2}{\rho_1^2}(n-1/4)^2/P_m = ctgP_m$, $P_m > 0$

$$\sum_{m=1}^{\infty} \frac{2(1+\pi^2(n-1/4)^2/\rho_1^2)}{P_m^2(1+S_s/S_c) + \pi^2(n-1/4)^2/\rho_1^2 + P_m^2 ctg^2 P_m} = 1$$

An interesting value of the outer boundary is the limit $\rho_1 \to \infty$. This is the case of a well in a very wide field, where the outer boundaries are so far from the well that they cannot be properly distinguished. For the limit $\rho_1 \to \infty$ formula (40) becomes an integral. One may denote $n\rho_1/\pi = \alpha$, $\rho_1/\pi = d\alpha$ to write this solution as,

$$\tilde{\phi}_{c\,\rho_1\to\infty} = -\frac{Q}{2\pi k_s H}\int_0^\infty \alpha J_0(\rho\alpha)\sum_{m=1}^{\infty}\frac{2(1-\exp(-P_m^2\tau))}{P_m^2(1+S_s/S_c)+\alpha^2+P_m^2 ctg^2 P_m}\,d\alpha \qquad (42)$$

where P_m is the m-th root of $(S_s/S_c)P_m - \alpha^2/P_m = ctg P_m$, $P_m > 0$

$$\sum_{m=1}^{\infty}\frac{2(1+\alpha^2)}{P_m^2(1+S_s/S_c)+\alpha^2+P_m^2 ctg^2 P_m} = 1$$

It is integrating to compare solution (41) with formula (42). They are similar for the case $\rho_1 = \pi$ is applied in (41). The only difference is that (41) is a summation and (42) an integral. Therefore since $K_0(\rho)$ nearly vanishes for $\rho > \pi$ one may consider (41) applied for the case $\rho_1 = \pi$, to be a good numerical approach to the integral form (42). The size of the step is then unity and may be that large, because the variable of formula (41) is shifted over the value 1/4. Furthermore, it is possible to simplify formula (42). As α is known as a function of P_m, one may substitute this function in the integral form. One obtains,

$$\tilde{\phi}_{c\,\rho_1\to\infty} = -\frac{Q}{2\pi k_s H}\sum_{m=1}^{\infty}\int_{P_m}^{m\pi} J_0\left(\rho\sqrt{\frac{S_s}{S_c}P^2 - P\,ctg P}\right)\frac{1-\exp(-P^2\tau)}{P}\,dP \qquad (43)$$

where, P_m satisfies $S_s/S_c \cdot P_m = ctg P_m$

$$K_0(\rho) = \sum_{m=1}^{\infty}\int_{P_m}^{m\pi} J_0\left(\rho\sqrt{\frac{S_s}{S_c}P^2 - P\,ctg P}\right)\frac{dP}{P}$$

This equation can also be derived directly by the theory of Laplace transformation.

It can be derived from formula (43) that the solution $\tilde{\phi}_{\rho\to\infty}$ behaves for large values of the time like a function of,

$$\rho \;;\; P_1^2\tau \qquad \text{where } S_s P_1/S_c = ctg P_1$$

P_1^2 can be fairly represented by $1/(S_s/S_c + 4/\pi^2)$. Therefore for the case $\rho_1 = \pi$ the following relationships are determined,

$$\frac{2\pi k_s H}{Q}\,\tilde{\phi}_c\left\{\left(\frac{k_c}{S_s + 4S_c/\pi^2}\frac{t}{h_0}\;;\; r\sqrt{\frac{k_c}{k_s}\frac{1}{h_0 H}}\;,\;\frac{S_c}{S_s}\right)\right\} \qquad \text{see Fig.3.}$$

$$\frac{2\pi k_s H}{Q}\,\tilde{\phi}_c\left\{\left(r\sqrt{\frac{k_c}{k_s}\frac{1}{h_0 H}}\;;\;\frac{k_c}{S_s + 4S_c/\pi^2}\frac{t}{h_0}\;,\;\frac{S_s}{S_c}\right)\right\} \qquad \text{see Fig.4.}$$

As for $r\sqrt{k_c/k_s h_0 H}$ a logarithmic scale is used. Therefore the value of the tangent at the curves will be a direct measure of the discharge through a cylindrical cross section around the well. Obviously for small values of the radius and not too small values of the time, the curves is parallel straight line, which means that the discharge will be nearly constant and equal to Q. As for the time, the basic parameter S_s+4S_c/π^2 is used. Actually one should expect this parameter to be $S_s+S_c/3$ from the theory in the previous section. However both parameters serve a different aim. $S_s+S_c/3$ characterizes the behaviour of consolidation in the total flow phenomenon. $S_s=4S_c/\pi^2$ characterizes the behaviour of consolidation alone for large values of the time variable $\frac{k_c}{S_s+4S_c/\pi^2}\frac{t}{h_0}$, one can apply the variable $\frac{k_c}{S_s+S_c/3}\frac{t}{h_0}$ by slightly shifting the determined curves.

One has now succeeded in determining one leg of the general solution for the average head over the height of the sandy layer. Now the other leg, lowering of the free surface, has to be studied. The basic equation to be investigated is the inverse equation (22), applied for the configuration of Fig.2.,

$$\frac{\partial^2\tilde{\phi}_s}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\tilde{\phi}_s}{\partial\rho}=\int_0^{\P}\frac{\partial\tilde{\phi}_s}{\partial\lambda}\exp(-(\P-\lambda))d\lambda \qquad (44)$$

where the variables r and t are replaced by ρ and $\P$. The boundary conditions are the same as before, during consolidation,

$$\lim_{\rho\to 0}\rho\frac{\partial\tilde{\phi}_s}{\partial\rho}=\frac{Q}{2\pi k_s H} \qquad \rho=\rho_1 \quad ; \quad \tilde{\phi}_s=0$$

It follows directly from equation (44) that the head at time $\P\to 0$ satisfies the equation,

$$\frac{\partial^2\tilde{\phi}_s}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\tilde{\phi}_s}{\partial\rho}=\tilde{\phi}_s$$

Therefore it holds that,

$$\lim_{\P\to\infty}\tilde{\phi}_s=\tilde{\phi}_0$$

Thus it turns out that the function $\tilde{\phi}_s$ is discontinuous for the time t=0 .

To solve equation (44) the same method can be used as for consolidation previously. Therefore one determines at first the steady flow equation after lowering of the free surface, the head in this case to be denoted by $\tilde{\phi}_\infty$,

$$\frac{\partial^2\tilde{\phi}_\infty}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\tilde{\phi}_\infty}{\partial\rho}=0$$

which can be easily solved giving,

$$\tilde{\phi}_\infty=\frac{Q}{2\pi k_s H}\ln\frac{\rho}{\rho_1} \qquad (45)$$

Again in order to obtain boundary conditions equal to zero one may substract the steady flow equation (44) and solve the problem for the variable,

$$\frac{Q}{2\pi k_s H}\phi = \tilde{\phi}_s - \tilde{\phi}_\infty \tag{46}$$

One has to solve then an equation of exactly the same type as (44)

$$\frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} = \int_0^{\P} \frac{\partial\phi}{\partial\lambda}\exp(-(\P-\lambda))d\lambda \tag{47}$$

But under the boundary conditions,

$$\lim_{\rho\to 0}\rho\frac{\partial\phi}{\partial\rho} = 0 \qquad \rho=\rho_1 \; ; \quad \phi = 0$$

The initial condition becomes,

$$\P = 0 \qquad \phi = -\ln\frac{\rho}{\rho_1}$$

Equation (47) also can be solved by the separation axiom. The solution of (47) can be supposed as before to be a function R of ρ times a function T of $\P$. As the operation with respect to the place coordinates is the same as during consolidation, the same function (35) for $R(\rho)$ will be obtained,

$$R = A\cdot J_0\left(\frac{\rho}{\rho_1}j_{0,n}\right) \tag{48}$$

The function $T(\P)$ canbe determined from an integral equation, with general solution for $\P>0$, which can be derived by the theory of Laplace transformation,

$$T = \frac{T_0}{1 + j_{0,n}^2/\rho_1^2}\exp\left(-\frac{j_{0,n}^2}{\rho_1^2 + j_{0,n}^2}\P\right) \tag{49}$$

where T_0 is the value of T for $\P=0$. It follows from (49) that the value of T for $\P\to 0$ is different from that for $\P=0$. This is due to the fact that the function $\tilde{\phi}_s$ is discontinuous for $t=0$.

By use of (48) and (49) the possible solution for ϕ can now be written in the form,

$$\phi = \sum_{n=1}^{\infty}\frac{a_n}{1+j_{0,n}^2/\rho_1^2}J_0\left(\frac{\rho}{\rho_1}j_{0,n}\right)\exp\left(-\frac{j_{0,n}^2}{\rho_1^2+j_{0,n}^2}\P\right) \tag{50}$$

where At_0 is denoted by a_n, Formula (50) satisfies the differential equation (47) and the boundary conditions for the sandy layer. The initial condition still has to be satisfied. Therefore the coefficients a_n have to be chosen in such a way that the following holds,

$$-\ln\frac{\rho}{\rho_1} = \sum_{n=1}^{\infty} a_n J_0\left(\frac{\rho}{\rho_1}j_{0,n}\right) \tag{51}$$

where of course from formula (49) for T the value T_0 had to be applied. It follows from the theory of Fourier-Bessel expansions that it holds then for the coefficients a_n that,

$$a_n = \frac{2}{J_1^2(j_{0,n})} \int_0^1 \lambda \ln \lambda J_0(\lambda j_{0,n}) d\lambda$$

It can be obtained easily by partial integration that,

$$a_n = \frac{2}{J_1^2(j_{0,n})} \frac{1}{j_{0,n}^2} \qquad (52)$$

One is now able to compose the solution for the average head over the height of the sandy layer for values of $\P > 0$ by meansof (45),(46),(50),(51) and (52),

$$\tilde{\Phi}_s = -\frac{Q}{2\pi k_s H} \sum_{n=1}^{\infty} \frac{2}{J_1^2(j_{0,n})} \frac{1}{j_{0,n}^2} J_0(\frac{\rho}{\rho_1} j_{0,n}) \; 1 - \frac{\exp(-j_{0,n}^2/(\rho_1^2 + j_{0,n}^2) \P)}{1 + j_{0,n}^2/\rho_1^2} \qquad (53)$$

with $\tilde{\Phi}_s = 0$ for $\P = 0$

For the limit $\P \to 0$ formula (53) will represent the solution (32) for $\tilde{\phi}_0$. Therefore the following Fourier-Bessel expansion will hold,

$$K_0(\rho) - \frac{K_0(\rho_1)}{I_0(\rho_1)} I_0(\rho) = \sum_{n=1}^{\infty} \frac{2}{J_1^2(j_{0,n})} \frac{1}{\rho_1^2 + j_{0,n}^2} J_0(\frac{\rho}{\rho_1} j_{0,n})$$

This result is used previously in this section to determine the coefficients a_n under the condition of consolidation.

Just as for the solution for consolidation alone, one may simplify solution (53) by the approximations,

$$j_{0,n} \sim \pi(n-1/4) \qquad J_1^2(j_{0,n}) \sim 2/\pi^2(n-1/4)$$

But in this case one has to allow the larger absolute error of about 5% of the unity of $2\pi k_s H/Q \cdot \tilde{\Phi}_s$, in order to write,

$$\tilde{\Phi}_s = -\frac{Q}{2\pi k_s H} \sum_{n=1}^{\infty} \frac{1}{n-1/4} J_0(\frac{\rho\pi}{\rho_1}(n-1/4))\{1 - \frac{\exp(-\pi^2(n-1/4)^2/(\rho_1^2 + \pi^2(n-1/4)^2)\P)}{1 + \pi^2/\rho_1^2 \cdot (n-1/4)^2}\} \qquad (54)$$

As the flow phenomenon inthis part of this section is due to lowering of the free surface in the clay, a very interesting variable to be determined is the head of the free surface. This can be determined from the relation it has with the average head in the sand, equation (14). From the inverse transformation of this equation and the solution for the average head (53), it can be derived that,

$$h_s = -\frac{Q}{2\pi k_s H} \sum_{n=1}^{\infty} \frac{2}{J_{1,n}^2(j_{0,n})} \frac{1}{j_{0,n}^2} J_0(\frac{\rho}{\rho_1} j_{0,n}) \{1 - \exp(-\frac{j_{0,n}^2}{\rho_1^2 + j_{0,n}^2} \P)\} \qquad (55)$$

An approximate value for this head follows from the inverse transform of (14) and the approximate solution (54),

$$h_s = -\frac{Q}{2\pi k_s H} \sum_{n=1}^{\infty} \frac{1}{n-1/4} J_0(\rho\frac{\pi}{\rho_1}(n-1/4))\ 1- \exp(- \frac{\pi^2 (n-1/4)^2}{\rho_1^2 + \pi^2 (n-1/4)^2} \P) \qquad (56)$$

For the same value of the boundary as used previously in the case of consolidation $\rho = \pi$, the following relationships are determined,

$$\frac{2\pi k_s H}{Q} \tilde{\phi}_s(\frac{k'_c}{n_c h_0} t \ ; \ r\sqrt{\frac{k'_c}{k_s} \frac{1}{h_0 H}}) \ ; \qquad \frac{2\pi k_s H}{Q} h_s(\frac{k'_c}{n_c h_o} t \ ; \ r\sqrt{\frac{k'_c}{k_s} \frac{1}{h_0 H}}) \qquad \text{see Fig.5.}$$

$$\frac{2\pi k_s H}{Q} \tilde{\phi}_s(r\sqrt{\frac{k'_c}{k_s} \frac{1}{h_0 H}} \ ; \ \frac{k'_c}{n_c h_0} t) \ ; \qquad \frac{2\pi k_s H}{Q} h_s(r\sqrt{\frac{k'_c}{k_s} \frac{1}{h_0 H}} \ ; \ \frac{k'_0}{n_c h_0} t) \qquad \text{See Fig.6.}$$

In Fig.5, one notices groups of two curves, ending up in the same point $t \to \infty$. The upper curve represents the head of the free surface in the clay, the lower the aveage head over the height of the sandy layer. As the curves for large values of the time behave like functions of $\exp(-9\P/25)$, in Fig.5, is drawn on exponential paper. One may notice that the drawdown for small values of ρ is relatively faster than for larger ones. As for $r\sqrt{k'_c/k_s h_0 H}$ just as in the previous case of consolidation, a logarithmic scale is used. Therefore here also the discharge can be read directly from the tangent at the curves for the head in the sand on the left side of Fig.6.

One has now succeeded in determining the solutions for consolidation $\tilde{\phi}_c$, and for lowering of the free surface in the clay $\tilde{\phi}_s$. According to the theory of the previous section one is able now to construct a formula for the behaviour of the general solution for the average head over the height of the sandy layer. This general solution is represented by (30),

$$\tilde{\phi} = \tilde{\phi}_c(\P) \qquad 0 \leq \P \leq \P_0$$

$$\tilde{\phi} = \tilde{\phi}_c(\P) + \tilde{\phi}_s(\P - \P_0) - \tilde{\Phi}_0 \qquad \P > \P_0 \qquad (57)$$

$$\P_0 = \frac{S}{n_c} \lim_{\P \to 0} (\frac{\partial \phi_s}{\partial \P} / \phi_s) \qquad S = S_s + S_c/3$$

Therefore the general solution can easily be composed from the already derived curves. For the relatively large value 0.1 for S/n_c the following relationship is determined,

$$\tilde{\Phi}\left(\frac{k'_c}{n_c h_0}t\ ;\ r\sqrt{\frac{k'_c}{k_s}\frac{1}{h_0 H}},\ S_s=0,\ S_c=0\right) \qquad \text{See Fig.7.}$$

Theoretically there will occur a bend in the curves due to the schematization of the auxiliary function $\tilde{\Phi}_*$, earlier in this section. However this bend turns out to be so small that by simply drawing the curves it will already be rounded off.
The representation of (57) means that the two types of curves, one for $\P > \P_0$, the other for $0 \leqq \P \leqq \P_0$, are close asymptotic expressions. The real value of the head arround $\P = \P_0$ has to be slightly lower than the representation suggests.

At last one can determine the head of the free surface for the general case. This can be done from the inverse transformation of (14) and formula (57). The result for values $\P > \P_0$ is,

$$h = \int_{\P_0}^{\P}\tilde{\Phi}_s(\lambda-\P_0)\exp(-(\P-\lambda))d\lambda - \int_0^{\P}(\tilde{\Phi}_0-\tilde{\Phi}_c)\exp(-(\P-\lambda))d\lambda + \int_0^{\P_0}\tilde{\Phi}_0\exp(-(\P-\lambda))d\lambda \qquad (58)$$

The second integral has already been discussed and determined for not too small values of the time. Its absolute value is equal to the absolute value of the third integral, but their signs are different. Therefore the head of the free surface turns out to be equal to the first integral, of which the solution is formula (55) or (56) applied for the variable $\P - \P_0$. As it was supposed that during consolidation the head of the free surface should not change, one may now write

$$\begin{aligned} h &= 0 && 0 \leqq \P \leqq \P_0 \\ h &= h_s(\P - \P_0) && \P > \P_0 \end{aligned} \qquad (59)$$

Just as in the case of the average head in the sand, the formula for the head of the free surface in the clay is obtained in the form of two asymptotic expression. The real value of the head around the time $\P = \P_0$ is slightly lower than the representation suggests. It is possible to get an impression of the value of the head for the time $\P = \P_0$. As $\P$ has a small number, one may integrate the solution $\tilde{\Phi}_c$ with respect to $\P$ over the timelapse $\P_0$; see (58) for $\P \to \P_0$. Integration can be done roughly by assuming the behaviour of the curves of Fig.3 to be logarithmic.
Even for a relatively large value 0.1 for S/n_c, the value of $2\pi k_s H/Q \cdot h$ is within a few percent of its unity.

One did succeed in applying the theory of the previous section to a certain configuration. It turned out that the processes of consolidation and lowering of the free surface hardly influence each other. The delay caused by consolidation in lowering the free surface is relatively of little interest. Besides the theory in this paper, the more general case of consolidation in the sandy layer alone and lowering of the free surface has been studied.

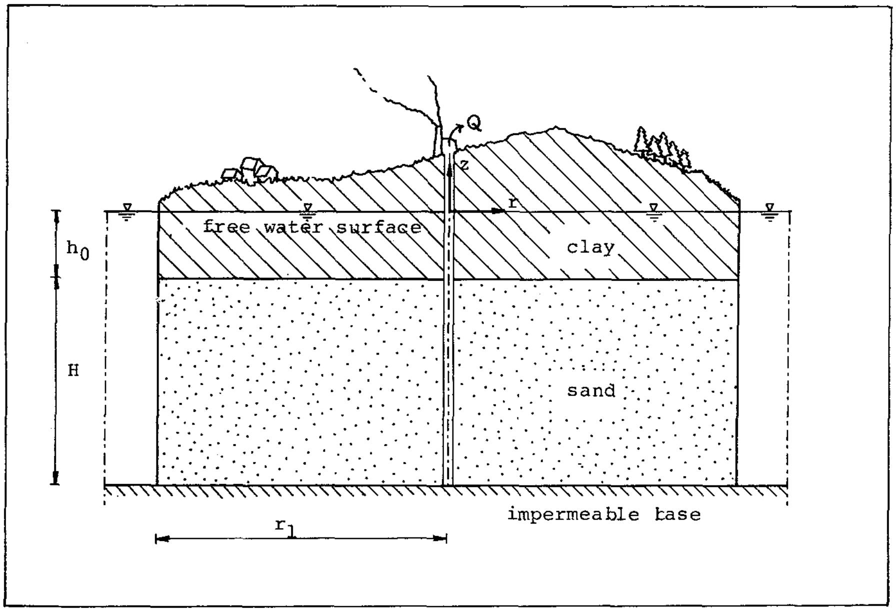

Fig.2 Configuration studied as a practical example.

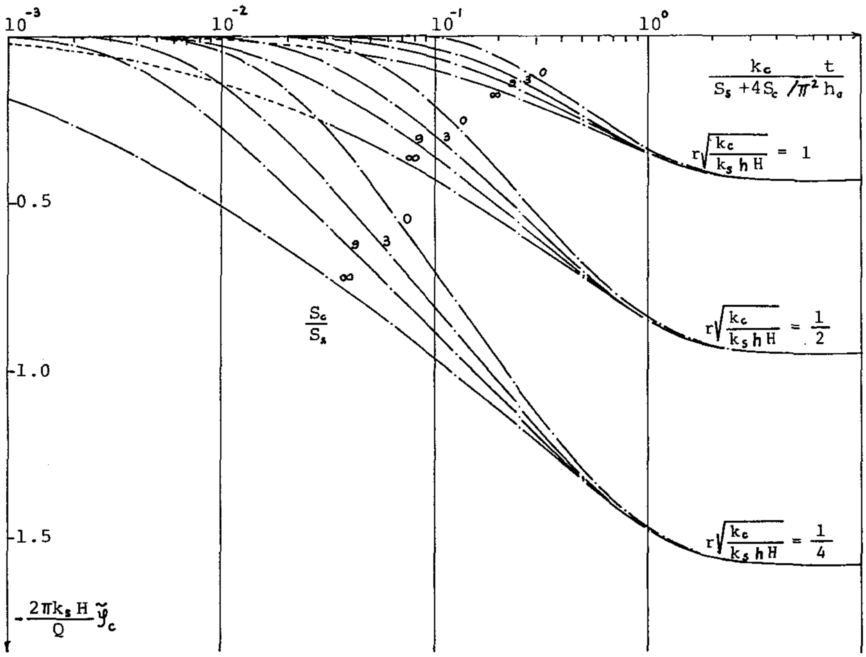

Fig.3 Drawdown in the sandy layer as function of the time due to consolidation alone.

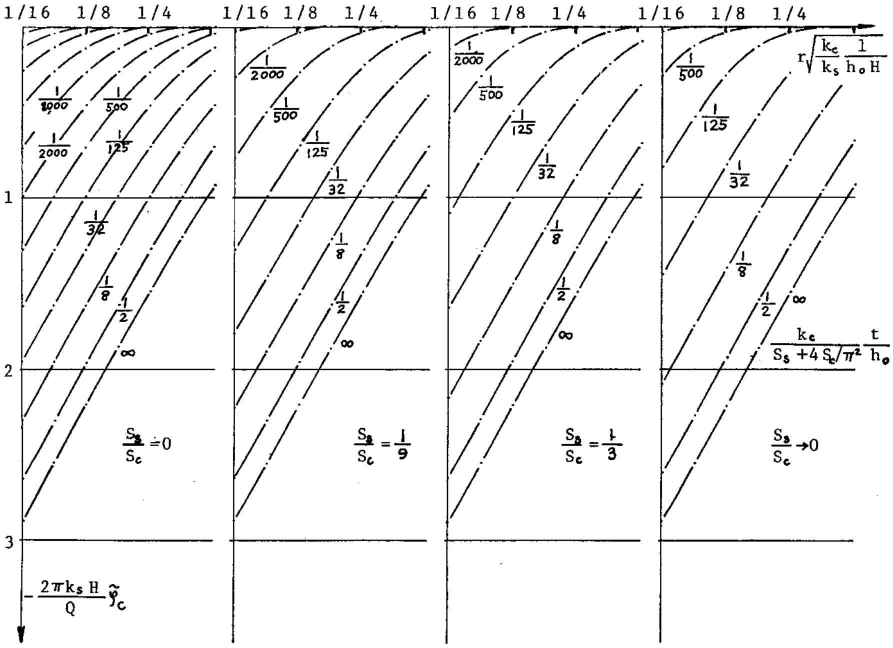

Fig.4 Drawdown in the sandy layer as function of the radius due to consolidation alone.

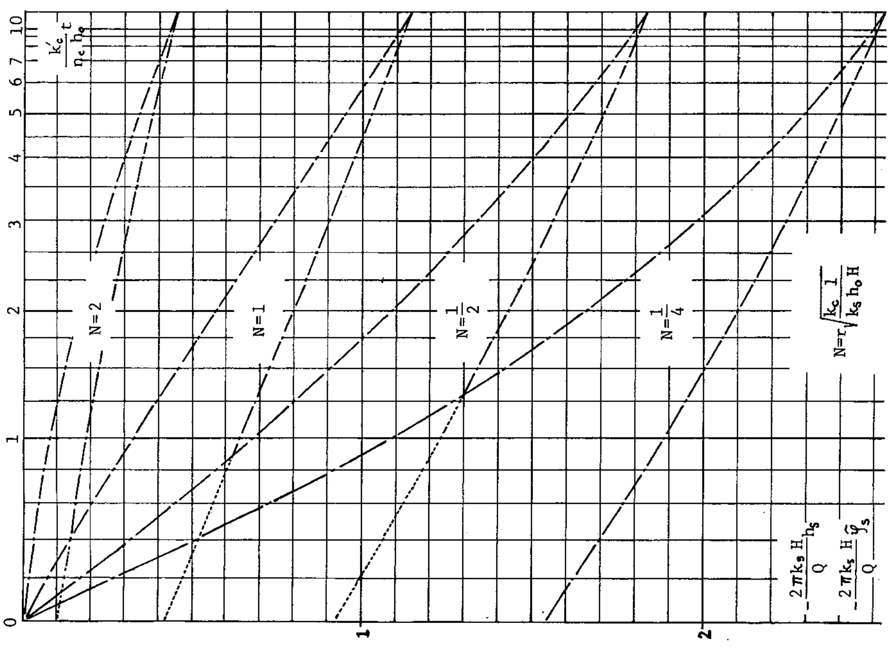

Fig.5 Drawdown in the clayey layer as well as the sandy layer as function of the time due to lowering of the free surface alone.

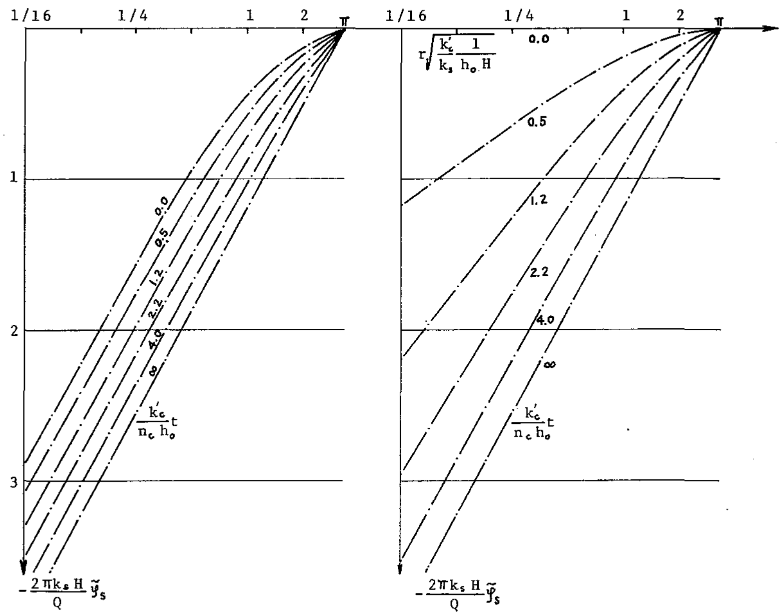

Fig.6 Drawdown in the sandy as well as the clay layer as function of the radius due to lowering of the free surface alone.

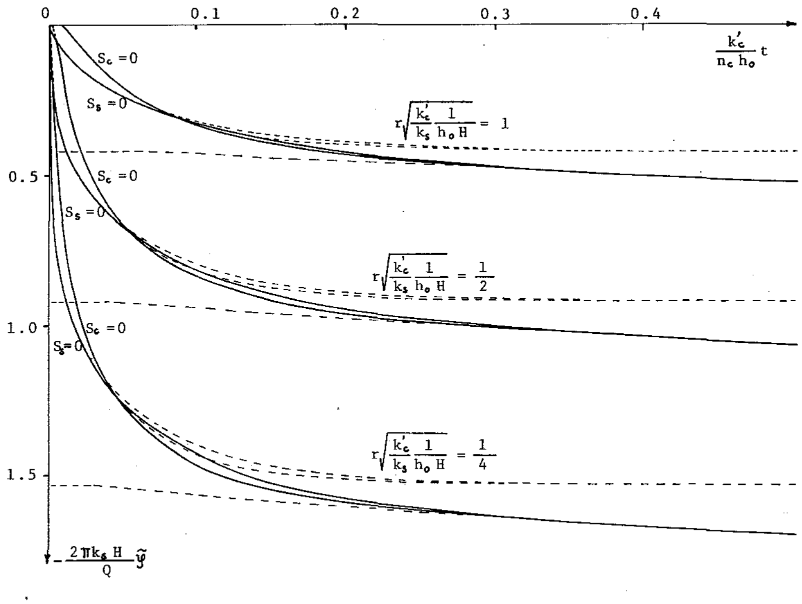

Fig.7 Drawdown in the sandy layer as function of the time due to consolidation as well as lowering of the free surface.

References

(1) N. S. Boulton: Analysis of data from non-equilibrium pumping tests allowing for delayed yield from storage, Proc. Inst. Civ. Engrs., 26 (1963).

(2) M. S. Hantush: Hydraulic of wells, New Mexico Institute of Mining and Technology, Socorro, New Mexico.

(3) G. N. Watson: A treatise on the theory of Bessel functions, Cambridge, The University Press (1966).

(4) J. B. Sellmeijer: Nonsteady flow in a two layer system, Laboratory of Geo-technological University (1972).

(5) I.Kono: Dewatering, Kashima-syuppan Ltd. (1970).

Nomenclatures

c consolidation coefficient

h_0 initial height of the free surface in the clay layer

k permeability coefficient

n porocity

H height of the sandy layer

Q discharge from a well

S storage coefficient

h head of the free surface in the clay layer

ϕ head

$\tilde{\phi}$ average head over the height of the sandy layer

$\tilde{\phi}_0$ steady flow solution after consolidation alone as well as the initial stage solution for lowering of the free surface only

$\tilde{\phi}_\infty$ steady flow solution after the nonsteady flow phenomenon has taken place completely

$\tilde{\phi}_1$ derivative of the solution for lowering of the free surface in the clay with respect to ¶ for the limit ¶ → 0

x, y, z, r place coordinate

ξ, η, Z, ρ auxiliary place coordinate

t time

τ ¶ auxiliary time coordinate

$¶_0$ characteristic for the delay in lowering of the free surface by consolidation

The indices c and s refer to the clay and the sand respectively.