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On Conformal Mapping onto Circular-Radial Slit Covering Surfaces and its Extremal Properties

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1. Let B be a domain on the z-plane of which the boundary C consists of a finite number of continua $C_1, \cdots, C_N$ $(N \geqq 1)$. Partition the boundary C into two disjoint sets

$$C' = \sum_{j=1}^{\lambda} C_j$$

and

$$C'' = \sum_{j=1}^{\mu} C_{\lambda+j}$$

$$(\lambda \geqq 0, \mu \geqq 0, \lambda + \mu = N),$$

where $C' = \phi$ or $C'' = \phi$ is permitted. Let z_j and ζ_k $(j = 1, \cdots, \iota;\ k = 1, \cdots, \kappa;\ \iota \geqq 1, \kappa \geqq 1)$ be arbitrarily preassigned $\iota + \kappa$ points in B, and m_j, and $n_k (j = 1, \cdots, \iota;\ k = 1, \cdots, \kappa)$ be arbitrarily preassigned positive integers under the condition

$$(1) \qquad p \equiv \sum_{j=1}^{\iota} m_j = \sum_{k=1}^{\kappa} n_k.$$

We shall conventionally agree to take as $\zeta_\kappa = \infty \in B$ through the present paper. Let $\mathfrak{F}_p$ be the class of functions $w = f(z)$ single-valued, analytic on B with the following properties;

(a) f has the only zeros z_j $(j = 1, \cdots, \iota)$ and the only poles ζ_k $(k = 1, \cdots, \kappa)$ with their orders m_j and n_k, respectively;

(b) The rotation number of the image of each C_j $(j = 1, \cdots, N)$ about $w = 0$ under f is equal to zero; i. e.

$$\nu_j(f) \equiv \frac{1}{2\pi} \int_{C_j^*} d \arg f = 0 \ (j = 1, \cdots, N),$$

where C_j^* $(j = 1, \cdots, N)$ are analytic Jordan curves homotopic to C_j in

$$B - \sum_{j=1}^{\iota} \{z_j\} - \sum_{k=1}^{\kappa} \{\zeta_k\}$$

and $\nu_j(f)$ $(j = 1, \cdots, N)$ are integers not depending on a particular choice of C_j^*;

$$(c) \qquad \left| \int_C \lg |f| \, d \arg f \right| < +\infty,$$

where the line integral means

$$\lim_{n \to \infty} \int_{\partial B_n} \lg |f| \, d \arg f$$

with an exhaustion $\{B_n\}_{n=1}^{\infty}$ of B;

(d) f satisfies the normalization condition

$$\lim_{z \to \infty} \frac{f(z)}{z^{n_\kappa}} = 1.$$

Since the rational function

$$R(z) \equiv \prod_{j=1}^{\iota} (z - z_j)^{m_j} \Big/ \prod_{k=1}^{\kappa-1} (z - \zeta_k)^{n_k}$$

belongs to $\mathfrak{F}_p$, we find that $\mathfrak{F}_p \neq \phi$.

Let $\mathfrak{G}_p$ be the subclass of $\mathfrak{F}_p$ which consists of functions $f(z)$ satisfying the condition:

(e) An arbitrary branch of $\arg f$ is constant on each component C_j $(j = \lambda + 1, \cdots, N)$, which means that for each decreasing sequence $\{\Omega_{jn}\}_{n=1}^{\infty}$ of ends defining C_j $(j = \lambda + 1, \cdots, N)$

$$\bigcap_{n=1}^{\infty} \overline{\arg f(\Omega_{jn})}$$

is reduced to a real value.

Let $\mathfrak{H}_p$ be the subclass of $\mathfrak{F}_p$ which consists of functions $f(z)$ of $\mathfrak{F}_p$ satisfying the condition:

(e′) $\lg |f|$ is constant on each component C_j $(j = 1, \cdots, \lambda)$, which means that for each decreasing sequence $\{\Omega_{jn}\}_{n=1}^{\infty}$ of ends defining C_j $(j = 1, \cdots, \lambda)$

$$\bigcap_{n=1}^{\infty} \overline{\lg |f(\Omega_{jn})|}$$

is reduced to a real value.

Here if $C'' = \phi$ or $C' = \phi$, $\mathfrak{G}_p$ or $\mathfrak{H}_p$, respectively, is identical to $\mathfrak{F}_p$. Let $\mathfrak{F}'_p$, $\mathfrak{G}'_p$ and $\mathfrak{H}'_p$ be the subclasses of $\mathfrak{F}_p$, $\mathfrak{G}_p$ and $\mathfrak{H}_p$, respectively, which consist of functions $f(z)$ of $\mathfrak{F}_p$, $\mathfrak{G}_p$ and $\mathfrak{H}_p$ satisfying the condition:

$$(f) \qquad \int_C \lg |f| \, d \arg f \leqq 0.$$

Clearly the rational function $R(z)$ belongs to $\mathfrak{F}'_p$. We shall also see that the other classes $\mathfrak{G}_p$, $\mathfrak{H}_p$, $\mathfrak{G}'_p$ and $\mathfrak{H}'_p$ are not vacuous (cf. REMARK of 2).

2. Let

$$(2) \quad J(f) = \int_C \lg |f| \, d \arg f - 2\pi \sum_{j=1}^{\iota} m_j \lg |f^{[m_j]}(z_j)|$$

$$- 2\pi \sum_{k=1}^{\kappa-1} n_k \lg |f^{[n_k]}(\zeta_k)|$$

for $f \in \mathfrak{F}_p$, where

$$f^{[m_j]}(z_j) \equiv \lim_{z\to z_j} \frac{f(z)}{(z-z_j)^{m_j}} = \frac{1}{m_j!} f^{(m_j)}(z_j) \quad (j=1, \cdots, \iota),$$

$$f^{[n_k]}(\zeta_k) \equiv \lim_{z\to\zeta_k} \frac{1}{(z-\zeta_k)^{n_k} f(z)} = \frac{1}{n_k!}\left[\left(\frac{1}{f(z)}\right)^{(n_k)}\right]_{z=\zeta_k} (k=1, \cdots, \kappa-1).$$

Then we obtain the following fundamental theorem.

THEOREM 1. (i) *There exists the unique element φ of $\mathfrak{G}_p$ and $\mathfrak{H}_p$ which maps B onto the p-sheeted covering surface of which the boundary consists of circular slits (the images of $C_1, \cdots, C_\lambda$) centred at the origin and radial slits (the images of $C_{\lambda+1}, \cdots, C_N$) emanating from the origin*;

(ii) *The function φ is the only element which simultaneously belongs to $\mathfrak{G}_p$ and $\mathfrak{H}_p$*;

(iii) *For every $f \in \mathfrak{G}_p$, the inequality*

$$J(\varphi) \leqq J(f)$$

holds. Here the equality sign appears if and only if $f \equiv \varphi$;

(iv) *For every $f \in \mathfrak{H}'_p$, the inequality*

$$J(\varphi) \geqq J(f)$$

holds. Here the equality sign appears if and only if $f \equiv \varphi$.

Proof. The domain B can always be conformally mapped onto the domain by a univalent function Φ satisfying the condition $\Phi(\infty) = \infty$, $\Phi'(\infty) = 1$ of which the boundary consists of analytic Jordan curves. Thus we may assume that so is the domain B. In fact, by the mapping Φ the functional $J(f)$ varies only an additive quantity

$$2\pi \sum_{j=1}^{\iota} m_j^2 \lg |\Phi'(z_j)| + 2\pi \sum_{k=1}^{\kappa-1} n_k^2 \lg |\Phi'(\zeta_k)|$$

independent of a particular choice of $f \in \mathfrak{F}_p$.

Construction of φ in (i). It is easy to find a solution u of the boundary value problem satisfying the conditions:

(A) u is single-valued harmonic on $B - \{z_j\}_{j=1}^{\iota} - \{\zeta_k\}_{k=1}^{\kappa}$ and has logarithmic singularities

$$u(z) = m_j \lg|z-z_j| + O(1) \quad \text{at } z_j \quad (i=1, \cdots, \iota),$$

$$u(z) = n_k \lg\frac{1}{|z-\zeta_k|} + O(1) \quad \text{at } \zeta_k \quad (k=1, \cdots, \kappa-1)$$

and

$$u(z) = n_\kappa \lg |z| + o(1) \quad \text{at } \zeta_\kappa = \infty;$$

(B) u is constant on each boundary component C_j $(j=1, \cdots, \lambda)$ and

$$\int_{C_j} \frac{\partial u}{\partial n} ds = 0 \quad (j=1, \cdots, \lambda),$$

where $\partial/\partial n$ denotes the inner normal derivative on C_j and ds does the line element of C_j;

(C) $\frac{\partial u}{\partial n} = 0$ along C_j $(j=\lambda+1, \cdots, N)$.

Let u^* be a conjugate harmonic function of u determined up to multiples of 2π such that

$$\lim_{z\to\infty}(u^*(z) - n_\kappa \arg z) = 2\nu\pi \ (\nu: \text{integers}),$$

and set $\varphi(z) = \exp(u + iu^*)$. Then it is easily verified that $\varphi(z)$ is the function satisfying the property (i) up to the uniqueness. The p-valency of φ is shown by the argument principle.

Proof of (iii) *and* (iv). Let f be an arbitrary element of $\mathfrak{G}_p$ or $\mathfrak{H}'_p$ and let

$$B_r = B - \sum_{j=1}^{\iota} \{|z-z_j| \leqq r\} - \sum_{k=1}^{\kappa-1} \{|z-\zeta_k| \leqq r\} - \{|z| \geqq 1/r\},$$

where r should be chosen suitably sufficiently small. Then, the image curves of $\{|z-z_j| = r\}$ $(j=1, \cdots, \iota)$, $\{|z-\zeta_k| = r\}$ $(k=1, \cdots, \kappa-1)$ and $\{|z| = 1/r\}$ under f surrounds about $w=0$ m_j-times $(j=1, \cdots, \iota)$, n_k-times $(k=1, \cdots, \kappa-1)$ and n_κ-times, respectively, and lies between circumferences

$$|w| = r^{m_j}|f^{[m_j]}(z_j)|(1+\delta(r)) \quad \text{and} \quad |w| = r^{m_j}|f^{[m_j]}(z_j)|(1-\delta(r)) \quad (j=1, \cdots, \iota),$$

$$|w| = \frac{1}{r^{n_k}|f^{[n_k]}(\zeta_k)|}(1+\delta(r)) \quad \text{and} \quad |w| = \frac{1}{r^{n_k}|f^{[n_k]}(\zeta_k)|}(1-\delta(r)) \quad (k=1, \cdots, \kappa-1),$$

and

$$|w|=\frac{1}{r^{n_\kappa}}(1+\delta(r)) \text{ and } |w|=\frac{1}{r^{n_\kappa}}(1-\delta(r)),$$

respectively, where the positive number $\delta(r)$ does not depend on $f\in\mathfrak{G}_p$ or $f\in\mathfrak{H}'_p$, and

$$\lim_{r\to 0}\delta(r)=0.$$

Therefore, using the Green's formula, we have

$$J(f)=D_{B_r}(\lg|f|)+\sum_{j=1}^{\iota}\int_{|z-z_j|=r}\lg|f|\,d\arg f+\sum_{k=1}^{\kappa-1}\int_{|z-\zeta_k|=r}\lg|f|\,d\arg f$$
$$-\int_{|z|=1/r}\lg|f|\,d\arg f-2\pi\sum_{t=1}^{\iota}m_j\lg|f^{[m_j]}(z_j)|-2\pi\sum_{k=1}^{\kappa-1}n_k\lg|f^{[n_k]}(\zeta_k)|$$
$$=D_{B_r}(\lg|f|)+2\pi\sum_{j=1}^{\iota}m_j\lg|r^{m_j}f^{[m_j]}(z_j)|+2\pi\sum_{k=1}^{\kappa-1}n_k\lg|r^{n_k}f^{[n_k]}(\zeta_k)|+2\pi n_\kappa\lg r^{n_\kappa}$$
$$-2\pi\sum_{j=1}^{\iota}m_j\lg|f^{[m_j]}(z_j)|-2\pi\sum_{k=1}^{\kappa-1}n_k\lg|f^{[n_k]}(\zeta_k)|+O(\delta(r))$$
$$=D_{B_r}(\lg|f|)+2\pi\sum_{j=1}^{\iota}m_j^2\lg r+2\pi\sum_{k=1}^{\kappa}n_k^2\lg r+O(\delta(r)),$$

where $D_{B_r}(\lg|f|)$ denotes the Dirichlet integral of $\lg|f|$ over B_r.

Let $f\in\mathfrak{G}_p$ and set $U=\lg|f|$, $u=\lg|\varphi|$ and $h=U-u$. Then, we have that

$$J(f)-J(\varphi)=D_{B_r}(\lg|f|)-D_{B_r}(\lg|\varphi|)+O(\delta(r))=D_{B_r}(U)-D_{B_r}(u)+O(\delta(r))$$
$$=2D_{B_r}(u,h)+D_{B_r}(h)+O(\delta(r)),$$

which yields, by $r\to 0$,

(3) $\quad J(f)-J(\varphi)=2D_B(u,h)+D_B(h).$

We shall show that

(4) $\quad D_B(u,h)=0.$

Let $\{B_n\}_{n=1}^{\infty}$ be an exhaustion of B such that $z_j\in B_1$ $(j=1,\cdots,\iota)$, $\zeta_k\in B_1$ $(k=1,\cdots,\kappa)$, C'' is a portion of the boundary ∂B_n of B_n for all n and $C'_n\equiv\partial B_n-C''$ consists of analytic Jordan curves C_{jn} $(j=1,\cdots,\lambda)$ homotopic to C_j, respectively. Let $u_n(z)$ $(n=1,2,\cdots)$ be the function on B_n which satisfies the conditions:

(A) u_n is single-valued harmonic on $B_n-\{z_j\}_{j=1}^{\iota}-\{\zeta_k\}_{k=1}^{\kappa}$ and has the logarithmic singularities

$$u_n(z)=m_j\lg|z-z_j|+O(1) \quad \text{at } z_j\ (j=1,\cdots,\iota),$$

$$u_n(z)=n_k\lg\frac{1}{|z-\zeta_k|}+O(1) \quad \text{at } \zeta_k\ (k=1,\cdots,\kappa-1)$$

and

$$u_n(z)=n_\kappa\lg|z|+O(1) \quad \text{at } \zeta_\kappa=\infty;$$

(B) $u_n=c_j$ on each component C_{jn} $(j=1,\cdots,\lambda)$, where c_j $(j=1,\cdots,\lambda)$ are the constant values which $u(z)$ takes on C_j, respectively;

(C) $$\frac{\partial u_n}{\partial n}=0 \quad \text{along } C''.$$

Set $u_n(z)=c_j$ on each ring domain of $B-\bar{B}_n$ adjacent to C_{jn} $(j=1,\cdots,\lambda)$. Then we can easily see that $\{u_n\}_{n=1}^{\infty}$ uniformly converges to u on B and thus

(5) $$\lim_{n\to\infty}D_B(u-u_n)=0.$$

Since

$$\int_{C_{jn}}\frac{\partial h}{\partial n}ds=\int_{C_{jn}}\frac{\partial U}{\partial n}ds-\int_{C_{jn}}\frac{\partial u}{\partial n}ds=0 \quad (j=1,\cdots,\lambda)$$

and

$$\frac{\partial h}{\partial n}=0 \quad \text{along } C'',$$

we find that

(6) $$D_{B_n}(u_n,h)=-\int_{\partial B_n}u_n\frac{\partial h}{\partial n}ds=0$$

for all n.

Further by the Schwarz's inequality,

(7) $$|D_B(u,h)-D_{B_n}(u_n,h)|\leqq|D_B(u-u_n,h)|\leqq\sqrt{D_B(u-u_n)\,D_B(h)}$$

holds. Our assertion (4) follows from (5), (6) and (7). Consequently, by (3) and (4) we have that

$$J(f)-J(\varphi)=D_B(h)\geqq 0.$$

The equality sign in the last inequality appears if and only if $h\equiv$ const. $=0$ and thus $f\equiv\varphi$, because of the normalization condition (d).

Next let $f\in\mathfrak{H}'_p$ and set $U=\lg|f|$, $u=\lg|\varphi|$ and $h=u-U$. Then we have that

$$\begin{aligned}J(\varphi)-J(f)&=D_{B_r}(\lg|\varphi|)-D_{B_r}(\lg|f|)+O(\delta(r))\\&=D_{B_r}(u)-D_{B_r}(U)+O(\delta(r))\\&=2D_{B_r}(U,h)+D_{B_r}(h)+O(\delta(r)),\end{aligned}$$

which yields, by $r\to 0$,

(8) $J(\varphi)-J(f)=2D_B(U,h)+D_B(h)$.

We shall show that

(9) $D_B(U,h)=-\int_C \lg|f|\,d\arg f$.

Let $\{B_n\}_{n=1}^{\infty}$ be an exhaustion of B such that $z_j\in B_1$ $(j=1,\cdots,\iota)$, $\zeta_k\in B_1$ $(k=1,\cdots,\kappa)$, C' is a portion of ∂B_n for all n and $C_n''\equiv\partial B_n-C'$ consists of analytic Jordan curves C_{jn} $(j=\lambda+1,\cdots,N)$ homotopic to C_j, respectively. Let $v_n(z)$ $(n=1,2,\cdots)$ be the function on B_n which satisfies the conditions:

(A) v_n is single-valued harmonic on $B_n-\{z_j\}_{j=1}^{\iota}-\{\zeta_k\}_{k=1}^{\kappa}$ and has the logarithmic singularities

$$v_n(z)=m_j\lg|z-z_j|+O(1)\quad\text{at } z_j\ (j=1,\cdots,\iota),$$

$$v_n(z)=n_k\lg\frac{1}{|z-\zeta_k|}+O(1)\quad\text{at } \zeta_k\ (k=1,\cdots,\kappa-1)$$

and

$$v_n(z)=n_\kappa\lg|z|+o(1)\quad\text{at } \zeta_\kappa=\infty;$$

(B) $v_n=$ const. on each component C_j $(j=1,\cdots,\lambda)$, and

$$\int_{C_j}\frac{\partial v_n}{\partial n}ds=0\quad(j=1,\cdots,\lambda);$$

(C) $\dfrac{\partial v_n}{\partial n}=0$ along C_{jn}

$$(j=\lambda+1,\cdots,N).$$

Extend v_n to B by setting $v_n=0$ on $B-\overline{B}_n$. For $n>m$ the equation

$$\begin{aligned}D_{B_m}&(v_m-v_1,\ v_n-v_1)\\&=-\int_{\partial B_1}(v_n-v_1)\frac{\partial}{\partial n}(v_m-v_1)\,ds\\&\quad-\int_{C_m''-C_1''}v_n\frac{\partial v_m}{\partial n}ds\\&=\int_{\partial B_1}v_1\frac{\partial v_m}{\partial n}ds=D_{B_m}(v_m-v_1)\end{aligned}$$

implies that

$$D_{B_m}(v_m-v_n)\leqq D_{B_n}(v_n-v_1)-D_{B_m}(v_m-v_1).$$

Thus $D_{B_n}(v_n-v_1)$ is increasing with n. Let v_0 be the function on B_1 which satisfies the conditions:

(A) v_0 is single-valued harmonic on $B_1-\{z_j\}_{j=1}^{\iota}-\{\zeta_k\}_{k=1}^{\kappa}$ and has the same logarithmic singularities as v_1 at z_j $(j=1,\cdots,\iota)$ and ζ_k $(k=1,\cdots,\kappa)$;

(B) $v_0=$ const. on ∂B_1.

Since, on setting $v_0\equiv 0$ on $B-\overline{B}_1$,

$$\begin{aligned}D_{B_n}&(v_0-v_1,\ v_n-v_1)\\&=-\int_{\partial B_1}v_1\frac{\partial v_n}{\partial n}ds=D_{B_n}(v_n-v_1),\end{aligned}$$

we find that

$$D_{B_n}(v_n-v_0)=D_{B_1}(v_0-v_1)-D_{B_n}(v_n-v_1).$$

Hence $D_{B_n}(v_n-v_1)$ is uniformly bounded and

$$v=\lim_{n\to\infty}v_n$$

exists on B with

(10) $$\lim_{n\to\infty}D_{B_n}(v-v_n)=0.$$

Clearly v is independent of the particular exhaustion $\{B_n\}$ of B and thus we see that

(11) $$v=u.$$

Set $h_n=v_n-U$ on B_n. Then since

$$U=\text{const.}\quad\text{on each } C_j\quad(j=1,\cdots,\lambda),$$

$$\int_{C_j}\frac{\partial h_n}{\partial n}ds=0,\ \int_{C_j}\frac{\partial U}{\partial n}ds=0\ (j=1,\cdots,\lambda)$$

and

$$\frac{\partial h_n}{\partial n}=-\frac{\partial U}{\partial n}\quad\text{along } C_n'',$$

we find that

(12) $$\begin{aligned}D_{B_n}&(U,\ h_n)\\&=-\sum_{j=1}^{\lambda}\int_{C_j}U\frac{\partial h_n}{\partial n}ds-\sum_{j=\lambda+1}^{N}\int_{C_{jn}}U\frac{\partial h_n}{\partial n}ds\\&=\sum_{j=\lambda+1}^{N}\int_{C_{jn}}U\frac{\partial U}{\partial n}ds=-\int_{\partial B_n}\lg|f|\,d\arg f.\end{aligned}$$

Further the inequality

(13) $$\begin{aligned}&|D_B(U,\ h)-D_{B_n}(U,\ h_n)|\\&\leqq|D_{B_n}(U,\ u-v_n)|+|D_{B-B_n}(U,h)|\end{aligned}$$

holds. Our assertion (9) follows from (10), (11), (12) and (13). Consequently, by (8), (9) and the condition (f) we have that

$$(14)\quad J(\varphi)-J(f) = -2\int_C \lg|f|\,d\arg f + D_B(h) \geqq 0.$$

The equality sign in the last inequality appears if and only if $h\equiv$ const. $=0$ and thus $f\equiv\varphi$, because of the normalization condition (d).

Proof of the uniqueness in (i). Let $\hat{\varphi}$ be another element of $\mathfrak{G}_p$ and $\mathfrak{H}_p$ with the same circular-radial slit mapping property as φ. Then by (iii) and (iv) we have that

$$J(\hat{\varphi}) = J(\varphi)$$

and thus

$$\hat{\varphi}\equiv\varphi.$$

Now (ii) is evident.

We should note that in THEOREM 1 the case $C'=\phi$ or $C''=\phi$ is permitted. Then we have the following corollary (cf. THEOREM 1 of [6]).

COROLLARY 1. (i) *There exists the unique element ψ of $\mathfrak{F}_p$ which maps B onto the p-sheeted covering surface of which the boundary consists of circular slits centred at the origin*;

(ii) *For every $f\in\mathfrak{F}_p$, the inequality*

$$J(\psi)\leqq J(f)$$

holds. Here the equality sign appears if and only if $f\equiv\psi$;

(iii) *There exists the unique element χ of $\mathfrak{F}_p$ which maps B onto the p-sheeted covering surface of which the boundary consists of radial slits emanating from the origin*;

(iv) *For every $f\in\mathfrak{F}'_p$, the inequality*

$$J(\chi)\geqq J(f)$$

holds. Here the equality sign appears if and only if $f\equiv\chi$.

REMARK. Let $D_j (j=1,\cdots,N)$ be the complement continua of B adjacent to C_j, respectively, and let

$$B^1 = B+\sum_{j=1}^{\lambda} D_j \text{ and } B^2 = B+\sum_{j=1}^{\mu} D_{\lambda+j}.$$

Let $\mathfrak{F}_p(B^1)$ and $\mathfrak{F}_p(B^2)$ be the class $\mathfrak{F}_p$ defined for the domains B^1 and B^2, respectively, in place of B. Apply the consequences (iii) and (i) of COROLLARY 1 to $\mathfrak{F}_p(B^1)$ and $\mathfrak{F}_p(B^2)$, respectively. Then we see that the restrictions to the domain B of the functions $\chi\in\mathfrak{F}_p(B^1)$ and $\psi\in\mathfrak{F}_p(B^2)$ of COROLLARY 1 belong to $\mathfrak{G}_p$ and $\mathfrak{H}_p$, respectively. Furthermore it is easily verified that the functions χ and ψ also belongs to $\mathfrak{G}'_p$ and $\mathfrak{H}'_p$. The above construction method is available for each domain conformally equivalent to B in place of B. Therefore we know that the both classes $\mathfrak{G}_p$ and $\mathfrak{H}_p$ have infinite numbers of elements other than the function φ of THEOREM 1.

3. Let

$$(15)\quad I(f) = \prod_{j=1}^{\iota} |f^{[m_j]}(z_j)|^{m_j} \prod_{k=1}^{\kappa-1} |f^{[n_k]}(\zeta_k)|^{n_k}$$

for $f\in\mathfrak{F}_p$. Then, we obtain the following theorem.

THEOREM 2. *Let φ be the function defined in* THEOREM 1.

(i) *For every $f\in\mathfrak{G}'_p$, the inequality*

$$I(\varphi)\geqq I(f)$$

holds. Here the equality sign appears if and only if $f\equiv\varphi$;

(ii) *For every $f\in\mathfrak{H}'_p$, the inequality*

$$I(\varphi)\leqq I(f)$$

holds. Here the equality sign appears if and only if $f\equiv\varphi$.

Proof. It is immediately seen that

$$(16)\quad \int_C \lg|\varphi|\,d\arg\varphi = 0$$

for φ of THEOREM 1 and thus $\varphi\in\mathfrak{G}'_p$ and $\varphi\in\mathfrak{H}'_p$. We note that

$$(17)\quad I(f) = \int_C \lg|f|\,d\arg f - 2\pi\lg I(f)$$

for any element f of $\mathfrak{G}'_p$ or $\mathfrak{H}'_p$.

Proof of (i). Let $f\in\mathfrak{G}'_p$. Then, by (f), (16), (17) and THEOREM 1,

$$-2\pi\lg I(\varphi) = J(\varphi)\leqq J(f)\leqq -2\pi\lg I(f)$$

and thus

$$I(\varphi)\geqq I(f).$$

Further, by THEOREM 1, the equality sign in the last inequality appears if and only if $f(z)\equiv\varphi(z)$.

Proof of (ii). Let $f\in\mathfrak{H}'_p$. Then, by (16) and (17), the equation

$$J(\varphi)-J(f)=-\int_C \lg|f|\,d\arg f + 2\pi(\lg I(f)-\lg I(\varphi))$$

holds. On the other hand, by (14), the equation

$$J(\varphi)-J(f)=-2\int_C \lg|f|\,d\arg f+D_B(h)$$

holds. Hence we have that

$$2\pi(\lg I(f)-\lg I(\varphi)) = -\int_C \lg|f|\,d\arg f+D_B(h)\geqq 0$$

and thus

$$I(\varphi)\leqq I(f).$$

The equality sign in the last inequality appears if and only if $h\equiv 0$ and thus $f(z)\equiv\varphi(z)$.

Similarly to COROLLARY 1 we have the following corollary of THEOREM 2 (cf. THEOREM 2 of [6]).

COROLLARY 2. (i) *Let ψ be the function defined in* (i) *of* COROLLARY 1. *Then for every $f\in\mathfrak{F}_p'$, the inequality*

$$I(\psi)\geqq I(f)$$

holds. Here the equality sign appears if and only if $f\equiv\psi$;

(ii) *Let χ be the function defined in* (ii) *of* COROLLARY 1. *Then for every $f\in\mathfrak{F}_p'$, the inequality*

$$I(\chi)\leqq I(f)$$

holds. Here the equality sign appears if and only if $f\equiv\chi$.

In the case $p=1$ in (1), we know that $\iota=\kappa=1$, $m_1=n_1=1$ and thus

$$I(f)=|f'(z_1)|.$$

Hence we have the following corollary of THEOREMS 1 and 2.

COROLLARY 3. (i) *There exists the unique element φ of $\mathfrak{G}_1$ and $\mathfrak{H}_1$ which univalently maps B onto the domain of which the boundary consists of circular slits (the images of $C_1,\cdots,C_\lambda$) centred at the origin and radial slits (the images of $C_{\lambda+1},\cdots,C_N$) emanating from the origin;*

(ii) *The function φ is the only element which simultaneously belongs to $\mathfrak{G}_1$ and $\mathfrak{H}_1$;*

(iii) *For every $f\in\mathfrak{G}_1'$, the inequality*

$$|\varphi'(z_1)|\geqq|f'(z_1)|$$

holds. Here the equality sign appears if and only if $f\equiv\varphi$;

(iv) *For every $f\in\mathfrak{H}_p'$, the inequality*

$$|\varphi'(z_1)|\leqq|f'(z_1)|$$

holds. Here the equality sign appears if and only if $f\equiv\varphi$.

4. Let $\mathfrak{F}_p''$, $\mathfrak{G}_p''$ and $\mathfrak{H}_p''$ be the subclasses of $\mathfrak{F}_p$, $\mathfrak{G}_p$ and $\mathfrak{H}_p$, respectively, which consist of functions $f(z)$ of $\mathfrak{F}_p$, $\mathfrak{G}_p$ and $\mathfrak{H}_p$ being *p-valent.*

LEMMA. $\mathfrak{F}_p''\subset\mathfrak{F}_p'$, $\mathfrak{G}_p''\subset\mathfrak{G}_p'$ and $\mathfrak{H}_p''\subset\mathfrak{H}_p'$.

Proof. Let $\{B_n\}_{n=1}^\infty$ be an exhaustion of B such that $z_j\in B_1(j=1,\cdots,\iota)$, $\zeta_k\in B_1(k=1,\cdots,\kappa)$ and such that ∂B_n consists of a finite number of analytic Jordan curves. Let $f(z)$ be an arbitrary element of $\mathfrak{F}_p''$ (, $\mathfrak{G}_p''$ or $\mathfrak{H}_p''$), and let F and F_n $(n=1,2,\cdots)$ be the image covering surfaces of B and B_n, respectively, by the mapping $w=f(z)$. We can take a sufficiently small positive number r such that ∂F_1 does not lie over $|w|\leqq r$ and $|w|\geqq 1/r$. Let F_r and F_{nr} $(n=1,2,\cdots)$ be the subsets of F and F_n, respectively, obtained by taking off from F and F_n the portions over $|w|\leqq r$ and $|w|\geqq 1/r$. Then, we find that

$$\begin{aligned}(18)\quad D_{F_r}(\lg|w|) &= \lim_{n\to\infty}\int_{\partial F_{nr}}\lg|w|\,d\arg w\\ &=\lim_{n\to\infty}\int_{\partial F_n}\lg|w|\,d\arg w-4\pi p\lg r\\ &=\int_C \lg|f|\,d\arg f-4\pi p\lg r.\end{aligned}$$

On the other hand,

$$\begin{aligned}(19)\quad D_{F_r}(\lg|w|)&\leqq p\,D_{\{r<|w|<1/r\}}(\lg|w|)\\ &=-4\pi p\lg r,\end{aligned}$$

for $f(z)$ is p-valent. By (18) and (19), we have that

$$\int_C \lg|f|\,d\arg f\leqq 0$$

and thus $f\in\mathfrak{F}_p'$ (, $f\in\mathfrak{G}_p'$ or $f\in\mathfrak{H}_p'$, resp.).

We note that $\varphi\in\mathfrak{G}_p''$ and $\varphi\in\mathfrak{H}_p''$ for the function φ defined in THEOREM 1. Then, by THEOREM 2 and LEMMA, we have immediately the following theorem.

THEOREM 3. *Let φ be the function defined in* THEOREM 1.

(i) *For every $f\in\mathfrak{G}_p''$, the inequality*

$$I(\varphi) \geq I(f)$$

holds. Here the equality sign appears if and only if $f \equiv \varphi$;

(ii) *For every* $f \in \mathfrak{H}_p''$, *the inequality*

$$I(\varphi) \leq I(f)$$

holds. Here the equality sign appears if and only if $f \equiv \varphi$.

We note that $\mathfrak{G}_1''$ (or $\mathfrak{H}_1''$) consists of all univalent functions $f(z)$ on B which satisfy the conditions

$$f(z_1) = 0,\ f(\infty) = \infty,\ f'(\infty) = 1$$

and (e) (or (e'), resp.) of **1.** Then we have the following corollary of THEOREM 3 (cf. [2], [3] and [4]).

COROLLARY 4. *Let* φ *be the function defined in* COROLLARY 3.

(i) *For every* $f \in \mathfrak{G}_1''$, *the inequality*

$$|\varphi'(z_1)| \geq |f'(z_1)|$$

holds. Here the equality sign appears if and only if $f \equiv \varphi$;

(ii) *For every* $f \in \mathfrak{H}_1''$, *the inequality*

$$|\varphi'(z_1)| \leq |f'(z_1)|$$

holds. Here the equality sign appears if and only if $f \equiv \varphi$.

If $C'' = \phi$ (or $C' = \phi$) in (i) (or (ii), resp.) of COROLLARY 4, the present consequences are reduced to the well-known classical results (cf. [1] and [8]).

REMARK. Each class $\mathfrak{F}_p''$ ($\mathfrak{G}_p''$ or $\mathfrak{H}_p''$) is a strict subclass of $\mathfrak{F}_p'$ ($\mathfrak{G}_p'$ or $\mathfrak{H}_p'$, resp.); i. e. $\mathfrak{F}_p'' \subsetneqq \mathfrak{F}_p'$ ($\mathfrak{G}_p'' \subsetneqq \mathfrak{G}_p'$ or $\mathfrak{H}_p'' \subsetneqq \mathfrak{H}_p'$). To see this, it is sufficient to show that there exists even the function of $\mathfrak{F}_p'$ ($\mathfrak{G}_p'$ or $\mathfrak{H}_p'$) of which the valence is *not bounded*. The detailed argument is omitted (cf. Example 1 of [6]). By the last assertion, we can infer that $\mathfrak{F}_p'$ ($\mathfrak{G}_p'$ or $\mathfrak{H}_p'$) is a class much larger than $\mathfrak{F}_p''$ ($\mathfrak{G}_p''$ or $\mathfrak{H}_p''$, resp.). THEOREM 2 (and COROLLARY 2) assert that φ (and ψ or χ) preserve the extremality with respect to the functional $I(f)$ even on such the classes $\mathfrak{G}_p'$ or $\mathfrak{H}_p'$ (and $\mathfrak{F}_p'$, resp.).

5. EXAMPLE 1. Does the function φ defined in THEOREM 1 preserve the maximality with respect to the functional $I(f)$ on the class $\mathfrak{G}_p$? The following example gives the negative answer for this question.

Let G be the whole w-plane slit along a circular arc

$$l' = \{w \mid |w| = 1,\ -\alpha \leq \arg w \leq \alpha\}$$

and a segment

$$l'' = \{w \mid \arg w = \pi,\ e^{-\rho} \leq |w| \leq e^{\rho}\}\ (\rho > 0),$$

and Δ' be the domain

$$\{w \mid e^{-\varepsilon} < |w| < e^{\varepsilon},\ -(\alpha+\varepsilon) < \arg w < \alpha+\varepsilon\}\quad (0<\alpha<\pi-\varepsilon,\ \varepsilon>0)$$

slit along l'. Let F be the covering surface over the w-plane obtained by the crosswise connection of Δ' and G along the common slit l'. Then F is a doubly-connected planar surface. Thus we can conformally map F onto the domain B of which the boundary consists of a circular slit C' (the image of $\partial\Delta' - l'$) centred at the origin and a radial slit C'' (the image of l'') emanating from the origin, and further may assume that the mapping function $z = g(w)$ satisfies the conditions

$$g(0) = 0,\ g(\infty) = \infty,\ g'(\infty) = 1.$$

The inverse function $w = f(z) \equiv g^{-1}(z)$ maps B onto F under the condition

$$f(0) = 0,\ f(\infty) = \infty,\ f'(\infty) = 1.$$

It is obvious that $f(z) \in \mathfrak{G}_1$. However $f(z) \notin \mathfrak{G}_1'$, for

$$\int_C \lg |f|\ d\arg f = \int_{\partial\Delta' - l'} \lg |w|\ d\arg w = 4\varepsilon(\alpha+\varepsilon) > 0.$$

Let B^* be the image domain of G by $g(w)$. Then we see that $\bar{B}^* - C'' \subset B$ and the restriction of $f(z)$ on B^* is the mapping function of B^* onto the domain G of which the boundary consists of the circular slit l' and the radial slit l''. Thus, by COROLLARY 4, we have that

$$|f'(0)| > 1.$$

On the other hand, $\varphi(z) \equiv z$ and thus $\varphi'(0) = 1$ for the present B. Consequently, we see that

$$|f'(0)| > \varphi'(0),$$

which rejects the maximality of $\varphi(z)$ with respect to $I(f)$ on the class $\mathfrak{G}_1$.

By an analogy of the present example, we can infer that the function φ of THEOREM 1 does not preserve the maximality with respect to the functional $I(f)$ on any class $\mathfrak{G}_p$.

6. EXAMPLE 2. Does the function φ defined in THEOREM 1 preserve the maximality (or minimality) with respect to the functional $J(f)$ (or $I(f)$, resp.) on the class $\mathfrak{H}_p$? The following example gives the negative answer for the both questions.

Let G, l' and l'' be the ones defined in EXAMPLE 1. Let Δ'' be the domain

$$\{w \mid e^{-(\rho+\varepsilon)} < |w| < e^{\rho+\varepsilon},\ \pi-\varepsilon < \arg w < \pi+\varepsilon\}\quad (0<\varepsilon<\pi)$$

slit along l''. Let F be the covering surface over w-plane obtained by the crosswise connection of Δ'' and G along the common slit l''. Then F is a doubly-connected planar surface. Thus we can conformally map F onto the domain B of which the boundary consists of a circular slit C' (the image of l') centred at the origin and a radial slit C'' (the image of $\partial\Delta'' - l''$) emanating from the origin, and further may assume that the mapping function $z=g(w)$ satisfies the conditions

$$g(0)=0,\ g(\infty)=\infty,\ g'(\infty)=1.$$

The inverse function $w=f(z)\equiv g^{-1}(z)$ maps B onto F under the condition

$$f(0)=0,\ f(\infty)=\infty,\ f'(\infty)=1.$$

It is obvious that $f(z)\in\mathfrak{H}_1$. However $f(z) \notin \mathfrak{H}_1'$, for

$$(20)\quad \int_C \lg |f|\, d\arg f = \int_{\partial\Delta''-l''} \lg |w|\, d\arg w = 4\pi(\rho+\varepsilon) > 0.$$

Let B^* be the image domain of G by $g(w)$. Then we see that $\bar{B}^* - C' \subset B$ and the restriction of $f(z)$ on B^* is the mapping function of B^* onto the domain G of which the boundary consists of the circular slit l' and the radial slit l''. Thus, by COROLLARY 4, we have that

$$|f'(0)| < 1.$$

On the other hand, $\varphi(z)\equiv z$ and thus $\varphi'(0)=1$ for the present B. Consequently, we see that

$$(21)\quad |f'(0)| < \varphi'(0),$$

which rejects the minimality of $\varphi(z)$ with respect to $I(f)$ on the class $\mathfrak{H}_1$. Further, by (20) and (21), we can also see that $\varphi(z)$ does not preserve the maximality with respect to $J(f)$ on the class $\mathfrak{H}_1$.

By an analogy of the present example, we can infer that the function φ of THEOREM 1 does not preserve the maximality (or minimality) with respect to the functional $J(f)$ (or $I(f)$, resp.) on any class $\mathfrak{H}_p$.

7. The present consequence suggests the possibility of an extension to the case of an infinitely-connected domain or an open Riemann surface of finite genus. We shall concern ourselves with the problem in the next paper.

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